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# SOME IDENTITIES OBTAINED BY USING THE CONCEPT OF EXPONENTIAL RIORDAN MATRICES

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Abstract. In this paper, we derive the some identities which are related to the multifactorial numbers and the Catalan numbers. We utilize the fundamental theorem of Riordan matrix to obtain those identities.

## 1. Introduction

Many combinatorial counting problems can be treated systematically using the Riordan matrix introduced by Shapiro, Getu, Woan, and Woodson [5]. In this paper, we use some elements from the exponential version of the Riordan matrix.

**Definition 1** ([1])**.** An *exponential Riordan matrix*, also denoted as an *e-Riordan matrix* is an infinite lower triangular matrix $R = [r_{n,k}]_{n,k\geq 0}$ whose $k$-th column has the exponential generating function $g(t)f(t)^k/k!$ where $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. Equivalently, $\ell_{n,k} = n![t^n]g(t)f(t)^k/k!$ where $[t^n]\sum_{i\geq 0} a_i t^i = a_n$. The matrix $R$ is denoted by $(g(t), f(t))$.

It is known [1] that if we multiply $R = (g(t), f(t))$ by a column vector $\mathbf{v} = (v_0, v_1, \ldots)^T$ corresponding to the exponential generating function $v(t) = \sum_{n\geq 0} v_n t^n/n!$, then the resulting column vector $R\mathbf{v} = (h_0, h_1, \ldots)^T$ has the exponential generating function $g(t)v(f(t)) = \sum_{n\geq 0} h_n t^n/n!$. This observation is known as the *fundamental theorem of Riordan matrix* (FTRM), and we write this as

$$(g(t), f(t))v(z) = g(t)v(f(t)).$$

The importance of the $e$-Riordan matrix is underlined by the fact that well-known combinatorial sequences such as the Stirling numbers of both kinds, Lah numbers, Bessel numbers, etc. can be expressed as $e$-Riordan matrices. Moreover, $e$-Riordan matrix methods give simple proofs of their identities. Thus the $e$-Riordan matrix has been studied combinatorially [1, 2, 3].

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In this paper, we find some identities which are related to multifactorial numbers (Theorems 2.1 and 2.2) and Catalan numbers (Corollary 2.3).

## 2. Main results

A common related notation is to use multiple exclamation points to denote a *multifactorial*, the product of integers in steps of two ($n!!$), three ($n!!!$) or more. The double factorial is the most commonly used variant, but one can similarly define the triple factorial ($n!!!$) and so on. In general, $k^{th}$ factorial, denoted by $n!^{(k)}$ is defined recursively as

$$n!^{(k)} = \begin{cases} 1 & \text{if } 0 \leq n < k; \\ n(n-k)!^{(k)} & \text{if } n \geq k, \end{cases} \tag{1}$$

where $k \in \mathbb{N}$. For instance, $(kn+1)!^{(k)} = \prod_{i=0}^{n}(k(n-i)+1)$ for $n \in \mathbb{N}$.

We define the two generating functions $F_m(t)$ and $H_m(t)$ by

$$F_m(t) = 1 - (1-mt)^{\frac{1}{m}} \quad \text{and} \quad H_m(t) = (1-mt)^{-\frac{1}{m}}.$$

**Theorem 2.1.** *For $n, k, m \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \left( \sum_{i=0}^{k} (-1)^{n-i} \binom{k}{i} \prod_{j=0}^{n-1} (i - jm) \right) = (m(n-1)+1)!^{(m)}.$$

*Proof.* By the binomial expansion,

$$(F_m(t))^k = \left(1 - (1-mt)^{\frac{1}{m}}\right)^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (1-mt)^{\frac{i}{m}}. \tag{2}$$

We note that the Taylor series expansion of $(1-mt)^{i/m}$ at $t=0$ gives

$$(1-mt)^{\frac{i}{m}} = 1 + \sum_{n\geq 1} (-1)^n \left( \prod_{j=0}^{n-1} (i-jm) \right) \frac{t^n}{n!}. \tag{3}$$

Now we consider the $e$-Riordan matrix $R = [r_{n,k}]_{n,k\geq 0} = (1, F_m(t))$. Then, by Definition 1, $r_{n,k} = n!/k![t^n](F_m(t))^k$. Thus, by (2) and (3),

$$r_{n,k} = \begin{cases} 1 & \text{if } n = k = 0; \\ \frac{1}{k!} \sum_{i=0}^{k} (-1)^{n-i} \binom{k}{i} \prod_{j=0}^{n-1} (i-jm) & \text{if } n \geq k \geq 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let $\mathbf{v} = (0!, 1!, 2!, 3!, \ldots)^T$ and $\mathbf{h} = (h_0, h_1, h_2, \ldots)^T = R\mathbf{v}$. Then $h_n = \sum_{k=0}^{n} k! h_{n,k}$. Since the exponential generating function $v(t)$ for $\mathbf{v}$ is

$$v(t) = \sum_{n\geq 0} n! \frac{t^n}{n!} = (1-t)^{-1},$$

by FTRM we obtain

$$\sum_{n\geq 0} h_n \frac{t^n}{n!} = (1, F_m(t))v(t) = v(F_m(t)) = (1-mt)^{-\frac{1}{m}} = H_m(t).$$

Therefore the Taylor series expansion of $H_m(t)$ at $t=0$ gives

$$\begin{aligned} H_m(t) = (1-mt)^{-\frac{1}{m}} &= 1 + \sum_{n\geq 1} \prod_{j=1}^{n} (m(n-j)+1) \frac{t^n}{n!} \\ &= 1 + \sum_{n\geq 1} (m(n-1)+1)!^{(m)} \frac{t^n}{n!}. \end{aligned}$$

Thus $h_n = (m(n-1)+1)!^{(m)}$ for $n \in \mathbb{N}$. Since $h_n = \sum_{k=0}^{n} k! r_{n,k}$, we obtain the desired result by (4). □

*Remark* 1. In the proof of Theorem 2.1, one can see

$$(1, F_m(t)) \begin{pmatrix} 0! \\ 1! \\ 2! \\ 3! \\ 4! \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1!^{(m)} \\ (m+1)!^{(m)} \\ (2m+1)!^{(m)} \\ (3m+1)!^{(m)} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ (m+1)1 \\ (2m+1)(m+1)1 \\ (3m+1)(2m+1)(m+1)1 \\ \vdots \end{pmatrix}.$$

By FTRM, it also can be written as

$$(1, F_m(t)) \frac{1}{1-t} = \frac{1}{1-F_m(t)} = H_m(t).$$

For instance, the first entries of Remark 1 for $m=2$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 15 & 15 & 6 & 1 & 0 & \\ 0 & 105 & 105 & 45 & 10 & 1 & \\ & & & \vdots & & & \end{pmatrix} \begin{pmatrix} 0! \\ 1! \\ 2! \\ 3! \\ 4! \\ 5! \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1!! \\ 3!! \\ 5!! \\ 7!! \\ 9!! \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 15 \\ 105 \\ 945 \\ \vdots \end{pmatrix}. \tag{5}$$

By FTRM, we obtain

$$\left(1, 1-(1-2t)^{\frac{1}{2}}\right)\frac{1}{1-t} = (1-2t)^{-\frac{1}{2}}.$$

The matrix which is obtained by deleting the first row and the first column of the matrix in (5) is the coefficient matrix of the reversed Bessel polynomials.

The explicit formula for $n$th Catalan number $C_n$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$ and its ordinarily generating function is

$$C(t) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t} = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots .$$

It is known [4] that the expansion of $C(t)^k$ for $k \in \mathbb{N}$ is

$$C(t)^k = \sum_{n\geq 0} \frac{k}{2n+k}\binom{2n+k}{n} t^n.$$

**Theorem 2.2.** *For $n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \frac{n!k}{2^{n-k}(2n-k)}\binom{2n-k}{n-k} = (2n-1)!!.$$

*Proof.* We note that $F_2(t) = 1-(1-2t)^{1/2} = tC(t/2)$. Let $R = [r_{n,k}]_{n,k\geq 0} = (1, F_2(t))$. Then

$$r_{n,k} = \frac{n!}{k!}[t^n](tC(t/2))^k = \frac{n!}{k!}[t^{n-k}]C(t/2)^k = \frac{n!k}{k!2^{n-k}(2n-k)}\binom{2n-k}{n-k}.$$

Hence, by Remark 1 for the case $m = 2$, we obtain

$$\sum_{k=0}^{n} k! r_{n,k} = \sum_{k=0}^{n} \frac{n!k}{2^{n-k}(2n-k)}\binom{2n-k}{n-k} = (2n-1)!!$$

for $n \in \mathbb{N}$. □

**Corollary 2.3.** *For $n \in \mathbb{N}$, we have the identity*

$$C_n = \sum_{k=1}^{n} \frac{2^k k}{n(n+1)}\binom{2n-k-1}{n-1}.$$

*Proof.* We note that $(2n-1)!! = \frac{(2n)!}{2^n n!}$. By Theorem 2.2, we have

$$\begin{aligned}
(2n-1)!! &= \sum_{k=0}^{n} \frac{n!k}{2^{n-k}(2n-k)} \binom{2n-k}{n-k} \\
\Rightarrow \quad \frac{(2n)!}{2^n n!} &= \sum_{k=0}^{n} \frac{k(2n-k-1)!}{2^{n-k}(n-k)!} \\
\Rightarrow \quad \frac{1}{n+1}\binom{2n}{n} &= \sum_{k=0}^{n} \frac{2^k k(2n-k-1)!}{(n+1)!(n-k)!} \\
\Rightarrow \quad C_n &= \sum_{k=1}^{n} \frac{2^k k}{n(n+1)} \binom{2n-k-1}{n-1}
\end{aligned}$$

which completes the proof. □

Various combinatorial interpretations and identities of Catalan numbers have been introduced. We checked whether the identity in Corollary 2.3 is on A000108 in the On-Line Encyclopedia of Integer Sequences (OEIS) [6] or not, but our identity is not there.

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