East Asian Math. J.
Vol. 39 (2023), No. 1, pp. 001–010
http://dx.doi.org/10.7858/eamj.2023.001

SOME FIXED POINT THEOREMS ON CONE S-METRIC SPACES USING IMPLICIT CONTRACTIVE CONDITIONS

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Abstract. In this paper, we introduce two kinds of implicit conditions and establish some fixed point theorems in cone S-metric spaces, which generalize the several existing results.

1. Introduction and Preliminaries

Banach contraction mapping principle in a metric space is one of the most useful result in nonlinear analysis. Many researchers have generalized and improved this result in two directions; one is to generalize its underlying (metric) space and the other is to generalize the contractive condition in various ways(see, for example [1, 3, 4, 7, 9, 10])

In 2007, Huang and Zhang [3] introduced the concept of cone metric, as a generalization of a usual metric, and proved some fixed point theorems for contractive mappings in normal cone metric spaces. A few years later, Sedghi et al. [10] introduced the concept of S-metric, a generalization of G-metric and D^*-metric, and obtained fixed point theorems in complete S-metric spaces under explicit contractive conditions. In 2017, Dhamodharan and Krishnakumar [2] introduced the concept of cone S-metric and obtained some fixed point theorems using a few contractive conditions in cone S-metric spaces.

On the other hand, since Popa [5, 6] employed an implicit contractive type condition instead of the usual explicit contractive conditions to obtain fixed point theorems, this direction of research produced a consistent literature on fixed point and common fixed point therorems in various spaces.

Recently, Saluja [7] obtained fixed point theorems in the setting of complete cone S-metric spaces under implicit contractive conditions which used in [11].

Motivated and inspired by the previous works, in this paper, we introduce some implicit conditions and establish some fixed point theorems in cone S-metric spaces, which generalize the several existing results.

First of all, we recall some basic notions of a cone and a partial ordering.

Received September 19, 2022; Accepted December 11, 2022.
2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 54C60.
Key words and phrases. fixed points, cone S-metric spaces.
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(pISSN 1226-6973, eISSN 2287-2833)

A nonempty subset P of a real Banach space E is called a cone if and only if
(P1) P is closed, $P \neq \{\mathbf{0}\}$;
(P2) $a, b \in \mathbb{R}$ with $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(P3) $x \in P$ and $-x \in P \Rightarrow x = \mathbf{0}$.
For a given cone $P \subset E$, we define a partial ordering '$\preceq$' with respect to P as follows; for $x, y \in E$, $x \preceq y$ if and only if $y - x \in P$. We shall note $x \ll y$ if and only if $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P. The cone P is called normal if there is a positive real number K such that $\mathbf{0} \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

Definition 1. [2] Let X be a nonempty set. Suppose that a mapping $S : X \times X \times X \to P$ satisfies the following;
(S1) $S(x, y, z) = \mathbf{0}$ if and only if $x = y = z$;
(S2) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z \in X$.
Then S is called a cone S-metric on M, and the set X with a cone S-metric S is called a cone S-metric space, denoted by (X, S).

Definition 2. Let (X, S) be a cone S-metric space. A sequence $\{x_n\}$ in X called a Cauchy sequence if for any $\varepsilon \succeq \mathbf{0}$, there exists $N \in \mathbb{N}$ such that $S(x_n, x_m, x_l) \preceq \varepsilon$ for each $n, m, l \geq N$.

Definition 3. The cone S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Following lemma is cone S-metric version of Lemma 2.5 in a S-metric spaces [10].

Lemma 1.1. *Let (X, S) be a cone S-metric space. Then, $S(x, x, z) = S(z, z, x)$ for all $x, z \in X$.*

Proof. From $(S2)$, we have

$$S(x, x, z) \preceq S(x, x, x) + S(x, x, x) + S(z, z, x) = S(z, z, x)$$

and similarly

$$S(z, z, x) \preceq S(z, z, z) + S(z, z, z) + S(x, x, z) = S(x, x, z).$$

By the property (P3) of a cone, we have $S(x, x, z) = S(z, z, x)$. □

2. Main Results

First of all, we introduce Implicit Relation 1 to obtain a fixed point theorem on cone S-metric spaces.

Implicit Relation 1. Let $\mathbb{F}$ be the set of all continuous functions $F : P^6 \to P$ consider the following properties;

$(F1)$ there exists $k \in [0, 1)$ such that for all $x, y, z \in P$, $y \preceq F(x, x, y, z, \mathbf{0}, \mathbf{0})$

with $z \preceq 2x + y$ implies that $y \preceq kx$,
$(F2)$ for all $y \in P$, $y \preceq F(y, \mathbf{0}, \mathbf{0}, y, y, y)$ implies that $y = \mathbf{0}$,
$(F3)$ $x_i \preceq y_i + z_i$ for all $x_i, y_i, z_i \in P (i = 1, 2, \cdots, 6)$,

$$F(x_1, x_2, \cdots, x_6) \preceq F(y_1, y_2, \cdots, y_6) + F(z_1, z_2, \cdots z_6).$$

Actually, for all $y \in P$, $F(\mathbf{0}, \mathbf{0}, 2y, y, \mathbf{0}, y) \preceq ky$ for some $k \in [0, 1)$.

Theorem 2.1. *Let X be a nonempty set with a complete cone S-metric $S : X \times X \times X \to P$, P a normal cone with normal constant K and $T : X \to X$ a mapping satisfies*

$$\begin{aligned} S(Tx, Tx, Ty) \preceq\ & F(S(x, x, y), S(x, x, Tx), S(y, y, Ty), S(x, x, Ty), S(y, y, Tx), \\ & S(Tx, Tx, y)) \qquad (1) \end{aligned}$$

for all $x, y \in X$ and some $F \in \mathbb{F}$. Then, we have the followings;
(a) If F satisfies $(F1)$, then T has a fixed point. Moreover, for $x_0 \in X$ and the fixed point x,

$$S(Tx_n, Tx_n, x) \preceq \frac{2k^n}{1-k} S(x_0, x_0, Tx_0).$$

(b) If F satisfies $(F2)$, then T has a unique fixed point.
(c) If F satisfies $(F3)$ and T has a fixed point x, then T is continuous at x.

Proof. (a) For $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. From (1), we have

$$\begin{aligned} & S(x_{n+1}, x_{n+1}, x_{n+2}) = S(Tx_n, Tx_n, Tx_{n+1}) \\ \preceq\ & F(S(x_n, x_n, x_{n+1}), S(x_n, x_n, Tx_n), S(x_{n+1}, x_{n+1}, Tx_{n+1}), \\ & S(x_n, x_n, Tx_{n+1}), S(x_{n+1}, x_{n+1}, Tx_n), S(Tx_n, Tx_n, x_{n+1})) \\ =\ & F(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \\ & S(x_n, x_n, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+1})) \\ =\ & F(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \\ & S(x_n, x_n, x_{n+2}), \mathbf{0}, \mathbf{0}). \end{aligned}$$

From the condition $(S2)$ and Lemma 1.1, we get

$$\begin{aligned} S(x_n, x_n, x_{n+2}) \preceq\ & S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1}) \\ =\ & 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned}$$

Since $S(x_n, x_n, x_{n+2}), S(x_n, x_n, x_{n+1})$ and $S(x_{n+1}, x_{n+1}, x_{n+2})$ satisfy the hypothesis of $(F1)$, there exists $k \in [0, 1)$ such that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \preceq kS(x_n, x_n, x_{n+1}).$$

Applying this method sequentially, we can obtain

$$kS(x_n, x_n, x_{n+1}) \preceq k^2 S(x_{n-1}, x_{n-1}, x_n) \preceq \cdots \preceq k^{n+1} S(x_0, x_0, x_1).$$

From the above inequality, we have

$$\begin{aligned}
S(x_n, x_n, x_m) &\preceq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\
&\preceq 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_m, x_m) \\
&\preceq 2k^n S(x_0, x_0, x_1) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_m, x_m, x_{n+2}) \\
&\preceq 2k^n S(x_0, x_0, x_1) + 2k^{n+1} S(x_0, x_0, x_1) + S(x_{n+2}, x_{n+2}, x_m) \\
&\preceq \cdots \\
&\preceq 2(k^n + k^{n+1} + \cdots + k^{m-1}) S(x_0, x_0, x_1) \\
&= 2\frac{k^n(1-k^{m-n})}{1-k} S(x_0, x_0, x_1) \\
&\preceq 2\frac{k^n}{1-k} S(x_0, x_0, x_1) \ \text{ for } \ n < m, \qquad (2)
\end{aligned}$$

which implies that

$$\|S(x_n, x_n, x_m)\| \leq 2\frac{k^n}{1-k} K \|S(x_0, x_0, x_1)\| \to 0 \ \text{ as } \ n, m \to \infty.$$

Therefore, we have $S(x_n, x_n, x_m) \to \mathbf{0}$ as $n, m \to \infty$ and thus $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. From (2), we get

$$S(x_{n+1}, x_{n+1}, x_m) \preceq 2\frac{k^{n+1}}{1-k} S(x_0, x_0, x_1).$$

By taking the limits as $m \to \infty$ in ths above inequality, we have

$$S(x_{n+1}, x_{n+1}, x) \preceq 2\frac{k^{n+1}}{1-k} S(x_0, x_0, x_1),$$

which implies that

$$S(Tx_n, Tx_n, x) \preceq 2\frac{k^{n+1}}{1-k} S(x_0, x_0, x_1).$$

Now, we show that x is a fixed point of T. From (1) and Lemma 1.1, we have

$$\begin{aligned}
& S(x_{n+1}, x_{n+1}, Tx) = S(Tx_n, Tx_n, Tx) \\
\preceq\ & F(S(x_n, x_n, x), S(x_n, x_n, Tx_n), S(x, x, Tx), \\
& S(x_n, x_n, Tx), S(x, x, Tx_n), S(Tx_n, Tx_n, x)) \\
=\ & F(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \\
& S(x_n, x_n, Tx), S(x, x, x_{n+1}), S(x_{n+1}, x_{n+1}, x)) \\
=\ & F(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \\
& S(x_n, x_n, Tx), S(x, x, x_{n+1}), S(x, x, x_{n+1})).
\end{aligned}$$

Since F is continuous, taking the limits as $n \to \infty$ in the above inequality, we have

$$S(x, x, Tx) \preceq F(\overline{0}, \overline{0}, S(x, x, Tx), S(x, x, Tx), \mathbf{0}, \mathbf{0}).$$

From the above inequality and $S(x,x,Tx) \preceq 2\cdot\mathbf{0} + S(x,x,Tx)$, F satisfies the condition $(F1)$ and thus, we obtain $S(x,x,Tx) \preceq k\cdot\mathbf{0} = \mathbf{0}$. By $(S1)$, we have $x = Tx$. Thus, x is a fixed point of T.

(b) Suppose that T has two distinct fixed points y and z in X. From (1) and Lemma 1.1, we obtain

$$\begin{aligned} & S(y,y,z) = S(Ty,Ty,Tz) \\ \preceq\ & F(S(y,y,z), S(y,y,Ty), S(z,z,Tz), \\ & S(y,y,Tz), S(z,z,Ty), S(Ty,Ty,Tz)) \\ =\ & F(S(y,y,z), S(y,y,y), S(z,z,z), \\ & S(y,y,z), S(z,z,y), S(y,y,z)) \\ =\ & F(S(y,y,z), \mathbf{0}, \mathbf{0}, S(y,y,z), S(y,y,z), S(y,y,z)). \end{aligned}$$

Since F satisfies the condition $(F2)$, $S(y,y,z) = \mathbf{0}$ and thus we have $y = z$. Therefore, T has a unique fixed point.

(c) Let x be a fixed point of T and $\{x_n\}$ be a convergent sequence in X with $x_n \to x$ as $n \to \infty$. From (1) and Lemma 1.1, we obtain

$$\begin{aligned} & S(x,x,Tx_n) = S(Tx,Tx,Tx_n) \\ \preceq\ & F(S(x,x,x_n), S(x,x,Tx), S(x_n,x_n,Tx_n), \\ & S(x,x,Tx_n), S(x_n,x_n,Tx), S(Tx,Tx,Tx_n)) \\ =\ & F(S(x,x,x_n), S(x,x,x), S(x_n,x_n,Tx_n), \\ & S(x,x,Tx_n), S(x_n,x_n,x), S(x,x,Tx_n)) \\ =\ & F(S(x,x,x_n), \overline{0}, S(Tx_n,Tx_n,x_n), \\ & S(Tx_n,Tx_n,x), S(x,x,x_n), S(Tx_n,Tx_n,x)). \end{aligned} \tag{3}$$

From $(S2)$ and Lemma 1.1, we have

$$\begin{aligned} S(Tx_n,Tx_n,x_n) &\preceq 2S(Tx_n,Tx_n,x) + S(x_n,x_n,x) \\ &= 2S(Tx_n,Tx_n,x) + S(x,x,x_n). \end{aligned} \tag{4}$$

Since F satisfies the condition $(F3)$, from (3) and (4), we have

$$\begin{aligned} S(x,x,Tx_n) \preceq\ & F(S(x,x,x_n), \mathbf{0}, \mathbf{0}, \mathbf{0}, S(x,x,x_n), \mathbf{0}) \\ & + F(\mathbf{0}, \mathbf{0}, 2S(Tx_n,Tx_n,x), S(Tx_n,Tx_n,x), \mathbf{0}, S(Tx_n,Tx_n,x)). \end{aligned}$$

From the condition $(F3)$, we obtain

$$\begin{aligned} S(x,x,Tx_n) \preceq\ & F(S(x,x,x_n), \mathbf{0}, \mathbf{0}, \mathbf{0}, S(x,x,x_n), S(Tx_n,Tx_n,x)) \\ & + kS(Tx_n,Tx_n,x) \\ \preceq\ & F(S(x,x,x_n), \mathbf{0}, \mathbf{0}, \mathbf{0}, S(x,x,x_n), S(Tx_n,Tx_n,x)) \\ & + kS(x,x,Tx_n) \text{ for some } k \in [0,1), \end{aligned}$$

which implies that

$$S(x,x,Tx_n) \preceq \frac{1}{1-k}F(S(x,x,x_n),\mathbf{0},\mathbf{0},\mathbf{0},S(x,x,x_n),\mathbf{0}) \to \mathbf{0} \text{ as } n\to\infty.$$

This shows that T is continuous at x. □

If we put $F(x_1,x_2,x_3,x_4,x_5,x_6) := \phi(x_1,x_2,x_3,x_4,x_5)$, then Theorem 2.1 can be modified as follows, which is the fixed point theorem in [7].

Theorem 2.2. [7] *Let T be a self-map on a complete cone S-metric space (X,S), P a normal cone with normal constant K and*

$$S(Tx,Tx,Ty) \preceq \phi(S(x,x,y),S(x,x,Tx),S(y,y,Ty),S(x,x,Ty),S(y,y,Tx))$$

for all $x,y\in X$ and some $\phi\in\psi$. Then, we have
(a) If ϕ satisfies the condition (A_1), then T has a fixed point. Moreover, for $x_0\in X$ and the fixed point x, we have

$$S(Tx_n,Tx_n,x) \preceq \frac{2k^n}{1-k}S(x_0,x_0,Tx_0).$$

(b) If ϕ satisfies (A_2), then T has a unique fixed point.
(c) If ϕ satisfies (A_3) and T has a fixed point x, then T is continuous at x.

By putting $F(x_1,x_2,x_3,x_4,x_5,x_6) = hx_1 (h\in(0,1))$ in Theorem 2.1, then the following Theorem 2.3 can be obtain as its corollary.

Theorem 2.3. [2] *Let X be a nonempty set with a S-cone metric $S : X\times X\times X \to (E,P)$, P a normal cone with normal constant K and $T : X\to X$ a mapping satisfies the following*

$$S(Tx,Tx,Ty) \quad \preceq \quad hS(x,x,y)$$

for all $x,y\in X$ and $h\in(0,1)$. Then T has a unique fixed point.

Now, we introdue another implicit relation as follows;

Implicit Relation 2. Let $\mathbb{G}$ be the set of all continuous functions $G : P^5 \to P$ consider the following properties;

$(G1)$ there exists $k\in[0,1)$ such that for all $x,y\in P$, $y\preceq G(x,y,y,x,\frac{4x+y}{3})$ implies that $x\preceq ky$,
$(G2)$ for all $y\in P$, $y\preceq G(\mathbf{0},\mathbf{0},\mathbf{0},y,\mathbf{0})$ implies that $y=\mathbf{0}$,
$(G3)$ for all $y\in P$, $y\preceq G(y,\mathbf{0},\mathbf{0},\mathbf{0},\frac{y}{3})$ implies that $y=\mathbf{0}$.

Theorem 2.4. *Let X be a nonempty set with a cone S-metric $S : X \times X \times X \to P$, P a normal cone with normal constant K and $T : X \to X$ a mapping satisfies the following*

$$\begin{aligned} S(Tx,Ty,Tz) \ &\preceq \ G(S(x,y,z), S(x,x,Tx), S(y,y,Ty), S(z,z,Tz), \\ &\quad \frac{1}{3}\{S(x,x,Ty) + S(z,zTy) + S(y,y,Tx)\}) \end{aligned} \tag{5}$$

for all $x, y \in X$ and some $G \in \mathbb{G}$. If G satisfies $(G1)$, $(G2)$ and $(G3)$, then T has a unique fixed point.

Proof. For $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. From (5), the condition $(S2)$ and Lemma 1.1, we have

$$\begin{aligned} & S(x_{n+1}, x_{n+1}, x_n) = S(Tx_n, Tx_n, Tx_{n-1}) \\ \preceq \ & G(S(x_n,x_n,x_{n-1}), S(x_n,x_n,Tx_n), S(x_n,x_n,Tx_n), S(x_{n-1},x_{n-1},Tx_{n-1}), \\ & \frac{1}{3}\{S(x_n,x_n,Tx_n) + S(x_{n-1},x_{n-1},Tx_n) + S(x_n,x_n,Tx_n)\}) \\ = \ & G(S(x_n,x_n,x_{n-1}), S(x_n,x_n,x_{n+1}), S(x_n,x_n,x_{n+1}), S(x_{n-1},x_{n-1},x_n), \\ & \frac{1}{3}\{S(x_n,x_n,x_{n+1}) + S(x_{n-1},x_{n-1},x_{n+1}) + S(x_n,x_n,x_{n+1})\}) \\ = \ & G(S(x_n,x_n,x_{n-1}), S(x_{n+1},x_{n+1},x_n), S(x_{n+1},x_{n+1},x_n), S(x_n,x_n,x_{n-1}), \\ & \frac{1}{3}\{2S(x_{n+1},x_{n+1},x_n) + S(x_{n+1},x_{n+1},x_{n-1})\}) \\ \preceq \ & G(S(x_n,x_n,x_{n-1}), S(x_{n+1},x_{n+1},x_n), S(x_{n+1},x_{n+1},x_n), S(x_n,x_n,x_{n-1}), \\ & \frac{1}{3}\{2S(x_{n+1},x_{n+1},x_n) + 2S(x_{n+1},x_{n+1},x_n) + S(x_n,x_n,x_{n-1})\}) \\ = \ & G(S(x_n,x_n,x_{n-1}), S(x_{n+1},x_{n+1},x_n), S(x_{n+1},x_{n+1},x_n), S(x_n,x_n,x_{n-1}), \\ & \frac{1}{3}\{4S(x_{n+1},x_{n+1},x_n) + S(x_n,x_n,x_{n-1})\}). \end{aligned}$$

Since G satisfies the condition $(G1)$, there exists $k \in [0, 1)$ such that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) \ &\preceq \ kS(x_n, x_n, x_{n-1}) \preceq k^2 S(x_{n-1}, x_{n-1}, x_{n-2}) \\ &\preceq \ \cdots \preceq k^{n+1} S(x_1, x_1, x_0). \end{aligned}$$

From the above inequality, we have

$$\begin{aligned}
S(x_n,x_n,x_m) &\preceq 2S(x_n,x_n,x_{n+1})+S(x_m,x_m,x_{n+1})\\
&= 2S(x_{n+1},x_{n+1},x_n)+S(x_{n+1},x_{n+1},x_m)\\
&\preceq 2k^{n+1}S(x_1,x_1,x_0)+2S(x_{n+1},x_{n+1},x_{n+2})+S(x_m,x_m,x_{n+2})\\
&\preceq 2k^{n+1}S(x_1,x_1,x_0)+2k^{n+2}S(x_1,x_1,x_0)+S(x_{n+2},x_{n+2},x_m)\\
&\preceq \cdots\\
&\preceq 2(k^{n+1}+k^{n+2}+\cdots+k^m)S(x_1,x_1,x_0)\\
&= 2\frac{k^{n+1}(1-k^{m-n})}{1-k}S(x_1,x_1,x_0)\\
&\preceq 2\frac{k^{n+1}}{1-k}S(x_1,x_1,x_0) \text{ for } n<m, \qquad (6)
\end{aligned}$$

which implies that

$$\|S(x_n,x_n,x_m)\| \le 2\frac{k^{n+1}}{1-k}K\|S(x_1,x_1,x_0)\| \to 0 \text{ as } n,m\to\infty.$$

Therefore, we have $S(x_n,x_n,x_m)\to\overline{0}$ as $n,m\to\infty$ and thus $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\lim_{n\to\infty} x_n = x$ for some $x\in X$. Now, we show that x is a fixed point of T. From (5) and Lemma 1.1, we have

$$\begin{aligned}
&S(x_{n+1},x_{n+1},Tx)=S(Tx_n,Tx_n,Tx)\\
\preceq\ & G(S(x_n,x_n,x),S(x_n,x_n,Tx_n),S(x_n,x_n,Tx_n),S(x,x,Tx),\\
&\frac{1}{3}\{S(x_n,x_n,Tx_n)+S(x,x,Tx_n)+S(x_n,x_n,Tx_n)\})\\
=\ & G(S(x_n,x_n,x),S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_{n+1}),S(x,x,Tx),\\
&\frac{1}{3}\{S(x_n,x_n,x_{n+1})+S(x,x,x_{n+1})+S(x_n,x_n,x_{n+1})\})\\
=\ & G(S(x_n,x_n,x),S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_{n+1}),S(x,x,Tx),\\
&\frac{1}{3}\{2S(x_n,x_n,x_{n+1})+S(x,x,x_{n+1})\}).
\end{aligned}$$

Since G is continuous, taking the limits as $n\to\infty$ in the above inequality, we have

$$S(x,x,Tx)\preceq G(\mathbf{0},\mathbf{0},\mathbf{0},S(x,x,Tx),\mathbf{0}).$$

Since G satisfies the condition $(G2)$, we obtain $S(x,x,Tx)\preceq k\cdot\overline{0}=\overline{0}$. Thus, x is a fixed point of T.

Suppose that T has two distinct fixed points y and z in X. From (5) and Lemma 1.1, we obtain

$$\begin{aligned} & S(y,y,z) = S(Ty,Ty,Tz) \\ \preceq \ & G(S(y,y,z), S(y,y,Ty), S(y,y,Ty), S(z,z,Tz), \\ & \frac{1}{3}\{S(y,y,Ty) + S(z,z,Ty) + S(y,y,Ty)\}) \\ = \ & G(S(y,y,z), S(y,y,y), S(y,y,y), S(z,z,z), \\ & \frac{1}{3}\{S(y,y,y) + S(z,z,y) + S(y,y,y)\}) \\ = \ & G(S(y,y,z), \mathbf{0}, \mathbf{0}, \mathbf{0}, \frac{1}{3}S(y,y,z)). \end{aligned}$$

Since G satisfies the condition $(G3)$, we have $S(y,y,z) = \overline{0}$. From $(S1)$, we have $y = z$. Thus, T has a unique fixed point. ☐

If P is a set of nonnegative real numbers and $G(x_1, x_2, x_3, x_4, x_5) := F(x_1, x_2, x_3, x_5)$, then Theorem 2.4 can be modified as follows, which is the result in [8].

Theorem 2.5. [8] *Let X be a nonempty set with a S-metric $S : X \times X \times X \to [0, \infty)$ and $T : X \to X$ a mapping satisfies the following*

$$\begin{aligned} S(Tx,Ty,Tz) \ \preceq \ & F(S(x,y,z), S(x,x,Tx), S(y,y,Ty), \\ & \frac{1}{3}\{S(x,x,Ty) + S(z,zTy) + S(y,y,Tx)\}) \end{aligned}$$

for all $x, y \in X$ and some $F \in F_S$. If F satisfies $(R1)$, $(R2)$ and $(R3)$, then T has a unique fixed point.

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