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RESEARCH ARTICLE

Using Concrete-representational-abstract Integrated Sequence to Teach Geometry to Students who Struggle

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Abstract

The concrete-representational-abstract integrated (CRA-I) sequence is an explicit approach for teaching students who struggle in mathematics. This approach is beneficial because it assists students in the development of conceptual understanding. This article describes how the approach is used in general as well as its use with a specific geometry concept, area of a rectangle. The author will describe why one might choose CRA-I and the steps needed for implementation. Finally, the CRA-I steps will be shown with a lesson series related to teaching the concept of area. The article will describe lesson activities, methods, materials, and procedures.

Keywords mathematics, concrete-representational-abstract integrated, explicit instruction

Evidence-based approaches to teaching mathematics to students who struggle include explicit instruction and the use of multiple representations (Zhang et al., 2020). Explicit instruction involves preparing the student for instruction in the target concept, teacher modeling with the target skill, teacher guidance as the student practices the new skill, and allowing the student to demonstrate the skill independently. The concrete-representational-abstract integrated (CRA-I) sequence is an explicit way of teaching using multiple representations. Multiple representations include objects that students can physically manipulate (concrete), visual representations such as pictures, student drawings, or number lines (representational), and symbols such as numbers and operational symbols (abstract).

Explicit CRA-I instruction provides students who struggle with experiences that assist them in understanding the meaning of mathematics and the language associated with

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it. Learning and remembering mathematical language or vocabulary is greater when students perform tasks during learning experiences and see an individual demonstrate actions associated with the language/vocabulary (Badinlou et al., 2017). For example, manipulation of concrete objects or drawing to solve algebraic equations involves physically moving items or drawing to combine, remove, or find the difference between quantities. The experience with objects and pictures/drawings allows students to see how the operation works. Seeing the operation in action or being a part of the action, accompanies verbal explanation using vocabulary and mathematical symbols. A teacher can tell a student that *equal* (=) means *same* or that *difference* means *how many numbers are between two quantities*. Concrete and representational experiences provide information that one processes physically and/or visually when a teacher uses groups of counters, a balance scale, and/or a number line. Vocabulary instruction within CRA-I ensures that the student receives information about vocabulary and symbols visually/kinesthetically as well as auditorily.

The first phase of CRA-I provides physical (concrete) and visual (representational) models while completing abstract tasks, but the phases change by removing concrete models followed by removing representational models. The rationale is to scaffold students' ability to work solely in the abstract. Students have repeated practice in which they process information and form conceptual understanding of the abstract with concrete and representational models. Next, they have repeated practice in which they process information and continue to build conceptual understanding of the abstract with representational models. This provides students with understanding of the abstract concept and associated vocabulary needed in order to work in the abstract. This understanding is important for solving problems, making decisions, and judging one's work as students engage in mathematical practices.

Successful mathematics learning includes engagement in mathematical practices. However, students who struggle may not have the foundational conceptual understanding to be successful. This occurs when students have some understanding of numbers, symbols, algorithms, or formulas, but they do not know how, why, and when to use them (Zwanch & Wilkins, 2021). For example, a student may quickly and accurately add and multiply single-digit numbers, but not understand the difference between the operations conceptually. This student would have difficulty engaging in problem solving or using operations in more complex ways. In contrast, CRA-I instruction provides experiences with multiple representations to develop students' understanding of abstract concepts so that they can engage in mathematical practices and fluently use numbers, symbols, formulas, and algorithms. The following sections will describe the CRA-I process and provide examples of its use in geometry.

I. WHAT IS CRA-I

CRA-I is a systematic approach to using multiple representations (Morano et al., 2020; Strickland & Maccini, 2013). It is systematic because the teacher organizes lesson materials in a particular order. First, the teacher provides explicit instruction using concrete objects, two-dimensional representations, and abstract symbols (e.g., equations). In each lesson, the students solve problems or complete mathematical tasks using all three types of representation (concrete, representational, and abstract). A phase one CRA-I lesson follows with an example showing how one would teach addition with single-digit numerals and sums to nine.

The learning sheet would present equations with addends to nine with sums 0-9. Students would use interlocking blocks as concrete objects, and number lines as visual representations with the length between each numeral equal to the length of each block. Students would learn how to solve given equations using both types of concrete and representational models. For example, they would combine two and three using blocks (concrete) to obtain a sum of five, and they would use a number line with a block set beside each numeral to show counting onto two with three to arrive at the sum of five. Both the blocks and the points on the number line show the sum. Once students can independently solve equations using both representations with accuracy (e.g., 80%-100% of sums correct), the teacher designs the next set of lessons according to the second phase.

The second phase of instruction fades the use of concrete objects, meaning that students only use visual representations without manipulating physical objects. Continuing along with the concept of addition, the learning sheet would present equations, pictorial representations of each number, and number lines (without blocks). Rather than physically manipulating blocks, the student would use the given pictures and draw sums. They would use number lines to count on, but without the blocks. This might involve marking the number line with an arc from one numeral to another while counting on. Students' mastery in phase one would have indicated to the teacher that students developed some understanding of the operation; continued success in phase two shows that students' understanding further develops.

The last phase of instruction fades concrete and representational models. Students complete equations using just numbers and symbols. During the abstract phase, the focus of instruction is fluency and mental strategies.

Assisting students in developing mental strategies, instruction using materials and procedures from the previous phases would show students how mental strategies work. For example, counting-on is a rudimentary mental strategy. Using the larger addend is more efficient (Bryant, 2021). In using the number line, the teacher can show students why counting on to the larger number is the most efficient approach (e.g., when solving 5 + 2, begin with five and count-on two more). A student is less likely to make an error with this approach (Tournaki, 2003). Counting-on involves counting the next number, and the number line explicitly shows this. According Nys and Content (2010), students who struggle frequently begin counting with the given number and consistently arrive at a sum that one less than the correct sum (e.g., when solving 7 + 2, the student counts *seven, eight* rather than *eight, nine*). Use of a number line and blocks shows this process and that there are fewer numbers to count when beginning with the larger addend.

Another mental strategy is using doubles plus one (4 + 3 is the same as 3 + 3 + 1). Using objects or drawings systematically, the teacher can show this strategy by systematically making sums. For example, when combining addends (three and four) with pictures or blocks, the teacher can place/draw three and pause while making four. When there are two groups of three (six), the teacher puts the last (fourth) picture/block off to the side of the other blocks/pictures. The teacher shows the composition of seven as 3 + 3 + 1. The teacher and students discuss their knowledge of doubles and the ease with which they can count one more to arrive at a sum. Practice with concrete and representational models gives students practice in using actions that will become mental strategies.

II. EXPLICIT INSTRUCTION AND CRA-I

Research on CRA-I for students who struggle includes explicit instruction (Flores & Hinton, 2022; Flores et al., 2022; Morano et al., 2020). There are five steps within explicit instruction (Bryant, 2021). The first step is preparing the student for the lesson; this may be called an advance organizer. The second step is teacher modeling. The third step is guided practice. The fourth step is independent practice. The last step is wrapping up the lesson; this may be called a post organizer. The contents of an explicit lesson plan are summarized in Table 1 and described in greater detail below.

Table 1. Components of an Explicit Lesson Plan

Advance Organizer – Prepare students for the lesson.	
Tasks	Review pre-requisite skills (e.g., multiplication prior to lesson that includes formula for area).
	State the lesson topic in terms students understand (e.g., reading the National Curriculum standard would not be terms easily understood by students).
	Tell students how the lesson topic can be used in their lives (e.g., use their hobbies, community experiences, and interests).
	Tell students about the lesson activities and what you expect them to do in each (e.g., answer questions, use objects, help solve problems, solve problems with the teacher's help)
Modeling – Teacher demonstrates tasks and keeps students engaged.	
Tasks	Complete items by physically showing what to do. Think aloud about the steps.
	Engage all students in each item by asking for assistance with tasks or asking for verbal responses for which they are proficient and competent based on prerequisite skills.
Guided Practice – Teacher and students carry out tasks together.	
Tasks	Give students portions of each task to complete.
	Provide support as students complete a task and fade your assistance.
	Increase the students' portion of task completion as they show competence.
Independent Practice – Students complete tasks without teacher assistance.	
Tasks	Give students tasks that require the same skills shown and practiced in the lesson.
	Observe students as they complete items, but do not assist.
	If there is an error in a completed item, provide feedback before student completes another.
Post Organizer – Teacher summarizes the lesson.	
Tasks	Tell about the important concepts in the lesson.
	Praise students for their efforts.
	Tell how current lesson will be connected to the next lesson on the topic.

Advance Organizer

This step ensures that the student is ready to begin learning the target concept or skill. According to Hudson and Miller (2006), this begins with a brief review a prerequisite skill. Using the addition example above, a pre-requisite skill is one-to-one correspondence and cardinality. These are counting skills in which the student shows that they count once or each time they touch one object, and the student reports the total as the last number counted. Since the first lesson involves counting physical objects, these foundational counting skills are necessary. The review should be brief, and the student completes it independently. The teacher would ask, *Will you count these, please?* Next, the teacher briefly tells about the target concept or skill to be learned, using language that students understand, and situates the target concept or skill. For example, *We have been adding numbers that have sums from 0-5. Remember that sum means the total amount. Today, we are going to add numbers that have sums to nine and we are going to use blocks and number lines to help find the sums.*

After telling students what they will learn, the teacher makes it relevant to their experiences in their daily lives. Students are more likely to be engaged in the lesson when they can imagine how the skill or concept could be used in their favorite activities. Continuing with the addition example, the teacher would talk about adding points, materials, ingredients, or food items that are related to the student's favorite sport, game, craft activity, cooking, or snack. It is very important to think about the student's actual preferences and experience rather than just giving your own example. For example, I personally enjoy basketball; however, this may not be relevant to the students I teach. Using a basketball scenario would not be engaging to students who do not like to or have experience playing/watching the sport. Using such an example could be counter-productive and students may be left thinking, *I don't like math and I don't like/know about basketball. If that is what adding is for, then I don't care about this.*

Finally, the last part of an advance organizer is telling students what to expect during the lesson. This is important because it lets students know what is coming and in what order. Students who struggle may be anxious about certain tasks. Telling students exactly what will happen, can ease that anxiety because they will know that the potentially difficult task will come third or maybe that task is not even part of this lesson. Be specific about what you want students to do. Here is an example. *We are going to add using blocks and number lines. First, I will show you how and you will repeat things that I say and answer questions with my help. Then, we are going to add numbers together and you will have a turn and I will have a turn. Last, you will use blocks and number lines to add numbers without my help.*

Modeling

The second step in explicit instruction is modeling or teacher demonstration. The teacher physically shows the task and talks aloud while completing it. This allows the students to hear the thinking process and see the physical actions involved in the task. It is important that the students are engaged, so while modeling, the teacher asks students to repeat information and tell about or do things they already know. Here is an example of asking students to repeat things. *I am using the number line to add. First, I show the largest addend on the number line. Which addend do I show on the number line?* The expected response is *Largest.* Here is an example of asking students to do something they already know. *I am going to combine the addends. I put down five and three. Count them with me.* The teacher and students count in unison to eight. It is important to move at a brisk pace; providing easy and relevant tasks will keep their attention. Otherwise, students' attention may be lost because sitting and watching may be boring.

Guided Practice

The third step in explicit instruction is guided practice. The teacher and the students carry out tasks together; all are actively involved in the task. Taking turns is one way to accomplish this. For example, break the task of solving an addition equation into parts. The teacher makes one addend with blocks and students make the other addend with blocks. The teacher puts down one part of the sum under/next to the equal symbol and the students put down the other part of the sum. They all count the sum together in unison. During the student's turn, the teacher can tell the student what to do, ask a question, or provide a reminder (Archer & Hughes, 2011). Examples of telling the student are *It is your turn; make the next group* or *Make the next group of six.* The second direction provides a greater level of assistance because it tells how many are in the group. Examples of asking a question are *How many will you put in the next group?* or *Make a group like mine with the next addend* (point to the addend)*; how many will you make?* The second question provides a prompt that will increase the likelihood of the correct response. An example of a reminder is *Remember how I made a group of three; you make one like that with the next addend*. Providing students with prompts or hints will assist students in completing tasks. As the teacher and the students complete items, it is optimal to fade prompts and give students more tasks within successive guided practice items. Guided practice allows the teacher to informally assess students' progress; the teacher gages how much or how little assistance students need. The teacher uses that information to decide whether more modeling or guided practice is needed or if the students are ready for the next step which is independent practice.

Independent Practice and Post Organizer

The fourth step is independent practice. The teacher gives the students tasks to complete without assistance. Using the addition example, the teacher would provide each student with several addition equations, blocks and a number line, and the students would use them to find the sums. The task complexity and characteristics are the same as those in modeling and guided practice; the only difference is that the teacher is not involved in completing the tasks. The teacher should observe students while they complete independent practice. It is helpful to notice an error and provide feedback sooner rather than later. For example, if the teacher notices an error in the first equation, stop the student and provide feedback using guided practice to correct the specific error. Then, ask the student to continue with the next equation so that the teacher can ensure that the student used the feedback.

The final step is the post organizer. This is a brief re-counting of the lesson highlights. The teacher can give general feedback about the students' overall performance. Last, the teacher connects this lesson to the future lesson. For example, *Today we added numbers with sums to nine using blocks and number lines. Sometimes we had trouble counting on the number line, but we were sure to make a dark mark at the largest addend, place our finger there, and we only counted after our finger moved once. Tomorrow, we are going add using pictures instead of the blocks.*

III. EXPLICIT CRA-I AND A GEOMETRY EXAMPLE OVERVIEW

Based on the author's experiences with students and CRA-I, here are ways to use CRA-I to teach students who struggle with the concept of area. Students may struggle because they do not understand the concept of square units. They may struggle because they need additional experiences with multiple representations. Since the National Curriculum calls for the use of multiple representations to teach the area of a quadrilateral and the related formula, here is how CRA-I can assist students who need remediation or supplementary instruction. Since students' previous experience with instruction according to the National Curriculum already included similar representations, the purpose of CRA-I is to bring those materials together and make explicit connections between representations and the abstract formula.

CRA-I is explicit and presents all the representations together, fading representations systematically to the abstract across three phases. Materials for phase one would be concrete, representational, and abstract. For example, the concrete materials would be plastic, square tiles (concrete) and learning sheets with simple word problems describing situations related to finding area, rectangles with white space in the middle. Plastic tiles would allow for concrete instruction (filling the space with tiles). Representational materials would be rectangles with equally spaced columns and rows (using same sized units as the plastic tiles). Explicit instruction would involve shading columns with a certain number of cells in each column. Both concrete and representational materials would be used to show multiplication of length and width, using the abstract formula.

The materials for phase two would continue using word problems with the rectangles with a grid system described above. Students would create their own grid system by drawing on a given rectangle. Supports for drawing one's one grid would include a straight edge and equally spaced marks along the length and width of the rectangle. The student uses the straight to draw a vertical or horizontal line each mark. Students may need additional supports for drawing their own square units which include marking each if the two lengths and widths in a rectangle. Students position the ruler vertically/horizontally on each of the first, second, third, etc. marks and draw a line. Another support might include printing the shape on graph paper or using an overlay with a printed grid from graph paper. This process of making one' own grid actively involves the student in creating equally sized (standard) square units. Explicit instruction would lead to the use of multiplication equations to find area.

Phase three would fade the representational materials, and students would solve word problems. To ensure that students understand the concept of area, these word problems would require other tasks, such as simple addition or subtraction. This would ensure that students understand the concept and can discriminate between tasks and operations, using mathematics practices.

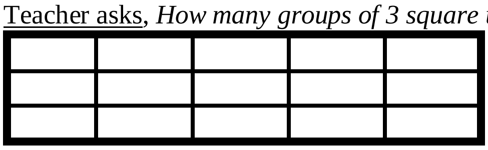
IV. CONCRETE, REPRESENTATIONAL, AND ABSTRACT LESSONS

As described above, the first set of CRA-I lessons present all the models in each lesson with concrete, representational and abstract. This section will describe each explicit lesson component in which students learn to find the area of a rectangle. Samples of the teacher's language and guidance and pictorial examples are included.

Advance Organizer

The teacher reviews a prerequisite skill, multiplication using arrays. The teacher describes what students will learn and how it is relevant to their experiences. The teacher explains what they will do during the lesson so that students know what is expected of them. See Figure 1.

Review Activity
Teacher asks, *How many groups of 3 square units fill this rectangle?* (expected response is five)



Teacher asks, *Show me a multiplication equation that shows five groups of three.*
Teacher asks, *How many square units are in the rectangle?*

Teacher Describes Target Concept
Teacher says, *Today we are going to solve word problems about the area of rectangles, or the space that fills the inside of the shape. We are going to use plastic tiles and pictures to help us do that*

Teacher Makes Target Concept Relevant
Teacher says, *You told me about decorating the room you share with your sister and painting one wall a different color. Today we will do something that helps in planning that project. Cans of paint tell how much space that one can of paint can cover. If you know the amount of space on the wall, you can buy the just the right amount of paint for your project. This may leave more money for the other things you want for decoration. You talked about pictures, but what are some other things on your wish list for your room? Finding the area of a rectangle, like your bedroom wall, can help us with projects that are fun.*

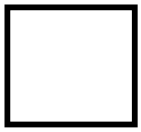
Teacher States Expectations for Lesson
Teacher says, *We are going to solve word problems about finding area using plastic tiles and pictures. I am going to show you how, and you will help. Then we are going to solve problems together, taking turns. Finally, you will solve problems using tiles and pictures by yourself. I want to you answer questions, help me, and actively participate in solving the word problems.*

Figure 1. Advance Organizer Example

Modeling

During modeling, the teacher directs the lesson activities while keeping the students engaged. The teacher fosters engagement by asking students to count, repeat information, and answer questions about concepts they have already mastered (e.g., single-digit multiplication). The teacher thinks aloud while manipulating the objects or using pictures to solve problems about finding area. The learning sheet for concrete instruction is shown in Figure 2. The teacher's behaviors for concrete instruction are described and shown in Figure 3. The teacher lays out plastic, square tiles on a diagram on the learning sheet. Students participate by repeating information, counting tiles aloud as the teacher makes equal-sized groups. The teacher explains the use and rationale for square units. The teacher brings students attention to the array of six columns of four tiles. Students say that the total can be found with multiplication. This is background knowledge that students already have. The teacher asks the students to identify the product (24) and explains that this is 24 square feet, the area inside of the rectangle. The teacher explains that finding the area involves multiplying the length and the width which they did using the square objects. This modeling activity connects lessons that students previously learned using the National Curriculum in one activity, connecting the concrete to the abstract formula. Further, the teacher is explicitly showing while keeping students engaged in the model.

Carpet tiles are square. They are 1 foot on each side. Each tile is one square foot.



There is a room that is in the shape of a rectangle. It is 4 feet wide and 6 feet in length. How many carpet tiles would cover the whole floor ?

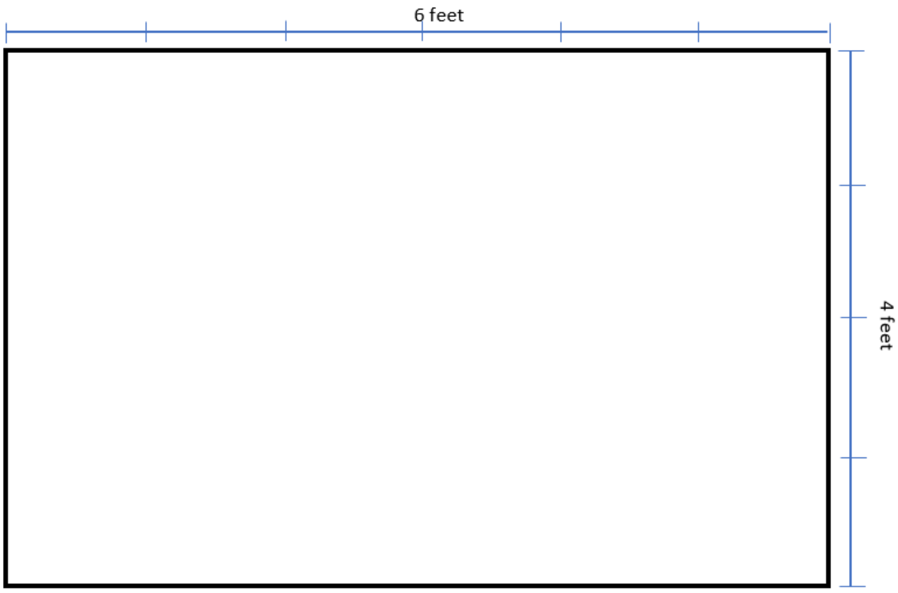
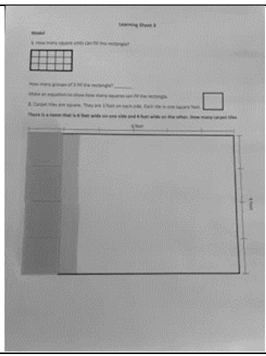


Figure 2. Learning Sheet Example for Modeling Concrete Instruction

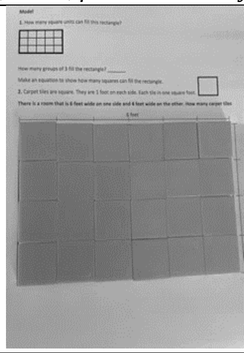
Example of Engagement with Students Repeating
Carpet tiles are square. Look at the picture. What shape are the carpet tiles? Yes, they are one foot on each side. Each tile is one square foot. The measurement for each tile is one square foot. How much does each tile measure?
I will read the word problem. There is a room that is 4 feet wide on one side and 6 feet long on the other side. How many carpet tiles will cover the whole floor? What does the problem ask me to find?

Example of Teacher Modeling with Concrete Materials
I will use these square tiles to represent the carpet tiles. I am going to put them out in groups.

Example of Keeping Students Engaged
Count with me as I make groups of tiles. One, two, three, four. How many tiles are this group?



Teacher makes a column of four plastic, square tiles on the diagram of the room. Students count as the teacher lays down blocks.



Teacher continues to make columns of four plastic, square tiles on the diagram of the room until it is filled. Students count as the teacher lays down blocks.

Teacher Makes Connection to Multiplication
I just made six groups of four tiles. There are six columns with four tiles in each column. We are going to find the total number of square-foot tiles.

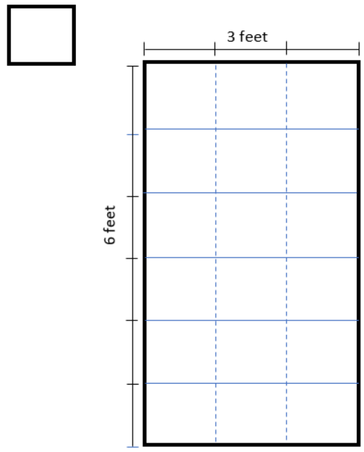
Example of Keeping Students Engaged by Asking About Previously Mastered Skills
When we combine things that are in groups of the same size, what is the operation?
We have 6 groups, and how many are in each group?
So, we multiply six and four. What is 6 x 4?
Do we have 24 tiles here? Do they fill up all the space in our rectangular room?
So, if we wanted to find the space inside of this rectangle, we would multiply the length of 6 feet and the width of 4 feet and we find that that area is 24 square feet. How many square feet do we have?

Figure 3. Teacher Modeling Behaviors for Concrete Instruction

The teacher demonstrates another example but uses a representational model. There is a word problem which the teacher solves using a picture. The teacher again, explains that they need to fill the space with square units. There is a rectangle that has measurements on the side and equally spaced square cells. The teacher shades the columns systematically to find the area. Just as with the concrete model, the teacher engages students, and makes the connection between shading the rectangle and multiplication. The teacher asks the students to find the product and explains the connection between their actions and the formula for area of a rectangle. Again, this activity brings multiple previous experiences together in one activity, connecting the representational model to the abstract with explicit modeling. See the example in Figure 4.

Learning Sheet Example for Representational Instruction

2. There is a closet that is 3 feet wide on one side and 6 feet in length. How many square feet is the floor of the closet?



Example of Engagement with Students Repeating

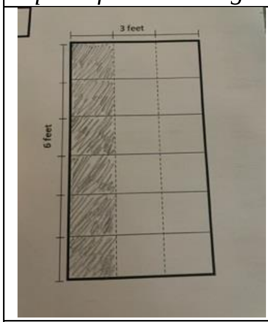
I will read the word problem. There is a closet that is 3 feet wide on one side and 6 feet in length. How many square feet is the floor of the closet?

Example of Teacher Modeling with Representational Materials

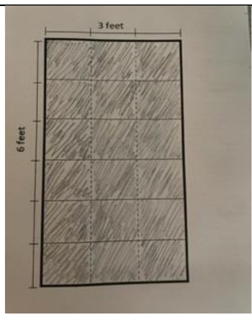
I look at the picture of the closet floor. The measurements are labeled, 3 feet wide and 6 feet in length. There are lines drawn at each foot, showing square feet. I will shade these square feet in groups.

Example of Keeping Students Engaged

Count the square feet with me as I shade this column. One, two, three, four, five six. How many square feet are this group?



Teacher makes a column of six shaded square feet. Students count as the teacher shades each.



Teacher continues to make columns of six shaded square feet unit the picture is filled. Students count as the teacher shades.

Teacher Makes Connection to Multiplication

I just made three groups of six. There are three columns with six square feet in each. We are going to find the total number of square feet.

Example of Keeping Engagement by Asking About Previously Mastered Skills and Skills Just Taught

When we combine things that are in groups of the same size, what is the operation?
We have 3 groups, and how many are in each group?
So, we multiply three and six. What is 3 x 6?
Do we have 18 filled squares here? Do they fill up all the space of our closet floor?
So, if we wanted to find the space inside of this closet, we would multiply the length of 6 feet and the width of 3 feet and we find that that area is 18 square feet. How many square feet do we have?

Figure 4. Teacher Modeling Behaviors for Representational Instruction

Guided Practice

After modeling concrete and representational examples, the teacher conducts guided practice. This is an important component of a lesson because the students have physical tasks to complete along with the teacher. They use the same concrete and representational models and actions observed when the teacher demonstrated.

The teacher and students would use word problems like those modeled and complete them together. For example, the teacher would read the word problem. She asks students to use the concrete objects (square tiles) to make a group, and the teacher and students count together as they assemble the tiles on the learning sheet. The teacher would take a turn and make another group of tiles and they would count together as they assemble the tiles on the learning sheet. They would continue until they filled the space. The teacher would talk about the number of groups (columns) made, and the students would tell the teacher how many were in each group. The students would write the corresponding multiplication equation with the product. They would talk about finding the area by multiplying the length and width. The teacher and the students would continue this process with a representational example in which they shade a picture associated with a word problem. If the students had difficulty responding or completing a task, the teacher would provide a hint or a prompt to assist them in responding. During guided practice, the students participate with the teacher's assistance, making the connection between concrete models, representational models, and the abstract formula. This is what differentiates CRA-I instruction from the student's previous experiences across separate lessons using the National Curriculum.

Independent Practice

After guided practice, the teacher would ask students to complete items like those used for modeling and guided practice. They would use concrete objects and pictorial representations to complete word problems independently without the teacher's assistance. The teacher may read word problems aloud if students have reading difficulties. The teacher observes the students as they complete problems. The teacher provides feedback to students as they complete items to correct errors and/or lets students know that they completed the items correctly. If a student makes an error, correction should involve modeling and guided practice so that the student understands the mistake and how to correct it.

Post Organizer

The last portion of a lesson is the post organizer in which the teacher highlights the important part so the lesson. A teacher might say, *Today we used objects and drawings to find the area or the space inside of a rectangle. The objects and pictures helped us fill the space with square units in groups that were the same size. We used those groups to make multiplication equations. We learned that we could find the area of a rectangle by multiplying the length and the width.* The teacher would end by giving general feedback about student's performance and telling what will be in the next lesson. The next lesson might involve continued practice with the same concept and materials, or it might move to the next phase of CRA-I in which the concrete items would be faded, and students would use pictures and make their own drawings.

V. REPRESENTATIONAL AND ABSTRACT LESSONS

The next phase of CRA-I instruction is using representational and abstract models. The concrete objects are faded or removed from instruction. The representational models shown in previous figures would be used. A lesson might also include a partially complete representation that students complete with their own drawing. For example, it would show a rectangle with measurements along the length and width (tick marks at each unit). Students would use those and a ruler/straight edge to draw lines showing the square units inside of the rectangle. They would use those to continue the steps used in previous lessons with representational instruction. See an example of the drawing process in Figure 5.

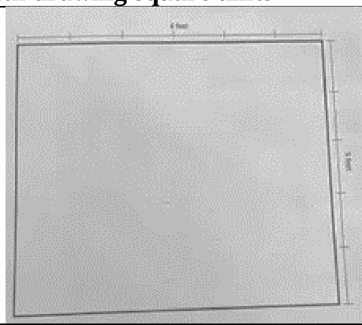
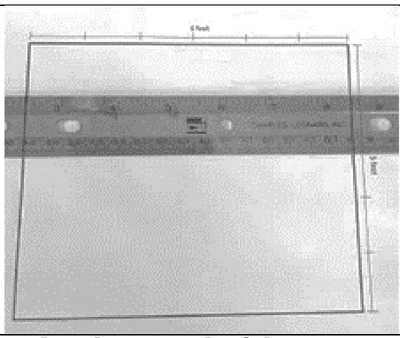
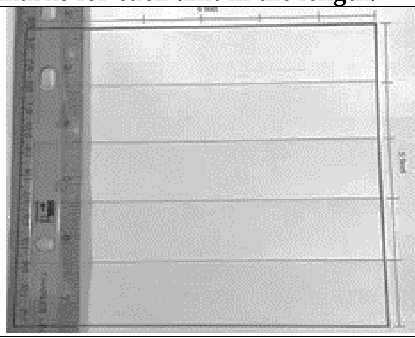
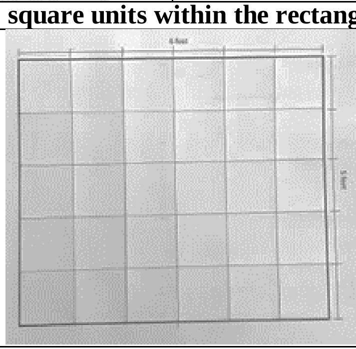
<table>
<tr><td colspan="2">Representational model shows a rectangle. Measurements along length and width have tick marks at each unit to assist with drawing square units</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td>Student uses ruler/straight edge to draw horizontal lines. Places ruler/straight edge at the tick marks for each unit in the width.</td><td>Student uses ruler/straight edge to draw vertical lines. Places ruler/straight edge at the tick marks for each unit in the length.</td></tr>
<tr><td></td><td></td></tr>
<tr><td colspan="2">Student has drawn each of the square units within the rectangle.</td></tr>
<tr><td colspan="2"></td></tr>
</table>

Figure 5. Representational Model with Drawings

Kim is painting one wall in her bedroom a bright color. A small can of paint says that it will cover 100 square feet. Kim's bedroom wall is 9 feet high and 10 feet in length. How many square feet will Kim need to paint her wall?

Paul is painting the rooms in his house that has two levels. He needs 6 cans of paint for the first floor and 7 cans of paint for the second floor. How many cans of paint will Paul need in all?

May will put new floor tiles in the hallway. Each tile is one square foot. If the hallway is 4 feet wide and 18 feet long, how many square-foot tiles will May need?

Jo's new office is 95 square feet. His old office was 80 square feet. How much bigger is Jo's new office compared to his old office?

Corey used 8 tiles to decorate a table and 6 tiles to decorate the seat that matches the table. How many tiles did Corey use?

Figure 6. Types of Word Problems for Discrimination at the Abstract Phase

VI. ABSTRACT LESSONS

The last phase of CRA-I involves just abstract representations. The students would learn how to solve word problems involving area using just the formula, length times width. To ensure that students are reading the word problems, attending to the question, and showing that they understand area, these lessons should involve other operations. If they only included word problems about area, a student might just approach each problem without using a thinking process, just finding the numbers and multiplying. Examples of word problems for abstract lesson are in Figure 6.

VII. WHY USE CRA-I

Now that you have seen an example lesson, it is important to highlight the benefits of such instruction. The CRA-I sequence provides students with hands-on and visual assistance in understanding the abstract calculation of area. All the representations used within the National Curriculum are included in the first phase. With the concrete objects and pictures, students can see the space inside of the rectangle and the rationale for using square units is shown. They can connect the concept of area to their knowledge about the multiplication operation. For students who struggle, they may have gaps in their multiplication understanding. CRA-I provides an opportunity to address those gaps in learning while teaching a more complex concept such as area. For example, students who struggle may not have a firm understanding of multiplicative reasoning. In contrast to additive reasoning, one multiplier acts on the other multiplier. Using arrays, physical objects (tiles), and shading, assists students in seeing this process in action. The teacher models and shows the process; the teacher keeps students engaged as they watch. Simply watching is not sufficient, so guided practice allows students to show multiplication with objects, pictures, and drawings. Guided practice allows the teacher to observe students' actions in showing the operation and to assess their understanding. Modeling and guided practice serve to prepare the student for independent work. While learning a geometric concept using CRA-I, the teacher can ensure that students understand foundational concepts such a multiplicative reasoning. CRA-I's explicit steps scaffold students' progress toward independence.

CRA-I assists students because they learn a concept in small steps. Teacher modeling and guidance increases the likelihood that students will be successful in completing those steps independently. This is particularly helpful for students who struggle in mathematics because they have an increased chance of being successful, building their confidence in their mathematical abilities. Students who struggle have many previous experiences with failure which leads to hesitance and apprehension. CRA-I's explicit nature provides students with learning scaffolds and predictable instruction. They will know what to expect when moving from one instructional step to another, increasing their confidence.

Another aspect of explicit instruction as included in CRA-I is the use of relevance. The examples in this article show real-life application to students' experiences. Choose examples that are relevant to your students' interests and experiences. Students who struggle in mathematics likely have past experiences with failure which may interfere with their enthusiasm for remedial instruction. Using word problems that show how a concept can assist them in pursuing their own interests may increase their engagement.

Explicit instruction may seem contrary to inquiry. However, explicit CRA-I will prepare students who struggle for inquiry because they will have the conceptual understanding with which to solve novel problems in different ways. For example, understanding the concept of area for rectangles can prepare students to investigate situations in which they find the area of other geometric shapes or surface area. They will have the conceptual understanding with which to make attempts at finding the area of other quadrilaterals or to find the surface area of rectangular prism.

Another benefit of CRA-I is the movement from one representation to another. Students begin with all representations together, mastering the skill with both concrete and representational models. When concrete is removed, students still have the previous experience with representational models with which to continue building their understanding. The representational models in this example became more complex, requiring students to make their own same-sized groups of square units. This further expands their understanding of the concept and helps students understand the language, *square unit* because they create their own square units.

CRA-I also supports students' language development. The first two phases of

CRA-I build students' language and vocabulary by providing physical and visual experiences in additional to verbal explanation. For example, the terms *square feet* or *square inches* can be seen as well as physically manipulated as students fill rectangles. Students process new information auditorily, kinesthetically, and visually. Students who struggle may view labeling of area using square units as arbitrary if they only received verbal explanations or did not have sufficient practice with multiple representation. CRA-I combines multiple representations, provides explicit and repeated practice so that students understand why the area of a rectangle isn't 20 inches, but 20 square inches.

VIII. CONCLUSIONS

CRA-I requires simple resources that are available in mathematics classrooms, especially since the National Curriculum includes the same types of representation and their use is a recommended practice in mathematics (McLeskey et al., 2022). Although the materials described above are common and commercially available, teachers could make them out of card stock. CRA-I does not take more time than other remedial approaches. Researchers have implemented lessons in 20-30 minutes, consistent with remedial intervention periods in schools (Flores et al., 2022; Morano et al., 2020; Strickland & Maccini, 2013). What makes CRA-I different from other instructional approaches is the inclusion of all representations together and the systematic fading to the abstract. Another difference is explicit instruction. Explicit instruction is a high leverage practice, which means that research has shown it to be highly effective, especially for students who struggle. Although this article only elaborates on one geometry concept, CRA-I can be applied to almost any mathematical concept as related lines of research have been effective for teaching number concepts, operations, fractions, and algebra (Bouck et al., 2018; Peltier et al., 2020).

CRA-I has been presented as a remedial approach to teaching mathematics. However, there may be times in which a teacher might use this within general instruction for all students. Perhaps CRA-I can be used when introducing a new concept or complex concept. The teacher can ensure that students see the foundational concepts in action and visually. A few lessons might provide preparation for inquiry in which students use what they learned during CRA-I lessons to produce novel solutions. CRA-I might be used to organize the way in which teachers present the representations prescribed in the National Curriculum.

There may be aspects of explicit instruction that can assist a teacher with informal assessment of student progress, saving time overall. For example, guided practice allows a teacher to see students work as they take a turn, or a teacher can note how many prompts were needed during students' turn. The teacher can take immediate action in re-teaching or using another example to correct student errors. This may be more efficient than waiting to score homework assignments or other independent tasks in which students make consistent errors. By that time, students may have repeatedly practiced errors, and re-teaching may be more involved.

In conclusion, CRA-I is an approach that requires few resources. It requires that teachers use their instructional time differently by modeling and guiding student practice. It is especially effective for students who struggle in mathematics because CRA-I combines representations and teacher support is faded as students move to independence. It is most important that students' eventual mastery of abstract tasks such as completing algebraic equations or using formulas and algorithms is based in conceptual understanding. CRA-I's systematicity and organization may assist teachers in a variety of situations to meet students' needs.

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