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PELL AND PELL-LUCAS NUMBERS WHICH ARE CONCATENATIONS OF TWO REPDIGITS

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Abstract. In this study, we search for Pell and Pell-Lucas numbers, which are concatenations of two repdigits and find these numbers to be only $12, 29, 70$ and $14, 34, 82$, respectively. We use Baker's Theory and Baker-Davenport basis reduction method while finding the solutions.

1. Introduction

Let $(P_k)_{k\geq 0}$ be the sequence of Pell numbers given by

$$P_0 = 0,\ P_1 = 1; P_k = 2P_{k-1} + P_{k-2} \text{ for } k \geq 2$$

and $(Q_k)_{k\geq 0}$ be the sequence of Pell-Lucas numbers given by

$$Q_0 = 2,\ Q_1 = 2;\ Q_k = 2Q_{k-1} + Q_{k-2} \text{ for } k \geq 2.$$

$\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. It is clear that $2 < \alpha < 3$, $-1 < \beta < 0$ and $\alpha\beta = -1$. Moreover, it is well known that

$$P_k = \frac{\alpha^k - \beta^k}{2\sqrt{2}} \text{ and } Q_k = \alpha^k + \beta^k.$$

The equalities are called the Binet formulas. The following inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \quad \text{for } n \geq 0 \tag{1}$$

and

$$\alpha^{n-1} \leq Q_n \leq 2\alpha^n \text{ for } n \geq 1 \tag{2}$$

can be proved by the induction method. A non-negative integer is called a base b-repdigit if its all digits are the same in base b. Let N be a non-negative

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integer. If N is a repdigit, then it is written as

$$N = \frac{d(10^m - 1)}{9} = \overline{\underbrace{d \cdots d}_{m \text{ times}}}$$

for some non-negative integers d, m with $0 \leq d \leq 9$ and $m \geq 1$. In [10], Faye and Luca have determined the largest repdigits in the Pell and Pell-Lucas numbers as $P_3 = 5$ as $Q_2 = 6$ respectively. If the form of M is

$$M = \overline{\underbrace{d_1...d_1}_{m_1 \text{ times}} \underbrace{d_2...d_2}_{m_2 \text{ times}} ... \underbrace{d_k...d_k}_{m_k \text{ times}}},$$

then it is said that M is a concatenations of k repdigits for some non-negative integers with $k \geq 1$, $0 \leq d_k \leq 9$, and $d_1 \geq 1$. The concatenations of two repdigits of different sequences have been studied by some authors. The solutions of equations of this type are given in [1] for Fibonacci numbers, in [6] for Padovan numbers, in [7] for Tribonacci numbers, and in [12] for Balancing numbers. In this study, we discussed the solutions of the Diophantine equations

$$P_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}} \underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}} \tag{3}$$

and

$$Q_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}} \underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}}, \tag{4}$$

where $d_1, m_1, m_2 \geq 1$ and $d_1, d_2 \in \{0, 1, ..., 9\}$. That is, we determined all Pell and Pell-Lucas numbers that are concatenations of two repdigits as $\{12, 29, 70\}$ and $\{14, 34, 82\}$, respectively. In Section 2, we give some definitions and lemmas to solve these Diophantine equations. In Section 3, we prove our main theorems by using a version of Matveev's result and Baker–Davenport basis reduction method.

2. Preliminaries

Assume that η is an algebraic number of degree d, the $\eta^{(i)}$ represent the conjugates of η, and minimal polynomial of η over $\mathbb{Z}$ is

$$a_0 x^d + a_1 x^{d-1} + ... + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x].$$

The logarithmic height of η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right). \tag{5}$$

The following properties can be found (see [5]):

$$h(\gamma \pm \eta) \leq \log 2 + h(\gamma) + h(\eta),$$

$$h(\gamma\eta^{\pm 1}) \leq h(\gamma) + h(\eta),$$

$$h(\eta^m) = |m| h(\eta),$$

and

$$h(a/b) = \log\left(\max\{|a|, b\}\right),$$

where $b \geq 1$ and $\gcd(a, b) = 1$.

The following lemma is owing to Matveev in [11] and also in [4]. Using this lemma, we find a large bound for the n in the equations (3) and (4).

Lemma 2.1. *Let $\gamma_1, \gamma_2, ..., \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D. Assume that $b_1, b_2, ..., b_t$ are rational integers and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not equal to zero. Then

$$|\Lambda| > \exp\left(T \cdot (1 + \log B)(1 + \log D) \cdot A_1 \cdot A_2 \cdot ... \cdot A_t\right),$$

where

$$T = -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2,$$

$$B \geq \max\{|b_1|, ..., |b_t|\}$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$$

for all $i = 1, ..., t$.

We use the following lemma given in [3] to reduce the bound found from the Lemma 2.1. Also, this lemma is a revision of the result given by Dujella and Pethő in [9]. Moreover, the result given in [9] is a revision of a lemma given by Baker and Davenport in [2].

Lemma 2.2. ([3], Lemma 1) *Assume that $A > 0, B > 1, \mu$ are some real numbers, and p/q is a convergent of the continued fraction of the irrational number γ such that $q > 6M$. Let $u, v, w, M \in \mathbb{Z}^+$, $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x, and $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

has no solution with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following two lemmas are given in [13] and [8], respectively.

Lemma 2.3. *Assume that* $a, x \in \mathbb{R}$. *If* $|x| < a$ *and* $0 < a < 1$, *then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

Lemma 2.4. *Assume that* M *is a positive integer,* N *is a nonnegative integer such that* $q_N > M$, τ *is an irrational number with* $\tau = [a_0; a_1, a_2, a_3, ...]$ *and for* $i = 0, 1, 2...$, $p_i/q_i := [a_0, a_1, .., a_i]$ *is the* i*-th convergents of the continued fraction expansion of* τ. *Put* $a(M) := \max\{a_i : i = 0, 1, 2, ..., N\}$. *Then the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2}$$

holds for all r *and* s *values where* $r > 0$ *and* $0 < s < M$.

3. Main Theorems

Theorem 3.1. *The only Pell numbers that are concatenations of two repdigits are* $12, 29, 70$.

Proof. Let $P_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}}\underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}}$. If we look at the first 64 Pell numbers, then it is seen that all solutions of this Diophantine equation are $P_n \in \{12, 29, 70\}$. If $d_1 = d_2$, the authors also showed that the biggest repdigit in the Pell numbers is $P_3 = 5$ in [10]. From now on, we assume that $n \geq 65$ and $d_1 \neq d_2$ in the equation (3). Now, let

$$P_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}}\underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}} = \underbrace{d_1 \cdots d_1}_{m_1 \text{ times}} \times 10^{m_2} + \underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}.$$

Then we have

$$P_n = \frac{d_1(10^{m_1}-1)}{9}10^{m_2} + \frac{d_2(10^{m_2}-1)}{9}, \tag{6}$$

i.e.,

$$\frac{\alpha^n - \beta^n}{2\sqrt{2}} = P_n = \frac{1}{9}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - d_2\right). \tag{7}$$

If (1) and (6) are combined, it follows that

$$10^{m_1+m_2-1} < P_n \leq \alpha^{n-1} < 10^{n-1}.$$

From here, we have $m_1 + m_2 < n$. On the other hand, the equation (7) can be rewritten as

$$\frac{9\alpha^n}{\sqrt{8}} - d_1 10^{m_1+m_2} = \frac{9\beta^n}{\sqrt{8}} - (d_1 - d_2)\, 10^{m_2} - d_2. \tag{8}$$

If we used the fact that $n \geq 65$, we obtain

$$\begin{aligned}\left|\frac{9\alpha^n}{\sqrt{8}} - d_1 10^{m_1+m_2}\right| &\leq \frac{9\left|\beta\right|^n}{\sqrt{8}} + \left|d_1 - d_2\right| 10^{m_2} + d_2 \\ &< 9\left|\beta\right|^n + 9 \cdot 10^{m_2} + 9 \\ &\leq (0.9)\, 10^{m_2} \left|\beta\right|^n + 9 \cdot 10^{m_2} + (0.9) 10^{m_2} \\ &= (0.9 \cdot \left|\beta\right|^n + 9.9) \cdot 10^{m_2},\end{aligned}$$

i.e.,

$$\left|\frac{9\alpha^n}{\sqrt{8}} - d_1 10^{m_1+m_2}\right| < (9.91) \cdot 10^{m_2} \tag{9}$$

from (8). If both sides of the inequality (9) are divided by $d_1 10^{m_1+m_2}$, we get

$$\left|\left(\frac{9}{d_1\sqrt{8}}\right)\alpha^n 10^{-m_1-m_2} - 1\right| \leq \frac{(9.91) \cdot 10^{m_2}}{d_1 10^{m_1+m_2}} < \frac{9.91}{10^{m_1}}. \tag{10}$$

Now, let's take $\Lambda_1 := \left(\frac{9}{d_1\sqrt{8}}\right)\alpha^n 10^{-m_1-m_2} - 1$, $\gamma_1 := 9/(d_1\sqrt{8})$, $\gamma_2 := \alpha$, $\gamma_3 := 10$, and $b_1 := 1$, $b_2 := n$, $b_3 := -m_1 - m_2$ to apply Lemma 2.1. Firstly, it should be examined whether the conditions necessary to use the Lemma 2.1 are ensure. The numbers γ_1, γ_2, and γ_3 are positive real numbers that are elements of the field $K = \mathbb{Q}(\sqrt{2})$. Since $[K : Q]= 2$, we obtain $D = 2$. Now, we will show that $\Lambda_1 \neq 0$. Assume that $\Lambda_1 = 0$. Then we get $\alpha^n = \frac{d_1\sqrt{8}\cdot 10^{m_1+m_2}}{9}$ and $\beta^n = \frac{-d_1\sqrt{8}\cdot 10^{m_1+m_2}}{9}$. Thus, it follows that $\alpha^n + \beta^n = 0 = Q_n$. This is impossible. Therefore $\Lambda_1 \neq 0$. Moreover, since

$$\begin{aligned}h(\gamma_1) = h(9/(d_1\sqrt{8}) &\leq h(9) + h(\sqrt{8}) + h(d_1) \\ &< 2\log 9 + \frac{\log 8}{2} < 5.44, \\ h(\gamma_2) = h(\alpha) &= \frac{\log \alpha}{2} < 0.45,\end{aligned}$$

and

$$h(\gamma_3) = h(10) = \log 10 < 2.3099,$$

we can choose $A_1 := 10.88$, $A_2 := 0.9$, and $A_3 := 4.62$. Since $m_1 + m_2 < n$, $B := n$ can be taken. Put $T = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2) \cdot (0.9) \cdot (4.62)$. Therefore, we have

$$(9.91) \cdot 10^{-m_1} > |\Lambda_1| > \exp\left(T \cdot (1 + \log n) \cdot (10.88)\right),$$

by using (10) and Lemma 2.1. It follows that

$$m_1 \log 10 < 4.39 \cdot 10^{13} \cdot (1 + \log n) + \log(9.91) \tag{11}$$

by a simple computation. If we reform the equation (7) as

$$\frac{9\alpha^n}{\sqrt{8}} - (d_1 \cdot 10^{m_1} - (d_1 - d_2))\, 10^{m_2} = \frac{9\beta^n}{\sqrt{8}} - d_2 \tag{12}$$

and take the absolute values of the equation (12), then it is seen that

$$\left|\frac{9\alpha^n}{\sqrt{8}} - (d_1 \cdot 10^{m_1} - (d_1 - d_2))\, 10^{m_2}\right| \leq \frac{9\,|\beta|^n}{\sqrt{8}} + d_2.$$

Since $d_2 \leq 9$, we obtain

$$\left|\frac{9\alpha^n}{\sqrt{8}} - (d_1 \cdot 10^{m_1} - (d_1 - d_2))\, 10^{m_2}\right| \leq 9\,|\beta|^n + 9 < 9.1. \tag{13}$$

Dividing both sides of (13) by $\frac{9\alpha^n}{\sqrt{8}}$, we give

$$\left|1 - \left(\frac{d_1 \cdot 10^{m_1} - (d_1 - d_2)}{9}\right)\sqrt{8}\alpha^{-n}10^{m_2}\right| \leq 2.86 \cdot \alpha^{-n}. \tag{14}$$

Assume that

$$\Lambda_2 := 1 - \left(\frac{d_1 \cdot 10^{m_1} - (d_1 - d_2)}{9}\right)\sqrt{8}\alpha^{-n}10^{m_2},$$

$$\gamma_1 := \left(\frac{d_1 \cdot 10^{m_1} - (d_1 - d_2)}{9}\right)\sqrt{8}, \gamma_2 := \alpha, \gamma_3 := 10,$$

and $b_1 := 1, b_2 := -n, b_3 := m_2$ to apply Lemma 2.1. The numbers γ_1, γ_2, and γ_3 are elements of the field $K = \mathbb{Q}(\sqrt{2})$ and they are positive real numbers. Since $[K : Q] = 2$, we have $D = 2$. It is clear that Λ_2 is nonzero. By using, the properties of the logarithmic height, we get

$$\begin{aligned} h(\gamma_1) &= h\left(\left(\frac{d_1(10^{m_1} - 1) + d_2}{9}\right)\sqrt{8}\right) \\ &\leq h(9) + h(d_1) + h(10^{m_1}) + h\,(d_2) + 2\log 2 + h(\sqrt{8}) \\ &\leq 3\log 9 + m_1 \log 10 + 2\log 2 + \frac{\log 8}{2} \\ &< 9.02 + m_1 \log 10, \end{aligned}$$

$$h(\gamma_2) = h(\alpha) = \frac{\log\alpha}{2} < 0.45,$$

and

$$h(\gamma_3) = h(10) = \log 10 < 2.31.$$

So, we can choose $A_1 := 18.04 + 2m_1 \log 10$, $A_2 := 0.9$, $A_3 := 4.62$ and $B := n$, since $m_2 < n - 1$. By using Lemma 2.1 and (14), we have

$$2.86 \cdot \alpha^{-n} > |\Lambda_2| > \exp(T \cdot (18.04 + 2m_1 \log 10) \cdot (1 + \log n)),$$

i.e.,

$$n\log\alpha - \log(2.86) < 4.04 \cdot 10^{12} \cdot (1 + \log n)\,(18.04 + 2m_1 \log 10)\,. \tag{15}$$

By using (11) and (15), we find $n < 2.02 \cdot 10^{30}$ thanks to easy calculation. We apply Lemma 2.2 to reduce the upper bound on n. Assume that

$$z_1 := \log 10 \cdot (m_1 + m_2) - \log\alpha \cdot n - \log\left(\frac{9}{d_1\sqrt{8}}\right).$$

We can write

$$|x| = \left|e^{-z_1} - 1\right| < \frac{9.91}{10^{m_1}} < 0.999$$

for $m_1 \geq 1$ from (10). By Lemma 2.3, we have the inequality

$$|z_1| = |\log(x+1)| < \frac{\log 1000}{0.999} \cdot \frac{9.91}{10^{m_1}} < \frac{68.53}{10^{m_1}}$$

for $a := 0.999$. Thus, we find

$$0 < \left|\frac{\log 10}{\log \alpha} \cdot (m_1 + m_2) - n - \left(\frac{\log(9/(d_1\sqrt{8}))}{\log \alpha}\right)\right| < (77.76) \cdot 10^{-m_1}. \tag{16}$$

Let's take

$$\gamma := \log 10 / \log \alpha \notin \mathbb{Q},$$

$$\mu := -\frac{\log(9/(d_1\sqrt{8}))}{\log \alpha},$$

and $m_1 + m_2 < M := 2.02 \cdot 10^{30}$ to use Lemma 2.2. Then it can be seen that the denominator of the 70th convergent of γ exceeds $6M$ by using a computer program. Also,

$$\epsilon := ||\mu q_{70}|| - M||\gamma q_{70}|| > 0.005$$

for $1 \leq d_1 \leq 9$. Let $A := 77.76$, $B := 10$ and $w := m_1$. Then, there is no solution to the inequality (16) for

$$m_1 \geq 36.56 > \frac{\log(Aq_{70}/\epsilon)}{\log B}.$$

So $m_1 \leq 36$. Replacing this upper bound for m_1 into (15), $n < 3.29 \cdot 10^{16}$ is found. We'll use Lemma 2.2 and Lemma 2.3 again to make n even smaller. Assume that

$$z_2 := \log 10 \cdot m_2 - \log \alpha \cdot n + \log\left(\frac{d_1 \cdot 10^{m_1} - (d_1 - d_2)}{9}\sqrt{8}\right).$$

From (14), it is seen that

$$|x| = |e^{z_2} - 1| < (2.86) \cdot \alpha^{-n} < 0.01 \text{ for } n \geq 65.$$

By Lemma 2.3, if we choose $a := 0.01$, we obtain

$$|z_2| = |\log(x+1)| < \frac{\log(100/99)}{0.01} \cdot \frac{(2.86)}{\alpha^n} < \frac{2.88}{\alpha^n}.$$

Hence, it can be seen that,

$$0 < \left|m_2 \cdot \frac{\log 10}{\log \alpha} - n + \frac{\log\left((d_1 10^{m_1} - (d_1 - d_2))\sqrt{8}/9\right)}{\log \alpha}\right| < 3.27 \cdot \alpha^{-n}. \tag{17}$$

Putting $\gamma := \dfrac{\log 10}{\log \alpha}$ and $m_2 < M := 3.29 \cdot 10^{16}$. Then, the denominator of the 45 th convergent of γ exceeds $6M$. Let

$$\mu := \frac{\log\left((d_1 10^{m_1} - (d_1 - d_2))\sqrt{8}/9\right)}{\log \alpha}.$$

Since $1 \leq m_1 \leq 36$, $d_1 \neq d_2$, $1 \leq d_1 \leq 9$ and $0 \leq d_2 \leq 9$, we find

$$\epsilon := ||\mu q_{45}|| - M||\gamma q_{45}|| > 0.0002$$

by using a computer program. In Lemma 2.2, let $A := 3.27$, $B := \alpha$, and $w := n$. Then, there is no solution to the inequality (17) for

$$n \geq 59.54 > \frac{\log(Aq_{45}/\epsilon)}{\log B}.$$

So $n \leq 59$. This is impossible since $n \geq 65$. $\square$

Theorem 3.2. *The only Pell-Lucas numbers that are concatenations of two repdigits are* $14, 34, 82$.

Proof. Let $Q_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}}\underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}}$. If we look at the first 94 Pell-Lucas numbers, then it is seen that all solutions of this Diophantine equation are $Q_n \in \{14, 34, 82\}$. If $d_1 = d_2$, the authors also showed that the biggest repdigit in the Pell numbers is $Q_2 = 6$ in [10]. From now on, we assume that $n \geq 95$ and $d_1 \neq d_2$ in the equation (4). Furthermore, the identity

$$Q_n \equiv 2, 4, 6, 8 (\mathrm{mod}\, 10)$$

is well known for $n \geq 0$. Thus, we take $d_2 = 2, 4, 6, 8$. Now, let

$$Q_n = \overline{\underbrace{d_1 \cdots d_1}_{m_1 \text{ times}}\underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}} = \underbrace{d_1 \cdots d_1}_{m_1 \text{ times}} \times 10^{m_2} + \underbrace{d_2 \cdots d_2}_{m_2 \text{ times}}.$$

Then we have

$$Q_n = \frac{d_1(10^{m_1} - 1)}{9} 10^{m_2} + \frac{d_2(10^{m_2} - 1)}{9} \tag{18}$$

and

$$\alpha^n + \beta^n = Q_n = \frac{1}{9}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - d_2\right). \tag{19}$$

Combining (2) and (18), we get

$$10^{m_1+m_2-1} < Q_n \leq 2\alpha^n < 10^{n+1}.$$

From this, we get $m_1 + m_2 < n + 2$. On the other hand, the equation (19) can be rewritten as

$$9\alpha^n - d_1 10^{m_1+m_2} = -9\beta^n - (d_1 - d_2)\, 10^{m_2} - d_2. \tag{20}$$

From (20), using the same arguments in (8), we have

$$\left|\left(\frac{9}{d_1}\right)10^{-m_1-m_2}\cdot\alpha^n-1\right|<\frac{9.91}{10^{m_1}}. \tag{21}$$

Now, let us apply Lemma 2.1 with $\Lambda_1 := \left(\frac{9}{d_1}\right)\alpha^n 10^{-m_1-m_2}-1, \gamma_1 := 9/d_1, \gamma_2 := \alpha, \gamma_3 := 10$ and $b_1 := 1, b_2 := n, b_3 := -m_1-m_2$. The numbers γ_1, γ_2, and γ_3 are elements of the field $K = \mathbb{Q}(\sqrt{2})$ and they are positive real numbers. Since $[K : Q] = 2$, we have $D = 2$. Assume that $\Lambda_1 = 0$. This is impossible since $\alpha^n = \frac{10^{m_1+m_2}\cdot d_1}{9}$ and α^n is irrational. So, $\Lambda_1 \neq 0$. Moreover, since $m_1+m_2 < n+2$,

$$h(\gamma_1) = h\left(\frac{9}{d_1}\right) \leq h(d_1)+h(9) < 2\log 9 < 4.4,$$

$$h(\gamma_2) = h(\alpha) = \frac{\log\alpha}{2} < 0.45$$

and

$$h(\gamma_3) = h(10) = \log 10 < 2.31$$

by (5), we can choose $B := n+2$, $A_1 := 8.8$, $A_2 := 0.9$ and $A_3 := 4.62$. By using (21) and Lemma 2.1, we have

$$10^{-m_1}\cdot(9.91) > |\Lambda_1| > \exp\left(T\cdot(1+\log(n+2))\cdot(8.8)\right),$$

where $T = -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)\cdot(0.9)\cdot(4.62)$. By a simple computation, it follows that

$$m_1\log 10 < 3.55\cdot 10^{13}\cdot(1+\log(n+2))+\log(9.91). \tag{22}$$

Reform the equation (19) as

$$9\alpha^n-(d_1 10^{m_1}-(d_1-d_2))\,10^{m_2} = -9\beta^n-d_2 \tag{23}$$

and taking absolute values of the equation (23), we find

$$|9\alpha^n-(d_1 10^{m_1}-(d_1-d_2))\,10^{m_2}| \leq 9\,|\beta|^n+d_2.$$

Thus it is written that

$$|9\alpha^n-(d_1 10^{m_1}-(d_1-d_2))\,10^{m_2}| \leq 9\,|\beta|^n+9 < 9.1. \tag{24}$$

If (24) is divided by $9\alpha^n$, it can be seen that

$$\left|1-\left(\frac{d_1 10^{m_1}-(d_1-d_2)}{9}\right)\alpha^{-n}10^{m_2}\right| \leq 1.02\cdot\alpha^{-n}. \tag{25}$$

Taking

$$\Lambda_2 := 1-\left(\frac{d_1 10^{m_1}-(d_1-d_2)}{9}\right)\alpha^{-n}10^{m_2},$$

$$\gamma_1 := \left(\frac{d_1 10^{m_1}-(d_1-d_2)}{9}\right), \gamma_2 := \alpha, \gamma_3 := 10,$$

$b_1 := 1, b_2 := -n$, and $b_3 := m_2$, we apply Lemma 2.1. The numbers γ_1, γ_2 and γ_3 are elements of the field $K = \mathbb{Q}(\sqrt{2})$. Also, they are positive real numbers.

$D = 2$ since $[K : Q] = 2$. Moreover, it can be easily seen that Λ_2 is not equal to zero. We get

$$\begin{aligned} h(\gamma_1) &= h\left(\frac{d_1(10^{m_1}-1)+d_2}{9}\right) \\ &\leq h(9) + h(d_1) + h(10^{m_1}) + h(d_2) + 2 \cdot \log 2 \\ &\leq 3 \cdot \log 9 + m_1 \cdot \log 10 + 2 \cdot \log 2 \\ &< 7.98 + m_1 \cdot \log 10, \end{aligned}$$

$$h(\gamma_2) = h(\alpha) = \frac{\log \alpha}{2} < 0.45$$

and

$$h(\gamma_3) = h(10) = \log 10 < 2.31$$

by using the properties of the logarithmic height. So, we can choose $A_1 := 15.96 + 2m_1 \cdot \log 10$, $A_2 := 0.9, A_3 := 4.62$ and $B := n + 1$ since $m_2 < n + 1$. By using (25) and Lemma 2.1, it can be shown that

$$1.02 \cdot \alpha^{-n} > |\Lambda_2| > \exp\left(T \cdot (1 + \log(n+1))(15.96 + 2m_1 \log 10)\right),$$

i.e.,

$$n \log \alpha - \log(1.02) < 4.04 \cdot 10^{12} \cdot (1 + \log(n+1))(15.96 + 2m_1 \log 10). \tag{26}$$

It can be seen that $n < 1.63 \cdot 10^{30}$ by using the inequalities (22) and (26). Here, we apply Lemma 2.2 to minimize the upper bound on n. Assume that

$$z_1 := \log 10 \cdot (m_1 + m_2) - \log \alpha \cdot n - \log\left(\frac{9}{d_1}\right).$$

It can be written that

$$|x| = \left|e^{-z_1} - 1\right| < \frac{9.91}{10^{m_1}} < 0.999$$

for $m_1 \geq 1$ from (21). If we take $a := 0.999$, we find

$$|z_1| = |\log(x+1)| < \frac{\log 1000}{0.999} \cdot \frac{9.91}{10^{m_1}} < \frac{68.53}{10^{m_1}}$$

by Lemma 2.3. From here, we get

$$0 < \left|\frac{\log 10}{\log \alpha} \cdot (m_1 + m_2) - n - \left(\frac{\log(9/d_1)}{\log \alpha}\right)\right| < (77.76) \cdot 10^{-m_1}. \tag{27}$$

To apply Lemma 2.2, let

$$\gamma := \frac{\log 10}{\log \alpha} \notin \mathbb{Q}, \mu := -\frac{\log(9/d_1)}{\log \alpha}$$

and $m_1 + m_2 < M := 1.63 \cdot 10^{30}$. Then the denominator of the 69th convergent of γ exceeds $6M$ and we obtain

$$\epsilon := ||\mu q_{69}|| - M||\gamma q_{69}|| \ > 0.06$$

for $1 \leq d_1 < 9$. Let $A := 77.76$, $B := 10$ and $w := m_1$. Thus, there is no solution of the inequality (27) for

$$m_1 \geq 34.82 > \frac{\log(Aq_{69}/\epsilon)}{\log B}.$$

So $m_1 \leq 34$. If $d_1 = 9$, then from (27), we have

$$0 < \left|(m_1+m_2)\frac{\log 10}{\log \alpha} - n\right| < (77.76) \cdot 10^{-m_1}.$$

If this inequality is divided by $m_1 + m_2$, we write

$$0 < \left|\frac{\log 10}{\log \alpha} - \frac{n}{m_1+m_2}\right| < \frac{77.76}{(m_1+m_2) \cdot 10^{m_1}}. \tag{28}$$

Assume that $m_1 \geq 35$. Then it can be seen that

$$\frac{10^{m_1}}{155.52} > 6.43 \cdot 10^{32} > n+2 > m_1+m_2.$$

So we obtain

$$\left|\frac{\log 10}{\log \alpha} - \frac{n}{m_1+m_2}\right| < \frac{77.76}{(m_1+m_2) \cdot 10^{m_1}} < \frac{1}{2 \cdot (m_1+m_2)^2}.$$

It can shown that the rational number $\frac{n}{m_1+m_2}$ is a convergent of $\gamma = \frac{\log 10}{\log \alpha}$ from the known properties of continued fraction. Assume that $\frac{p_r}{q_r}$ is r-th convergent of the continued fraction of γ and $\frac{n}{m_1+m_2}$ is equal to $\frac{p_t}{q_t}$ for some t. Then it follows that $q_{68} > 2 \cdot 10^{31} > n+2 > m_1+m_2$, so $t \in \{0, 1, 2, ..., 67\}$ and $a_M = \max\{a_i | i = 0, 1, 2, ..., 67\} = 52$. We obtain

$$\left|\gamma - \frac{p_t}{q_t}\right| > \frac{1}{(a_M+2)(m_1+m_2)^2} = \frac{1}{54 \cdot (m_1+m_2)^2}$$

by Lemma 2.4. Thus, from (28) and the above inequality, we get

$$\frac{77.76}{(m_1+m_2) \cdot 10^{m_1}} > \frac{1}{54 \cdot (m_1+m_2)^2}.$$

This shows that

$$\frac{7.776}{10^{34}} \geq \frac{77.76}{10^{m_1}} > \frac{1}{54 \cdot (m_1+m_2)} > \frac{1}{1.08 \cdot 10^{33}},$$

a contradiction. Therefore $m_1 \leq 34$. Replacing this upper bound for m_1 into (26), we get $n < 3.09 \cdot 10^{16}$. If

$$z_2 := \log 10 \cdot m_2 - \log \alpha \cdot n + \log\left(\frac{d_1 10^{m_1} - (d_1 - d_2)}{9}\right)$$

is taken, from (25), it is seen that

$$|x| = |e^{z_2} - 1| < (1.02) \cdot \alpha^{-n} < 0.01$$

for $n \geq 95$. Taking $a := 0.01$, by Lemma 2.3, we have

$$|z_2| = |\log(x+1)| < \frac{\log(100/99)}{0.01} \cdot \frac{(1.02)}{\alpha^n} < \frac{1.03}{\alpha^n}.$$

From here, we can say

$$0 < \left| \frac{\log 10}{\log \alpha} \cdot m_2 - n + \frac{\log\left((d_1 10^{m_1} - (d_1 - d_2))/9\right)}{\log \alpha} \right| < 1.17 \cdot \alpha^{-n}. \tag{29}$$

Taking $\gamma := \dfrac{\log 10}{\log \alpha}, \mu := \frac{\log((d_1 10^{m_1} - (d_1 - d_2))/9)}{\log \alpha}$ and $m_2 < M := 3.09 \cdot 10^{16}$, it can be seen that q_{58}, the denominator of the 58 th convergent of γ exceeds $6M$. It can be shown that

$$\epsilon := ||\mu q_{58}|| - M||\gamma q_{58}|| > 0$$

for $1 \leq m_1 \leq 34$, $d_1 \neq d_2$, $1 \leq d_1 \leq 9$ and $d_2 = 2, 4, 6, 8$. In Lemma 2.2, we can take $A := 1.17$, $B := \alpha$, and $w := n$. Thus, we can say that there is no solution of the inequality (29) for

$$n \geq 91.48 > \frac{\log(A \cdot q_{58}/\epsilon)}{\log B}.$$

So, $n \leq 91$. This is impossible since $n \geq 95$. □

References

[1] A. Alahmadi, A. Altassan, F. Luca, and H. Shoaib, *Fibonacci numbers which are concatenations of two repdigits*, Quaestiones Mathematicae **4** (2021), no. 2, 281–290.

[2] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), no. 1, 129–137.

[3] J. J. Bravo, C. A. Gomez, and F. Luca, *Powers of two as sums of two k-Fibonacci numbers*, Miskolc Math. Notes **17** (2016), no. 1, 85–100.

[4] Y. Bugeaud, M. Mignotte, and S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Ann. of Math. **163** (2006), no. 3, 969–1018.

[5] Y. Bugeaud, *Linear Forms in Logarithms and Applications*, IRMA Lectures in Mathematics and Theoretical Physics, 28, Zurich: European Mathematical Society, 2018.

[6] M. Ddamulira, *Padovan numbers that are concatenations of two repdigits*, Mathematica Slovaca **71** (2021), no. 2, 275–284.

[7] M. Ddamulira, *Tribonacci numbers that are concatenations of two repdigits*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), no. 4, 203.

[8] M. Ddamulirai, *On the x-coordinates of Pell equations that are products of two Padovan numbers*, Integers **20** (2020), no. A70, 20 pp.

[9] A. Dujella and A. Pethò, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 3, 291–306.

[10] B. Faye and F. Luca, *Pell and Pell-Lucas numbers with only one distinct digit*, Ann. Math. Inform. **45** (2015), 55–60.

[11] E. M. Matveev, *An Explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 6, 125-180 (Russian); Translation in Izv. Math. **64** (2000), no. 6, 1217–1269.

[12] S. G. Rayaguru and G. K. Panda, *Balancing numbers which are concatenations of two repdigits*, Bol. Soc. Mat. Mex. **26** (2020), 911–919.

[13] B. M. M. de Weger, *Algorithms for Diophantine Equations*, CWI Tracts 65, Stichting Maths. Centrum, Amsterdam, 1989.

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