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(k,m)-TYPE SLANT HELICES FOR THE NULL CARTAN CURVE WITH THE BISHOP FRAME IN $\mathbf{E}_1^4$

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Abstract. In this paper, we obtain (k,m)-type slant helices for a null Cartan curve with the Bishop frame in Minkowski space E_1^4.

1. Introduction

Minkowski space (or Minkowski space-time E_1^4), having an important place in mathematical physics, is a combination of Euclidean 3-space and time (as the fourth dimension). Both physicists and geometers have been working on the Minkowski space, which has become more interesting especially with Einstein's relativity theory. The curves theory in Euclidean and Minkowski spaces occupies a large place. In Euclidean 3-space (or Minkowski 3-space), a helix is defined as a curve whose tangent lines make a constant angle with a fixed direction, and whose curvature and torsion are nonzero. Based on this definition, definitions of other types of the helices such as general helix, slant helix or null helix are given. If the principal normal vector field of a curve makes a constant angle with a fixed direction, that curve is called a slant helix. A null helix in Minkowski 3-space is a null curve with constant lightlike curvature, [13, 19].

There are lots of studies on slant helices and null helices, [1, 6, 8, 10, 11, 12, 15, 28]. The frames constructed on a curve γ are tools to determine the characteristic features of the curve. One of them, Frenet frame $\{T, N, B\}$, consists of tangent, principal normal and binormal vectors of a regular curve in E^3, respectively. Through these vectors, the curvatures (κ and τ) of the curve can obtained. In 1975, the frame $\{T, N_1, N_2\}$ of a regular curve γ in E^3 was defined by Bishop and it was called Bishop frame or relatively parallel adapted frame, [4]. One of the tools used where the Frenet frame of a curve does not work is the Bishop frame. In this frame, since the vector fields N_1 and N_2 are col-linear with the tangent vector field T at every point of the curve γ. The normal vector fields N_1 and N_2 are called relatively parallel vector fields. The Bishop frame works well even at points where the first Frenet

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curvature function κ of a curve vanishes, where the Frenet frame does not work. The studies on various Bishop frames such as type-1, type-2, N-type in 3 or 4 dimensional Euclidean or Minkowski spaces continue to produce solutions for single point curves, [7, 9, 14, 16, 18, 21, 22, 24, 26, 30, 29]. Besides, the null Cartan curve and its Bishop frame in Minkowski spaces E_1^3 and E_1^4 are studied in [2, 3, 5, 6, 10, 11, 12, 15, 16, 17, 20, 27]. In this study, we obtained (k, m) −type slant helices for a null Cartan curve with the Bishop frame in E_1^4.

2. Preliminaries

Minkowski space-time E_1^4 is the real vector space E^4 equipped with the standard flat metric $\langle ., . \rangle$ defined by

$$\left\langle \tilde{\xi}, \xi \right\rangle = -\tilde{\xi}_1\xi_1 + \tilde{\xi}_2\xi_2 + \tilde{\xi}_3\xi_3 + \tilde{\xi}_4\xi_4,$$

for any two vectors $\tilde{\xi} = \left(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4\right)$ and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ in E_1^4. $\langle ., . \rangle$ is an indefinite metric, so there are three cases for any vector $\tilde{\xi} \in E_1^4$: spacelike $\left(\left\langle \tilde{\xi}, \tilde{\xi} \right\rangle > 0\right)$ or $\tilde{\xi} = 0$, timelike $\left(\left\langle \tilde{\xi}, \tilde{\xi} \right\rangle < 0\right)$ or null (lightlike) $\left(\left\langle \tilde{\xi}, \tilde{\xi} \right\rangle = 0\right)$, [25]. $\left\|\tilde{\xi}\right\| = \sqrt{\left|\left\langle \tilde{\xi}, \tilde{\xi} \right\rangle\right|}$ is called as norm of the vector $\tilde{\xi} \in E_1^4$. Similarly, a curve $\gamma : I \to E_1^4$ has the characterization that all its tangent vectors have. Besides, a curve with the parameterization specified by the pseudo arc function

$$s(t) = \int_0^t \sqrt{\|\gamma''(u)\|}du$$

is called a null Cartan curve, [5]. We know that the first and second curvature functions of a curve in 3-dimensional spaces have a important role in defining the physical and geometric properties of that curve. For null Cartan curves in E_1^4, these quantities are also defined as the first, second and third Cartan curvatures and these curvatures are denoted by k_1, k_2, k_3, respectively. The Frenet frame of the non-geodesic null Cartan curve γ consists of the orthonormal vectors $\{T, N, B_1, B_2\}$. The Frenet frame equations are as follows [23]:

$$\begin{aligned}
&\bigtriangledown_T T = k_1 N,\\
&\bigtriangledown_T N = -k_2 T + k_1 B_1,\\
&\bigtriangledown_T B_1 = -k_2 N + k_3 B_2,\\
&\bigtriangledown_T B_2 = k_3 T,
\end{aligned}$$

where $k_1 = 1$, k_2 and k_3 are arbitrary functions. Hence there are the following equalities for the Frenet frame $\{T, N, B_1, B_2\}$ of null Cartan curve γ:

$$\begin{gathered}
\langle T, T\rangle = \langle T, N\rangle = \langle T, B_2\rangle = \langle N, B_1\rangle = \langle N, B_2\rangle = \langle B_1, B_1\rangle = \langle B_1, B_2\rangle = 0,\\
\langle N, N\rangle = \langle B_2, B_2\rangle = 1,\\
\langle T, B_1\rangle = -1.
\end{gathered}$$

According to the Cartan curvatures k_1, k_2, k_3, two type of Bishop frames are defined. In this study, we use the following Bishop frame with $k_1 = 0$, k_2, k_3 be arbitrary functions. The Bishop frame of the null Cartan curve γ consists of the orthonormal vectors $\{T_1, N_1, N_2, N_3\}$. The Bishop frame equations are as follows [23]:

$$\begin{aligned} \nabla_{T_1} T_1 &= \sigma_2 T_1 + \sigma_1 N_1 - \sigma_3 N_3, \\ \nabla_{T_1} N_1 &= \sigma_1 N_2, \\ \nabla_{T_1} N_2 &= -\sigma_2 N_2, \\ \nabla_{T_1} N_3 &= -\sigma_3 N_2, \end{aligned} \tag{1}$$

where the first, second and third Bishop curvatures are given the following equations, respectively:

$$\begin{aligned} \sigma_1 &= \sin\theta, \\ \sigma_2 &= \frac{k_3 - \theta''}{\theta'}, \\ \sigma_3 &= \cos\theta, \end{aligned}$$

where $\theta' \neq 0$ and

$$2\theta' \left(\theta''' - k_3'\right) + 2\theta'' \left(k_3 - \theta''\right) + \theta'^4 - \left(k_3 - \theta''\right)^2 - 2k_2\theta'^2 = 0.$$

Here, we also have

$$\begin{gathered} \langle T_1, T_1\rangle = \langle T_1, N_1\rangle = \langle T_1, N_3\rangle = \langle N_1, N_2\rangle = \langle N_1, N_3\rangle = \langle N_2, N_2\rangle = \langle N_2, N_3\rangle = 0, \\ \langle N_1, N_1\rangle = \langle N_3, N_3\rangle = 1, \\ \langle T_1, N_2\rangle = -1. \end{gathered}$$

Definition 2.1. *Let γ be a unit speed regular curve in E_1^4 and $\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4\}$ be the Frenet frame of γ. If there is a non-zero fixed vector field $U \in E_1^4$ satisfying $\langle \Upsilon_k, U\rangle = \lambda$ and $\langle \Upsilon_m, U\rangle = \mu$, (where λ, μ are constant), then γ is called $(k, m)-$ type slant helix, for $1 \leq k, m \leq 4$, $k \neq m$. The fixed vector U is on axis of $(k, m)-$type slant helix,* [3].

Here if we denote $\Upsilon_1 = T, \Upsilon_2 = N, \Upsilon_3 = B_1, \Upsilon_4 = B_2$, we can write $U = \omega_1 T + \omega_2 N + \omega_3 B_1 + \omega_4 B_2$, where $\omega_i = \omega_i(s)$ are differentiable functions of s.

3. *(k,m)*$-$Type Slant Helices For The Null Cartan Curve With The Bishop Frame in $\mathbf{E_1^4}$

Theorem 3.1. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(1, 2)-$type slant helix and σ_1, σ_2, $\sigma_3 \neq 0$, the following relation is available:*

$$c_1\sigma_2 + c_2\sigma_1 = c\sigma_3, \tag{2}$$

here $c, c_1, c_2 \in \mathbb{R}$.

Proof. Let γ be a $(1,2)-$type slant helix. There are $c_1, c_2 \in \mathbb{R}$, such that

$$\langle T_1, \upsilon\rangle = c_1, \tag{3}$$

$$\langle N_1, \upsilon\rangle = c_2. \tag{4}$$

If we take the derivative of both sides of (3), use the expressions (1), (3) and (4), we get

$$\langle N_3, \upsilon\rangle = \frac{c_1\sigma_2 + c_2\sigma_1}{\sigma_3}. \tag{5}$$

Similarly, if we take the derivative of both sides of (4) and use (1), we get

$$\langle N_2, \upsilon\rangle = 0. \tag{6}$$

If we take the derivative of both sides of (5) and consider the expression (6), we get

$$\left(\frac{c_1\sigma_2 + c_2\sigma_1}{\sigma_3}\right)' = 0$$

or the expression (2) . ☐

Theorem 3.2. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(1,3)-$type slant helix and $\triangle = \sigma_1\sigma_3' - \sigma_1'\sigma_3 \neq 0$, the following relations are available:*

$$\langle N_1, \upsilon\rangle = \frac{\sigma_3\left(-\sigma_1^2 + \sigma_2\sigma_3\right)}{\sigma_1\sigma_3' - \sigma_1'\sigma_3}c_3 \tag{7}$$

and

$$\langle N_3, \upsilon\rangle = \frac{\sigma_1\left(-\sigma_1^2 + \sigma_2\sigma_3\right)}{\sigma_1\sigma_3' - \sigma_1'\sigma_3}c_3, \tag{8}$$

here $c_3 \in \mathbb{R}$.

Proof. Let γ be a $(1,3)-$type slant helix. There are $c_1, c_3 \in \mathbb{R}$, such that

$$\langle T_1, \upsilon\rangle = c_1, \tag{9}$$

$$\langle N_2, \upsilon\rangle = c_3. \tag{10}$$

If we take the derivative of both sides of (9), use the expressions (1) and (9), we get

$$\sigma_1\langle N_1, \upsilon\rangle - \sigma_3\langle N_3, \upsilon\rangle = -c_1\sigma_2. \tag{11}$$

Similarly, if we take the derivative of both sides of (10) and use the expression (10), we have

$$\sigma_2 = 0. \tag{12}$$

If we substitute (12) in (11) and take the derivative of the resulting expression, we get

$$\sigma_1'\langle N_1, \upsilon\rangle - \sigma_3'\langle N_3, \upsilon\rangle = \left(-\sigma_1^2 + \sigma_2\sigma_3\right)c_3.$$

Since the expressions (11) and (12) are a grammar system and the equation

$$\triangle = \begin{vmatrix} \sigma_1 & \sigma_3 \\ \sigma_1' & \sigma_3' \end{vmatrix} \neq 0$$

is given in the hypothesis, the solution of this system is:

$$\triangle_1 = \begin{vmatrix} 0 & \sigma_3 \\ -\left(\sigma_1^2 + \sigma_2\sigma_3\right) c_3 & \sigma_3' \end{vmatrix} = \sigma_3 \left(\sigma_1^2 + \sigma_2\sigma_3\right) c_3$$

and

$$\triangle_2 = \begin{vmatrix} \sigma_1 & 0 \\ \sigma_1' & -\left(\sigma_1^2 + \sigma_2\sigma_3\right) c_3 \end{vmatrix} = -\sigma_1 \left(\sigma_1^2 + \sigma_2\sigma_3\right) c_3.$$

So, from the equations

$$\langle N_1, \upsilon \rangle = \frac{\triangle_1}{\triangle} \qquad \text{and} \qquad \langle N_3, \upsilon \rangle = \frac{\triangle_2}{\triangle},$$

we obtain the expressions (7) and (8) . ☐

Theorem 3.3. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(1,4)-$type slant helix and σ_1, σ_2, $\sigma_3 \neq 0$, the following relation is available:*

$$-c_1\sigma_2 + c_4\sigma_3 = c\sigma_1, \tag{13}$$

here $c, c_1, c_4 \in \mathbb{R}$.

Proof. Let γ be a $(1,4)-$type slant helix. There are $c_1, c_4 \in \mathbb{R}$, such that

$$\langle T_1, \upsilon \rangle = c_1, \tag{14}$$

$$\langle N_3, \upsilon \rangle = c_4. \tag{15}$$

If we take the derivative of both sides of (14), use the expressions (1) and (15), we get

$$\langle N_1, \upsilon \rangle = \frac{-c_1\sigma_2 + c_4\sigma_3}{\sigma_1}. \tag{16}$$

Similarly, if we take the derivative of both sides of (15) and use (1), we get

$$\langle N_2, \upsilon \rangle = 0. \tag{17}$$

And, if we take the derivative of both sides of (16), use the expressions (1) and (17), we get

$$\left(\frac{-c_1\sigma_2 + c_4\sigma_3}{\sigma_1}\right)' = 0$$

or the expression (13). ☐

Theorem 3.4. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(2,3)-$type slant helix and σ_1, σ_2, $\sigma_3 \neq 0$, the following relation is available:*

$$\sigma_1 = \sigma_2 = 0. \tag{18}$$

Proof. Let γ be a $(2,3)-$type slant helix. There are $c_2, c_3 \in \mathbb{R}$, such that

$$\langle N_1, \upsilon\rangle = c_2, \tag{19}$$

$$\langle N_2, \upsilon\rangle = c_3. \tag{20}$$

If we take the derivative of both sides of the expressions (19) and (20), use the expressions (1), (19) and (20), we get the expression (18) . ☐

Theorem 3.5. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(2,4)-$type slant helix and $\langle N_2, \upsilon\rangle = constant$, the following relation is available:*

$$\sigma_1 = \sigma_2 = \sigma_3 = 0.$$

Proof. Let γ be a $(2,3)-$type slant helix. There are c_2, $c_4 \in \mathbb{R}$, such that

$$\langle N_1, \upsilon\rangle = c_2, \tag{21}$$

$$\langle N_3, \upsilon\rangle = c_4. \tag{22}$$

If we take the derivative of both sides of the expressions (21) and (22), use (1) and consider the equation $\langle N_2, \upsilon\rangle = constant$, we get $\sigma_1 = \sigma_3 = 0$. On the other hand, if we take the derivative of both sides of $\langle N_2, \upsilon\rangle = constant$ and use (1), we get $\sigma_2 = 0$. ☐

Theorem 3.6. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(3,4)-$type slant helix, the following relation is available:*

$$\sigma_2 = \sigma_3 = 0. \tag{23}$$

Proof. Let γ be a $(3,4)-$type slant helix. So, there are c_3, $c_4 \in \mathbb{R}$, such that

$$\langle N_2, \upsilon\rangle = c_3, \tag{24}$$

$$\langle N_3, \upsilon\rangle = c_4. \tag{25}$$

If we take the derivative of both sides of the expressions (24) and (25), use the expressions (1) and (24), we get the expression (23). ☐

Theorem 3.7. *Let γ be a null Cartan curve with $k_1 = 1$, $k_3 = 0$ and the arbitrary function k_2 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If the curve γ is a type-1 slant helix and $\sigma_1 = \sigma_2 \neq 0$, γ is a type-2 also slant curve.*

Proof. Let γ be a type-1 slant helix. There is $c_1 \in \mathbb{R}$, such that

$$\langle T_1, \upsilon\rangle = c_1. \tag{26}$$

If we take the derivative of both sides of (26), consider the hypothesis, we get

$$\langle N_1, \upsilon\rangle = c_1.$$

So, γ is a type-2 slant curve. ☐

Theorem 3.8. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary function k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(1,2)-$type slant helix and $\sigma_1 = \sigma_2 = \sigma_3 \neq 0$, γ is also a type-4 slant curve.*

Proof. Let γ be a $(1,2)-$type slant helix. We have the expressions (3) and (4). If we take the derivative of both sides of (3), use the expressions (1), (3) and (4) we get the expression (5). If we consider $\sigma_1 = \sigma_2 = \sigma_3 \neq 0$, from the expression (5), we get

$$\langle N_3, \upsilon \rangle = constant.$$

So, γ is a type-4 slant curve. □

Theorem 3.9. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary function k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(1,4)-$type slant helix and $\sigma_1 = \sigma_2 = \sigma_3 \neq 0$, γ is also a type-2 slant curve.*

Proof. Let γ be a $(1,4)-$type slant helix. The expressions (14) and (15), and therefore the expression (16) are obtained. If we substitute $\sigma_1 = \sigma_2 = \sigma_3$ in the expression (16), we get

$$\langle N_1, \upsilon \rangle = constant.$$

So, γ is a type-2 slant curve. □

Theorem 3.10. *Let γ be a null Cartan curve with $k_1 = 1$ and the arbitrary function k_2, k_3 in E_1^4 and the Bishop frame of γ be $\{T_1, N_1, N_2, N_3\}$. If γ is a $(2,4)-$type slant helix and $\sigma_1 \neq 0$, $\sigma_3 \neq 0$, then $\sigma_2 = 0$ or $\sigma_2 \neq 0$.*

Proof. Let γ be a $(2,4)-$type slant helix. The expressions (21) and (22) are obtained. If we take the derivative of both sides of the expressions (21) and (22), use the Bishop frame and consider $\sigma_1 \neq 0$, $\sigma_3 \neq 0$, we get

$$\langle N_2, \upsilon \rangle = 0. \tag{27}$$

If we take the derivative of both sides of (27) and use the Bishop frame, we get $\sigma_2 = 0$ or $\sigma_2 \neq 0$. □

4. Conclusions and Discussion

The Bishop frame of a null Cartan curve with null Cartan curvature in Minkowski space E_1^4 is defined in two ways, [27]. In this study, the theories on $(k,m)-$type slant helices are studied, especially considering the Bishop frame of a null Cartan curve with $k_1 = 1$ and the arbitrary functions k_2, k_3. But $(k,m)-$type slant helices of the Bishop frame of a null Cartan curve with $k_1 = 1$, $k_3 = 0$ and arbitrary function k_2 is still an open problem.

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