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EXISTENCE OF POSITIVE SOLUTION FOR THE SECOND ORDER DIFFERENTIAL SYSTEMS WITH INTEGRAL BOUNDARY CONDITIONS.

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Abstract. This paper is concerned with the existence of positive solutions to the second order differential systems with strongly coupled integral boundary value conditions. The fixed point index theorems are used for the main results.

1. Introduction

The main problem of this paper is motivated from the existence of positive radial solutions to the following nonlocal boundary value system :

$$\begin{cases} \Delta u + h_1(|x|)f_1(u(x), v(x)) = 0, & x \in E_{r_0}, \\ \Delta v + h_2(|x|)f_2(u(x), v(x)) = 0, & x \in E_{r_0}, \\ u(x) \to 0, \quad v(x) \to 0, & \text{if } |x| \to \infty, \\ u(x) = \int_{E_{r_0}} l_1(|y|)u(y) + l_2(|y|)v(y)dy, & \text{if } |x| = r_0, \\ v(x) = \int_{E_{r_0}} l_3(|y|)u(y) + l_4(|y|)v(y)dy, & \text{if } |x| = r_0, \end{cases} \tag{1}$$

where $E_{r_0} = \{x \in \mathbb{R}^N : |x| \geq r_0 \text{ for } r_0 > 0, N \geq 3\}$, $h_i \in C((r_0, \infty), (0, \infty))$ is such that $\int_{r_0}^{\infty} rh_i(r)dr < \infty$, $f_i \in C([0, \infty)^2, [0, \infty))$ for $i = 1, 2$, and $l_j \in L^1((r_0, \infty))$ is a nonnegative function satisfying $0 < w_N r_0^{N-2} \int_{r_0}^{\infty} rl_j(r)dr < 1$ for each $j = 1, 2, 3, 4$, when w_N is the surface area of unit sphere in $\mathbb{R}^N$.

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Such differential equations with an integral boundary condition arise in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma phenomena([2] and [3]). One may refer to [1]∼[6] and [8]∼[10] for integral boundary value problems and the references therein.

Note that the change of variables $r = |x|$ and $t = (\frac{r}{r_0})^{2-N}$ transforms (1) into:

$$\begin{cases} u''(t) + a_1(t)f_1(u(t), v(t)) = 0, & t \in (0,1), \\ v''(t) + a_2(t)f_2(u(t), v(t)) = 0, & t \in (0,1), \\ u(0) = 0 = v(0), \\ u(1) = \int_0^1 g_1(s)u(s) + g_2(s)v(s)ds, \\ v(1) = \int_0^1 g_3(s)u(s) + g_4(s)v(s)ds \end{cases} \tag{2}$$

with

$$a_i(t) = \left(\frac{1}{N-2}\right)^2 r_0^2 t^{\frac{-2(N-1)}{N-2}} h_i\left(r_0 t^{\frac{-1}{N-2}}\right),$$
$$g_i(t) = w_N \left(\frac{1}{N-2}\right) r_0^N t^{\frac{-2(N-1)}{N-2}} l_i\left(r_0 t^{\frac{-1}{N-2}}\right),$$

where $a_i \in C((0,1), [0,\infty))$ such that $\int_0^1 s(1-s)a_i(s)ds < \infty$ for $i \in \{1,2\}$ and a nonnegative function $g_i \in L^1(0,1)$ is such that $0 < \int_0^1 sg_j(s)ds < 1$ for each $j \in \{1,2,3,4\}$. We know that the existence of positive solutions for the system (2) guarantees the existence of positive radial solutions for (1). Hence we focus on the system (2) to investigate solutions for (1).

Throughout this paper, we assume the following hypothesis;

$(H1)$ $\left(1 - \int_0^1 sg_1(s)ds\right)\left(1 - \int_0^1 sg_4(s)ds\right) - \int_0^1 sg_2(s)ds \int_0^1 sg_3(s)ds > 0.$

$(H2)$ There exist constants λ_{ij}, μ_{ij} with $0 < \lambda_{ij} \le \mu_{ij}$, $\sum_{j=1}^2 \lambda_{ij} > 1$ *for* $i, j \in \{1,2\}$ such that for $t \in (0,1)$, $u, v \in (0,\infty)$, and $i \in \{1,2\}$,

$$c^{\mu_{i1}} f_i(u,v) \le f_i(cu, v) \le c^{\lambda_{i1}} f_i(u,v), \text{ if } 0 < c \le 1, \tag{3}$$
$$c^{\mu_{i2}} f_i(u,v) \le f_i(u, cv) \le c^{\lambda_{i2}} f_i(u,v), \text{ if } 0 < c \le 1. \tag{4}$$

Remark 1. (i) (3) and (4) imply

$$c^{\lambda_{i1}} f_i(u,v) \le f_i(cu, v) \le c^{\mu_{i1}} f_i(u,v), \text{ if } c \ge 1 \text{ for } i \in \{1,2\}, \tag{5}$$

and

$$c^{\lambda_{i2}} f_i(u,v) \le f_i(u, cv) \le c^{\mu_{i2}} f_i(u,v), \text{ if } c \ge 1 \text{ for } i \in \{1,2\}, \tag{6}$$

respectively. Conversely, (5) implies (3) and (6) implies (4).

(ii) (3) and (4) imply

$$f_i(u_1, u_2) \le f_i(v_1, v_2), \text{ if } 0 < u_j \le v_j, \text{ for } i, j \in \{1, 2\}. \tag{7}$$

This paper is organized as follows. In Section 2, we shall give some preliminary results and lemmas to prove our main results. In Section 3, the main result, Theorem 3.1, is proven.

2. Preliminaries

We set up the operator for problem (2). Let $E := C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be the Banach space with the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$. Let us denote

$$A := \begin{pmatrix} 1 - \int_0^1 sg_1(s)ds & -\int_0^1 sg_2(s)ds \\ -\int_0^1 sg_3(s)ds & 1 - \int_0^1 sg_4(s)ds \end{pmatrix},$$

then by $(H1)$, $\det A \neq 0$ and $a_{ij} > 0$ for all $i, j \in \{1, 2\}$, where

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Define

$$P := \{(u,v) \in E \mid u(t) \ge \gamma t \|(u,v)\|, \ v(t) \ge \gamma t \|(u,v)\|, t \in [0,1]\},$$

where

$$\rho = \max\left\{1 + \int_0^1 q_1(\tau)d\tau, 1 + \int_0^1 q_2(\tau)d\tau, \int_0^1 q_3(\tau)d\tau, \int_0^1 q_4(\tau)d\tau\right\},$$

$$\nu = \min\left\{\int_0^1 \tau(1-\tau)q_j(\tau)d\tau \mid j = 1,2,3,4\right\} \text{ and } 0 < \gamma = \frac{\nu}{\rho} < 1$$

with $q_1(\tau) := a_{11}g_1(\tau) + a_{12}g_3(\tau)$, $q_2(\tau) := a_{21}g_2(\tau) + a_{22}g_4(\tau)$, $q_3(\tau) := a_{11}g_2(\tau) + a_{12}g_4(\tau)$, and $q_4(\tau) := a_{21}g_1(\tau) + a_{22}g_3(\tau)$. Clearly, P is a cone of E and we define $S_1,\ S_2 : P \to Q = \{u \in C([0,1],\mathbb{R}) \mid u(t) \ge 0, t \in [0,1]\}$ by

$$S_1(u,v)(t) := \int_0^1 (H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s)))ds,$$

$$S_2(u,v)(t) := \int_0^1 (H_2(t,s)a_2(s)f_2(u(s),v(s)) + tK_2(s)a_1(s)f_1(u(s),v(s)))ds,$$

where

$$\begin{aligned} H_1(t,s) &= G(t,s) + t\int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau, \\ H_2(t,s) &= G(t,s) + t\int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau, \\ K_1(s) &= \int_0^1 G(\tau,s)(a_{11}g_2(\tau) + a_{12}g_4(\tau))d\tau, \\ K_2(s) &= \int_0^1 G(\tau,s)(a_{21}g_1(\tau) + a_{22}g_3(\tau))d\tau, \end{aligned}$$

and

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

Now we define an operator $S : P \to Q\times Q$ by

$$S(u,v)=(S_1(u,v),S_2(u,v)).$$

Remark 2. It is easy to check that

$$t(1-t)s(1-s)\le G(t,s)=G(s,t)\le s(1-s),\ t,s\in[0,1]. \tag{8}$$

From (8), we have

$$H_i(t,s)\le \rho s(1-s),\ K_i(s)\le \rho s(1-s),\ i\in\{1,2\}, \tag{9}$$

$$H_i(t,s)\ge \nu ts(1-s),\ K_i(s)\ge \nu s(1-s),\ i\in\{1,2\}. \tag{10}$$

For $(u,v)\in P$, let c be a positive number such that $c>\max\{\|(u,v)\|,1\}$. From (3), (4) and (7), we have

$$f_i(u(t),v(t))\le f_i(c,c)\le c^{\mu_{i1}+\mu_{i2}}f_i(1,1),\ i\in\{1,2\}. \tag{11}$$

By (9) and (11), we have

$$\begin{aligned}
&S_i(u,v)(t)=\int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s))ds+t\int_0^1 K_1(s)a_2(s)f_2(u(s),v(s))ds\\
&\le \rho\int_0^1 s(1-s)a_1(s)f_1(u(s),v(s))ds+\rho t\int_0^1 s(1-s)a_2(s)f_2(u(s),v(s))ds\\
&\le \rho c^{\mu_{11}+\mu_{12}}\int_0^1 s(1-s)a_1(s)f_1(1,1)ds+\rho t c^{\mu_{21}+\mu_{22}}\int_0^1 s(1-s)a_2(s)f_2(1,1)ds.
\end{aligned}$$

Thus S is well defined on P and it is notice that S is completely continuous, by standard argument and if $(u,v)\in P$ is a fixed point of S, then (u,v) is a positive solution of differential system (2).

Lemma 2.1. *Assume that* $(H1)$ *and* $(H2)$ *hold. Then* $S(P)\subset P$.

Proof. From (10), for $t,s\in[0,1]$, we know

$$K_i(s)\ge \gamma\rho s(1-s),\ H_i(t,s)\ge \gamma\rho ts(1-s),\ i\in\{1,2\}. \tag{12}$$

Then by (9) and (12), we have for $\tau,t,s\in[0,1]$,

$$H_i(t,s)\ge \gamma tH_j(\tau,s),\ K_i(s)\ge \gamma H_j(\tau,s),\ H_i(t,s)\ge \gamma tK_j(s),\ i,j\in\{1,2\}. \tag{13}$$

For $(u,v)\in P$ and $t,\tau\in[0,1]$, by using (13), we have

$$\begin{aligned}
&S_1(u,v)(t)=\int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s))ds+t\int_0^1 K_1(s)a_2(s)f_2(u(s),v(s))ds\\
&\ge \gamma t\int_0^1 H_1(\tau,s)a_1(s)f_1(u(s),v(s))ds+\gamma t\tau\int_0^1 K_1(s)a_2(s)f_2(u(s),v(s))ds\\
&=\gamma tS_1(u,v)(\tau),
\end{aligned}$$

and

$$S_1(u,v)(t) = \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s))ds + t\int_0^1 K_1(s)a_2(s)f_2(u(s),v(s))ds$$
$$\geq \gamma t\tau \int_0^1 K_2(s)a_1(s)f_1(u(s),v(s))ds + \gamma t\int_0^1 H_2(\tau,s)a_2(s)f_2(u(s),v(s))ds$$
$$= \gamma t S_2(u,v)(\tau).$$

Then $S_1(u,v)(t) \geq \gamma t\|S_1(u,v)\|_\infty$ and $S_1(u,v)(t) \geq \gamma t\|S_2(u,v)\|_\infty$ and thus

$$S_1(u,v)(t) \geq \gamma t\|(S_1(u,v), S_2(u,v))\|.$$

In the same way, we obtain that $S_2(u,v)(t) \geq \gamma t\|(S_1(u,v), S_2(u,v))\|$. Therefore $S(P) \subset P$.

□

To show the existence of a positive solution of (2), we need the following lemmas for fixed point index argument in [7].

Lemma 2.2. *Let X be a Banach space, P a cone in X. For $r > 0$, define $P_r = \{x \in P : \|x\| < r\}$. Assume that $T : \bar{P}_r \to P$ is a compact map such that $Tx \neq x$ for all $x \in \partial P_r$. If $\|x\| \leq \|Tx\|$ for all $x \in \partial P_r$, then*

$$i(T, P_r, P) = 0.$$

Lemma 2.3. *Let X be a Banach space, P a cone in X and Ω bounded open in X. Let $0 \in \Omega$ and $T : P \cap \bar{\Omega} \to P$ be condensing. Suppose that $Tx \neq \nu x$ for all $x \in P \cap \partial\Omega$ and all $\nu \geq 1$. Then*

$$i(T, P \cap \Omega, P) = 1.$$

3. Main Result

Theorem 3.1. *Assuming that $(H1)$ and $(H2)$ hold, the differential system (2) has at least one positive solution.*

Proof. Choose a constant $R > 0$ such that

$$R > \max\{\frac{1}{\gamma} + 1,\ (\sigma\gamma^{\lambda_{11}+\lambda_{12}})^{-\frac{1}{\lambda_{11}+\lambda_{12}-1}},\ (\sigma\gamma^{\lambda_{21}+\lambda_{22}})^{-\frac{1}{\lambda_{21}+\lambda_{22}-1}}\},$$

where $\sigma = \frac{\nu}{4}\int_0^1 (\gamma s)^{\mu_{11}+\mu_{12}} s(1-s)a_1(s)f_1(1,1)ds > 0$. For real constant $r > 0$, define $\Omega_r = \{(u,v) \in P \mid \|(u,v)\| < r\}$. We may suppose that $S(u,v) \neq (u,v)$ for $(u,v) \in \partial\Omega_R$ since otherwise the proof is done. For $(u_1,v_1) \in \partial\Omega_R$, by the definition of P and the choice of R,

$$u_1(s) \geq \gamma s\|(u_1,v_1)\| \geq \gamma s\|u_1\|_\infty,\ v_1(s) \geq \gamma s\|(u_1,v_1)\| \geq \gamma s\|v_1\|_\infty \tag{14}$$

and

$$\|u_1\|_\infty \geq u_1(1) \geq \gamma\|(u_1,v_1)\| = \gamma R > 1,\ \|v_1\|_\infty \geq \gamma\|(u_1,v_1)\| > 1. \tag{15}$$

By using (3) $\sim$ (6) and (15), it is easy to check that for $s \in [0,1]$,

$$f(\gamma s\|u_1\|_\infty, \gamma s\|v_1\|_\infty) \geq (\gamma s)^{\mu_{11}+\mu_{12}}\|u_1\|_\infty^{\lambda_{11}}\|v_1\|_\infty^{\lambda_{12}} f(1,1). \tag{16}$$

By (7), (14) $\sim$ (16), we have, for $t \in [\frac{1}{4}, 1]$,

$$\begin{aligned} S_1(u_1,v_1)(t) &\geq \int_0^1 H_1(t,s)a_1(s)f_1(u_1(s),v_1(s))ds \\ &\geq \frac{\nu}{4}\int_0^1 s(1-s)a_1(s)f_1(\gamma s\|u_1\|_\infty, \gamma s\|v_1\|_\infty)ds \\ &\geq \frac{\nu}{4}\|u_1\|_\infty^{\lambda_{11}}\|v_1\|_\infty^{\lambda_{12}}\int_0^1 s(1-s)(\gamma s)^{\mu_{11}+\mu_{12}}a_1(s)f_1(1,1)ds \\ &= \sigma\|u_1\|_\infty^{\lambda_{11}}\|v_1\|_\infty^{\lambda_{12}} \\ &\geq \sigma(\gamma\|(u_1,v_1)\|)^{\lambda_{11}}(\gamma\|(u_1,v_1)\|)^{\lambda_{12}} \\ &= \sigma\gamma^{\lambda_{11}+\lambda_{12}}R^{\lambda_{11}+\lambda_{12}}. \end{aligned}$$

Since $1-(\lambda_{11}+\lambda_{12})<0$, $\sigma\gamma^{\lambda_{11}+\lambda_{12}} \geq R^{1-(\lambda_{11}+\lambda_{12})}$ and we obtain

$$\begin{aligned} \|S(u_1,v_1)\| &\geq \|S_1(u_1,v_1)\|_\infty \\ &\geq \sigma\gamma^{\lambda_{11}+\lambda_{12}}R^{\lambda_{11}+\lambda_{12}} \\ &\geq R^{1-(\lambda_{11}+\lambda_{12})}R^{\lambda_{11}+\lambda_{12}} \\ &= R = \|(u_1,v_1)\|. \end{aligned}$$

By Lemma 2.2, we have

$$i(S,\Omega_R,P)=0. \tag{17}$$

Next, we claim that

$$S(u,v) \neq \tau(u,v), \text{ for all } (u,v)\in\partial\Omega_r, \ \tau \geq 1, \tag{18}$$

where

$$0<r<\min\{\frac{1}{2}, \delta^{-\frac{1}{\lambda-1}}\}, \ \lambda=\min\{\lambda_{11}+\lambda_{12}, \ \lambda_{21}+\lambda_{22}\}>1,$$

$$\delta=\rho\left(\int_0^1 s(1-s)a_1(s)f_1(1,1)ds+\int_0^1 s(1-s)a_2(s)f_2(1,1)ds\right).$$

Otherwise, there exist $(u_2,v_2)\in\partial\Omega_r$ and $\bar{\tau}\geq 1$ such that

$$S(u_2,u_2)=\bar{\tau}(u_2,v_2). \tag{19}$$

Without loss of generality, we assume that $\|u_2\|_\infty \geq \|v_2\|_\infty$ and we know

$$u_2(s)\leq\|u_2\|_\infty=\|(u_2,v_2)\|=r<1, \ v_2(s)\leq\|v_2\|_\infty\leq\|u_2\|_\infty=r<1 \tag{20}$$

By using (3), (4), (7), (9), (19) and (20), it follows that

$$\begin{aligned}
&\bar{\tau}u_2(t) = S_1(u_2,v_2)(t)\\
\leq\ & \int_0^1 H_1(t,s)a_1(s)f_1(u_2(s),v_2(s))ds + \int_0^1 K_1(s)a_2(s)f_2(u_2(s),v_2(s))ds\\
\leq\ & \rho\int_0^1 s(1-s)a_1(s)f_1(r,r)ds + \rho\int_0^1 s(1-s)a_2(s)f_2(r,r)ds\\
\leq\ & \rho r^{\lambda_{11}+\lambda_{12}}\int_0^1 s(1-s)a_1(s)f_1(1,1)ds + \rho r^{\lambda_{21}+\lambda_{22}}\int_0^1 s(1-s)a_2(s)f_2(1,1)ds\\
\leq\ & \delta r^{\lambda},\ t\in[0,1].
\end{aligned}$$

Consequently,

$$r = \|u_2\|_\infty < \bar{\tau}\|u_2\|_\infty \leq \delta r^{\lambda},$$

namely

$$r \geq \delta^{-\frac{1}{\lambda-1}},$$

which is a contradiction. Hence (18) is true and by Lemma 2.3, we have

$$i(S,\Omega_r,P) = 1. \tag{21}$$

By (17), (21) and the properties of the fixed point index, we have

$$i(S,\Omega_R\setminus\overline{\Omega_r},P) = i(S,\Omega_R,P) - i(S,\Omega_r,P) = -1.$$

Thus S has at least one fixed on $\Omega_R\setminus\overline{\Omega_r}$. This means that differential system (2) has at least one positive solution. The proof is complete. □

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