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LOCAL CALDERÓN-ZYGMUND ESTIMATES FOR PARABOLIC EQUATIONS WITH DUAL DATA

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Abstract. We establish the interior L^q regularity estimates for spatial gradient of weak solutions to nonlinear parabolic equations with the inhomogeneity which is given by the divergence and nondivergence data.

1. Introduction

In this paper, we investigate the interior regularity properties of the solutions to inhomogeneous nonlinear parabolic equations of the form:

$$u_t - \operatorname{div} \mathbf{a}(Du) = g - \operatorname{div} F \quad \text{in } \ \Omega_T, \tag{1}$$

where $\Omega_T := \Omega \times (0,T)$ is a space-time cylinder over a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$ and T is a positive constant, $u = u(x,t)$ is a real valued function, u_t is the time derivative of u, $Du = D_x u \in \mathbb{R}^n$ is the spatial gradient of u, and $g : \Omega_T \to \mathbb{R}$ and $F : \Omega_T \to \mathbb{R}^n$ are some given functions. The nonlinearity $\mathbf{a}$ is supposed to satisfy the conditions:

$$|\mathbf{a}(\xi)| \leqslant L|\xi|, \ |D_\xi \mathbf{a}(\xi)| \leqslant L \tag{2}$$

and

$$D_\xi \mathbf{a}(\xi)\, \eta \cdot \eta \geqslant \nu |\eta|^2 \tag{3}$$

for any $\xi, \eta \in \mathbb{R}^n$ and for some constants ν, L with $0 < \nu \leqslant 1 \leqslant L$. The main aim of this paper is to show the validity of the following implication:

$$g \in L^{q_*}_{\mathrm{loc}}, \ F \in L^q_{\mathrm{loc}} \ \Rightarrow \ Du \in L^q_{\mathrm{loc}} \ \text{ for any } q \geqslant 2, \tag{4}$$

where $q_* := \frac{(n+2)q}{n+2+q}$ with the corresponding Calderón-Zygmund type estimates (see (5)). The result in (4) ultimately means that the given functions g and F at least have the integrability properties in (4) in order to obtain the L^q integrability of the spatial gradient Du.

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In the stationary case, L^q regularity theory for the spatial gradient of the solution with Calderón-Zygmund type estimates has been studied by Iwaniec [9] for the p-Laplacian equations when $p \geqslant 2$ and by DiBenedetto and Manfredi [7] for the p-Laplacian systems with $1 < p < \infty$. More general nonlinear elliptic problems have been treated by Caffarelli and Peral [5] and Acerbi and Mingione [1]. Similar parabolic problems that are related to our equation (1) have been considered for parabolic p-Laplacian systems by Acerbi and Mingione [2]. In particular, they developed a new approach that avoids heavy harmonic analysis tools and uses covering and comparison arguments on the super level set of solutions allowing to treat intrinsic cylinders. For more general equations and systems we refer to [3, 4, 6]. Our proof is based on the approach of Acerbi and Mingione but it is not necessary to adopt the intrinsic geometry viewpoint because we only consider the case $p = 2$. The key idea in proving our results is to derive Calderón-Zygmund type estimates of solution to the modified equation which is involved with the solution to the heat equation with the inhomogeneity g given in (1). It could make the proof steps simpler.

For the equation (1), we are dealing with the weak solution u, which is defined as a function $u \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ satisfying

$$\int_{\Omega_T} -u\varphi_t + \mathbf{a}(Du)\cdot D\varphi \, dz = \int_{\Omega_T} g\varphi + F \cdot D\varphi \, dz$$

for every $\varphi \in C_0^\infty(\Omega_T)$. Note that the existence of the weak solution u can be shown in the case that the inhomogeneity $g - \mathrm{div} F$ belongs to the dual space $L^2(0,T;W^{-1,2}(\Omega))$, see for instance [11].

Our main result is the following:

Theorem 1.1. *Let $q \geqslant 2$ and $u \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ be a weak solution of* (1). *Suppose $g \in L^{q_*}_{loc}(\Omega_T)$ and $F \in L^q_{loc}(\Omega_T)$ where $q_* := \frac{(n+2)q}{n+2+q}$. Then we have $Du \in L^q_{loc}(\Omega_T)$ with the estimate*

$$\left(\fint_{Q_r(z_0)} |Du|^q \, dz \right)^{\frac{1}{q}} \leqslant c\left[\left(\fint_{Q_{2r}(z_0)} |Du|^2 \, dz \right)^{\frac{1}{2}} + r\left(\fint_{Q_{2r}(z_0)} |g|^{q_*} \, dz \right)^{\frac{1}{q_*}} + \left(\fint_{Q_{2r}(z_0)} |F|^q \, dz \right)^{\frac{1}{q}} \right] \tag{5}$$

for any $Q_{2r}(z_0) \Subset \Omega_T$, where a constant $c > 0$ depends on $n, \nu, L,$ and q.

2. Comparison estimates

We start this section with some standard notations. A parabolic cylinder $Q_r(z_0)$ where $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ is denoted as $Q_r(z_0) := B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$, where $B_r(x_0)$ is the open ball in $\mathbb{R}^n$ with the center x_0 and radius $r > 0$. We denote $\partial_{\mathrm{p}} Q_r(z_0) := (B_r(x_0) \times \{t = t_0 - r^2\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0 + r^2))$ as the parabolic boundary of $Q_r(z_0)$. When no confusion arises, omitting the reference point, we simply write $Q_r = Q_r(z_0)$. For an integrable function $f : U \to \mathbb{R}^m$ with $U \subset \mathbb{R}^N$, we define $\fint_U f\,dz := \frac{1}{|U|}\int_U f\,dz$, where $|U|$ is the Lebesgue measure of U in $\mathbb{R}^N$. To simplify the notation, the letter c will denote a positive universal constant which may vary at each appearance throughout the paper.

Let $Q_\rho = Q_\rho(z_0) \Subset \Omega_T$. As mentioned in the introduction, for given g in the equation (1), we consider the following Dirichlet problem for heat equations:

$$\begin{cases} v_t - \Delta v &= g \quad \text{in} \quad Q_\rho, \\ v &= 0 \quad \text{on} \quad \partial_{\mathrm{p}} Q_\rho. \end{cases} \tag{6}$$

According to the standard L^γ regularity theory, it is well known that

$$\fint_{Q_\rho} |v_t|^\gamma + |D^2 v|^\gamma\,dz \leqslant c \fint_{Q_\rho} |g|^\gamma\,dz$$

holds for some constant $c = c(n,\gamma) > 0$. Then by virtue of Sobolev's inequality, we see that

$$\left(\fint_{Q_\rho}\left(\frac{|Dv|}{\rho}\right)^{\gamma^*} dz\right)^{\frac{1}{\gamma^*}} \leqslant c\left(\fint_{Q_\rho} |v_t|^\gamma + |D^2 v|^\gamma\,dz\right)^{\frac{1}{\gamma}} \leqslant c\left(\fint_{Q_\rho} |g|^\gamma\,dz\right)^{\frac{1}{\gamma}} \tag{7}$$

for any $1 < \gamma < n+2$ where $\gamma^* := \frac{(n+2)\gamma}{n+2-\gamma}$, which implies that if $g \in L^\gamma(Q_\rho)$, then $Dv \in L^{\gamma^*}(Q_\rho)$.

On the other hand, the problem (6) can be rewritten as

$$\begin{cases} v_t - \operatorname{div}(Dv) &= g \quad \text{in} \quad Q_\rho, \\ v &= 0 \quad \text{on} \quad \partial_{\mathrm{p}} Q_\rho. \end{cases}$$

Combining this problem with (1), we then discover that

$$u_t - \operatorname{div}\mathbf{a}(Du) = \partial_t v - \operatorname{div}(Dv) - \operatorname{div}F \text{ in } Q_\rho,$$

which implies

$$(u-v)_t - \operatorname{div}\mathbf{a}(D(u-v) + Dv) = -\operatorname{div}(Dv + F) \text{ in } Q_\rho.$$

Setting $w := u - v$, we observe that w satisfies

$$w_t - \operatorname{div}\mathbf{a}(Dw + Dv) = -\operatorname{div}(Dv + F) \text{ in } Q_\rho \tag{8}$$

in the weak sense. Then we obtain the following comparison estimates:

Lemma 2.1. *Let h be any weak solution to the homogeneous problem*

$$\begin{cases} h_t - \operatorname{div}\mathbf{a}(Dh) &= 0 \quad \text{in} \quad Q_\rho, \\ h &= w \quad \text{on} \quad \partial_{\mathrm{p}} Q_\rho, \end{cases} \tag{9}$$

where w solves the equation (8). Then for any $\varepsilon > 0$, there exists a small $\delta = \delta(n,\nu,L,\varepsilon) > 0$ such that if

$$\fint_{Q_\rho} |Dw|^2 + \frac{1}{\delta}(|F|^2 + |Dv|^2)\,dz \leqslant 1, \tag{10}$$

then

$$\fint_{Q_\rho} |Dw - Dh|^2 \, dz \leqslant \varepsilon.$$

Moreover, we have

$$\|Dh\|_{L^\infty(Q_{\frac{\rho}{2}})} \leqslant c_{Lip}$$

for some $c_{Lip} = c_{Lip}(n, \nu, L) \geqslant 1$*.*

Proof. We take $w - h$ as a test function in (8) and (16) using Steklov averages to obtain

$$\begin{aligned} &\fint_{Q_\rho} w_t(w-h)\, dz + \fint_{Q_\rho} \mathbf{a}(Dw + Dv) \cdot D(w-h)\, dz \\ &= \fint_{Q_\rho} (Dv + F) \cdot D(w-h)\, dz \end{aligned}$$

and

$$\fint_{Q_\rho} h_t(w-h)\, dz + \fint_{Q_\rho} \mathbf{a}(Dh) \cdot D(w-h)\, dz = 0.$$

Combining two previous estimates, we then have

$$\begin{aligned} &\fint_{Q_\rho} (w-h)_t(w-h)\, dz + \fint_{Q_\rho} \big(\mathbf{a}(Dw + Dv) - \mathbf{a}(Dh)\big) \cdot (Dw - Dh)\, dz \\ &= \fint_{Q_\rho} (Dv + F) \cdot (Dw - Dh)\, dz. \end{aligned}$$

Since

$$\begin{aligned} &\big(\mathbf{a}(Dw + Dv) - \mathbf{a}(Dh)\big) \cdot (Dw - Dh) \\ &= (\mathbf{a}(Dw + Dv) - \mathbf{a}(Dw)) \cdot (Dw - Dh) + (\mathbf{a}(Dw) - \mathbf{a}(Dh)) \cdot (Dw - Dh), \end{aligned}$$

it follows that

$$\begin{aligned} &\fint_{Q_\rho} (w-h)_t(w-h)\, dz + \fint_{Q_\rho} \big(\mathbf{a}(Dw) - \mathbf{a}(Dh)\big) \cdot (Dw - Dh)\, dz \\ &= \fint_{Q_\rho} (Dv + F) \cdot (Dw - Dh)\, dz - \fint_{Q_\rho} (\mathbf{a}(Dw + Dv) - \mathbf{a}(Dw)) \cdot (Dw - Dh)\, dz. \end{aligned} \tag{11}$$

Note that

$$\begin{aligned} \int_{Q_\rho} (w-h)_t(w-h)\, dz &= \int_{Q_\rho} \frac{1}{2}\frac{\partial}{\partial t}(w-h)^2\, dz \\ &= \frac{1}{2}\int_{B_r} (w-h)^2\, dx\bigg|_{t=r^2} - \frac{1}{2}\int_{B_r} (w-h)^2\, dx\bigg|_{t=-r^2} \geqslant 0. \end{aligned}$$

Noting that the ellipticity condition (3) means the monotonicity condition

$$|\xi - \eta|^2 \leqslant c(\nu)\, \big(\mathbf{a}(\xi) - \mathbf{a}(\eta)\big) \cdot (\xi - \eta)$$

for any $\xi, \eta \in \mathbb{R}^n$, we see that

$$|Dw - Dh|^2 \leqslant c(\nu)\, \big(\mathbf{a}(Dw) - \mathbf{a}(Dh)\big) \cdot (Dw - Dh).$$

Moreover, from (2) we derive

$$\begin{aligned}\big|\mathbf{a}(Dw+Dv)-\mathbf{a}(Dw)\big| &\leqslant \left|\int_0^1 \frac{\partial}{\partial\tau}\mathbf{a}(Dw+\tau Dv)\,d\tau\right| \\ &\leqslant \int_0^1 \big|D_\xi\mathbf{a}(Dw+\tau Dv)\big|\,d\tau|Dv| \leqslant L|Dv|.\end{aligned}$$

Then we have

$$\begin{aligned}c(\nu)\big|\mathbf{a}(Dw+Dv)-\mathbf{a}(Dw)\big||Dw-Dh| &\leqslant c(\nu,L)\,|Dv||Dw-Dh| \\ &\leqslant \kappa_1|Dw-Dh|^2+c(\kappa_1,\nu,L)|Dv|^2\end{aligned}$$

for any $\kappa_1>0$, by Young's inequality.

On the other hand, for the first term on the right hand side of (11), using Young's inequality, we have

$$\begin{aligned}&\fint_{Q_\rho}(Dv+F)\cdot(Dw-Dh)\,dz \\ &\leqslant \kappa_2\fint_{Q_\rho}|Dw-Dh|^2\,dz+c(\kappa_2)\fint_{Q_\rho}|Dv|^2+|F|^2\,dz\end{aligned}$$

for any $\kappa_2>0$.

Therefore from (11) we obtain

$$\fint_{Q_r}|Dw-Dh|^2\,dz \leqslant \kappa\fint_{Q_\rho}|Dw-Dh|^2\,dz+c(\kappa)\fint_{Q_\rho}|Dv|^2+|F|^2\,dz$$

for any $\kappa>0$, and choose $\kappa=\frac{1}{2}$ to derive

$$\fint_{Q_\rho}|Dw-Dh|^2\,dz \leqslant c\fint_{Q_\rho}|Dv|^2+|F|^2\,dz \leqslant c\delta$$

by the assumption (10). We now choose $\delta\in(0,1)$ so small that $c\,\delta\leqslant\varepsilon$ to discover

$$\fint_{Q_\rho}|Dw-Dh|^2\,dz\leqslant\varepsilon.$$

Moreover, we infer

$$\fint_{Q_\rho}|Dh|^2\,dz \leqslant \fint_{Q_\rho}|Dw-Dh|^2\,dz+\fint_{Q_\rho}|Dw|^2\,dz\leqslant\varepsilon+1,$$

where the assumption (10) was used in the last inequality, and then it follows that

$$\|Dh\|_{L^\infty(Q_{\frac{\rho}{2}})}\leqslant c\Big(\fint_{Q_\rho}|Dh|^2\,dz\Big)^{\frac{1}{2}}\leqslant c_{Lip}$$

for some $c_{Lip}=c_{Lip}(n,\nu,L)>0$. □

3. Proof of Theorem 1.1

The following is technical lemma that will be used in the proof of our main theorem.

Lemma 3.1 (Lemma 6.1 in [8])**.** *Let $\psi : [R_1, R_2] \to [0,\infty)$ be a bounded function. Suppose that for any ρ_1 and ρ_2 with $0 < R_1 \leqslant \rho_1 < \rho_2 \leqslant R_2$,*

$$\psi(\rho_1) \leqslant \vartheta\psi(\rho_2) + \frac{\alpha}{(\rho_2-\rho_1)^\kappa} + \beta$$

where $\alpha > 0$ and $\beta \geqslant 0$, $\kappa > 0$ and $\vartheta \in [0,1)$. Then there exists $c = (\vartheta,\kappa) > 0$ such that

$$\psi(R_1) \leqslant c(\vartheta,\kappa)\left[\frac{\alpha}{(R_2-R_1)^\kappa} + \beta\right].$$

Now we prove our main Theorem 1.1. Its proof is divided into three steps.

Step 1. (Covering by stopping time argument)

Fix any $Q_{2r} = Q_{2r}(z_0) \Subset \Omega_T$. Recalling (8), we consider a weak solution w to

$$w_t - \operatorname{div}\mathbf{a}(Dw + Dv) = -\operatorname{div}(Dv + F) \text{ in } Q_{2r}, \tag{12}$$

where v solves

$$\begin{cases} \partial_t v - \Delta v &= g \quad \text{in} \quad Q_{2r}, \\ v &= 0 \quad \text{on} \quad \partial_{\mathrm{p}} Q_{2r}. \end{cases}$$

We let $Q_\rho = Q_\rho(z_0)$ for any $\rho \in (0, 2r]$. For $\rho > 0$ and $\lambda > 0$, we define the super level set

$$E(\rho,\lambda) := \{z \in Q_\rho : |Dw(z)| > \lambda\}.$$

From now on for simplicity, we write $\Phi(z) := \left(|F(z)|^2 + |Dv(z)|^2\right)^{\frac{1}{2}}$. We also define

$$\lambda_0 := \left(\fint_{Q_{2r}} |Dw|^2 + \frac{1}{\delta}\Phi^2\, dz\right)^{\frac{1}{2}} \geqslant 1, \tag{13}$$

where $\delta \in (0,1)$ will be chosen later, depending only on n, ν, L, and q (see below from (20)).

Let $r \leqslant r_1 < r_2 \leqslant 2r$ and consider any λ satisfying

$$\lambda \geqslant \lambda_1 := \left(\frac{64r}{r_2 - r_1}\right)^{\frac{n+2}{2}} \lambda_0. \tag{14}$$

We notice that $Q_\rho(\tilde{z}) \subset Q_{r_2} \subset Q_{2r}$ for any $\tilde{z} = (\tilde{x},\tilde{t}) \in E(r_1,\lambda)$ and all $0 < \rho < r_2 - r_1$. Then we have the Vitali type covering lemma of the super level set $E(r_1,\lambda)$ as follows:

Lemma 3.2. *For each $r \leqslant r_1 < r_2 \leqslant 2r$ and $\lambda \geqslant \lambda_1$, there exist $z_i \in E(r_1,\lambda)$ and $\rho_i \in \left(0, \frac{r_2-r_1}{32}\right)$, $i = 1,2,3,\cdots$, such that the parabolic cylinders $Q_{\rho_i}(z_i)$ are mutually disjoint,*

$$E(r_1,\lambda)\setminus\mathcal{N} \;\subset\; \bigcup_{i=1}^{\infty} Q_{8\rho_i}(z_i)$$

for some Lebesgue measure zero set $\mathcal{N}$,

$$\fint_{Q_{\rho_i}(z_i)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz = \lambda^2,$$

and

$$\fint_{Q_{\rho}(z_i)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz < \lambda^2 \quad \text{for all } \rho \in (\rho_i, r_2-r_1].$$

Proof. The proof of this lemma is the same as that in [10, Lemma 3.1] (see also [2]) but we provide the proof for reader's convenience.

For $\tilde z \in E(r_1,\lambda)$ and $\frac{r_2-r_1}{32} \leqslant \rho \leqslant r_2-r_1$, we have

$$\begin{aligned}\fint_{Q_{\rho}(\tilde z)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz &\leqslant \frac{|Q_{2r}|}{|Q_\rho(\tilde z)|}\fint_{Q_{2r}} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz\\ &= \frac{|Q_{2r}|\lambda_0^2}{|Q_\rho(\tilde z)|} = \left(\frac{2r}{\rho}\right)^{n+2}\lambda_0^2\\ &\leqslant \left(\frac{64r}{r_2-r_1}\right)^{n+2}\lambda_0^2 = \lambda_1^2 \leqslant \lambda^2\end{aligned}$$

by (13) and (14). Moreover, the parabolic Lebesgue differentiation theorem yields that, for almost every $\tilde z \in E(r_1,\lambda)$,

$$\lim_{\rho\to 0^+}\fint_{Q_{\rho}(\tilde z)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz \geqslant |Dw(\tilde z)|^2 > \lambda^2.$$

Since the map $\rho \mapsto \fint_{Q_{\rho}(\tilde z)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz$ is continuous, there exists $\rho_{\tilde z} \in \left(0, \frac{r_2-r_1}{32}\right)$ such that

$$\fint_{Q_{\rho_{\tilde z}}(\tilde z)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz = \lambda^2$$

and

$$\fint_{Q_{\rho}(\tilde z)} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz < \lambda^2 \quad \text{for all } \rho \in (\rho_{\tilde z}, r_2-r_1].$$

Therefore we apply Vitali's covering lemma for $\{Q_{\rho_{\tilde z}}(\tilde z) : \tilde z \in E(r_1,\lambda)\}$ to complete the proof. □

From this lemma, setting $Q_i^{(j)} := Q_{2^j\rho_i}(z_i)$, $j = 0,1,2,3,4,5$, we obtain that

$$|Q_i^{(0)}| \leqslant \frac{2}{\lambda^2}\int_{Q_i^{(0)}\cap\{|Dw|^2>\frac{\lambda^2}{4}\}} |Dw|^2\,dz + \frac{2}{\delta\lambda^2}\int_{Q_i^{(0)}\cap\{\Phi^2>\frac{\delta\lambda^2}{4}\}} \Phi^2\,dz \tag{15}$$

and

$$\fint_{Q_i^{(5)}} |Dw|^2 + \frac{1}{\delta}\Phi^2 \, dz < \lambda^2.$$

Step 2. (Estimates of supper-level sets)

We consider the following rescaled functions:

$$\mathbf{a}_\lambda(\xi) := \frac{1}{\lambda}\mathbf{a}(\lambda\xi) \quad \text{for } \xi \in \mathbb{R}^n,$$

$$w_{\lambda,i}(z) := \frac{1}{8\rho_i\lambda}w(Z_i), \; v_{\lambda,i}(z) := \frac{1}{8\rho_i\lambda}v(Z_i), \text{ and } F_{\lambda,i}(z) := \frac{1}{\lambda}F(Z_i)$$

for $z = (x,t) \in Q_4(0)$, where $Z_i = z_i + \big(8\rho_i x, (8\rho_i)^2 t\big)$. Then it is clear that $\mathbf{a}_\lambda(\xi)$ satisfies (2) and (3) with $\Omega_T = Q_4(0)$. Moreover, we observe that $w_{\lambda,i}$ is a weak solution to

$$\frac{\partial}{\partial t}w_{\lambda,i} - \operatorname{div}\mathbf{a}_\lambda(Dw_{\lambda,i} + Dv_{\lambda,i}) = -\operatorname{div}(Dv_{\lambda,i} + F_{\lambda,i}) \;\text{ in } Q_4(0).$$

Therefore we have

$$\fint_{Q_4(0)} |Dw_{\lambda,i}|^2 + \frac{1}{\delta}\Phi_{\lambda,i}^2 \, dz = \frac{1}{\lambda^2}\fint_{Q_i^{(5)}} |Dw|^2 + \frac{1}{\delta}\Phi^2 \, dz < 1,$$

where $\Phi_{\lambda,i} := \big(|Dv_{\lambda,i}|^2 + |F_{\lambda,i}|^2\big)^{\frac{1}{2}}$.

Now let $\varepsilon > 0$ be sufficiently small, which will be chosen in (20) below. Then considering a weak solution $\widetilde{h}_{\lambda,i}$ to

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{h}_{\lambda,i} - \operatorname{div}\mathbf{a}(D\widetilde{h}_{\lambda,i}) = 0 & \text{in } Q_4(0), \\ \widetilde{h}_{\lambda,i} = w_{\lambda,i} & \text{on } \partial_{\mathrm{p}}Q_4(0), \end{cases} \tag{16}$$

Lemma 2.1 provides that there exist $\delta = \delta(n,\mu,L,\varepsilon) > 0$ and $c_{Lip} = c_{Lip}(n,\nu,L) \geqslant 1$ such that

$$\fint_{Q_2(0)} |Dw_{\lambda,i} - D\widetilde{h}_{\lambda,i}|^2 dz \leqslant \varepsilon \quad \text{and} \quad \|D\widetilde{h}_{\lambda,i}\|_{L^\infty(Q_1(0))} \leqslant c_{Lip}.$$

Here we remark that both δ and c_{Lip} are independent of λ and i. Setting

$$h_{\lambda,i}(z) = h_{\lambda,i}(x,t) := 8\rho_i\lambda\,\widetilde{h}_{\lambda,i}\left(\frac{x-y_i}{8\rho_i}, \frac{t-\tau_i}{(8\rho_i)^2}\right),$$

where $z_i = (y_i, \tau_i)$, we therefore obtain

$$\fint_{Q_i^{(3)}} |Dw - Dh_{\lambda,i}|^2 dz \leqslant \varepsilon\lambda^2 \quad \text{and} \quad \|Dh_{\lambda,i}\|_{L^\infty(Q_i^{(3)})} \leqslant c_{Lip}\lambda. \tag{17}$$

Next, for any $\lambda \geqslant \lambda_1$, we consider the upper-level sets $E(r_1, c_{Lip}\lambda)$. By Lemma 3.2, the collection $\{Q_i^{(3)}\}$ covers $E(r_1,\lambda)\setminus\mathcal{N}$ with $|\mathcal{N}| = 0$. It is obvious that $E(r_1, 2c_{Lip}\lambda)\setminus\mathcal{N} \subset E(r_1,\lambda)\setminus\mathcal{N}$. For $z \in Q_i^{(3)}$ such that $|Dw(z)| > 2c_{Lip}\lambda$, we note that

$$\begin{aligned}|Dw(z)|^2 &\leqslant |Dw(z) - Dh_{\lambda,i}(z)|^2 + |Dh_{\lambda,i}(z)|^2\\ &\leqslant |Dw(z) - Dh_{\lambda,i}(z)|^2 + c_{Lip}^2\lambda^2\\ &< |Dw(z) - Dh_{\lambda,i}(z)|^2 + \frac{1}{4}|Dw(z)|^2,\end{aligned}$$

and then

$$|Dw(z)|^2 < 2|Dw(z) - Dh_{\lambda,i}(z)|^2.$$

Then it follows from (17) that

$$\begin{aligned}\int_{E(r_1,2c_{Lip}\lambda)} |Dw|^2\,dz &\leqslant \sum_{i=1}^{\infty}\int_{Q_i^{(3)}\cap\{|Dw|>2c_{Lip}\lambda\}} |Dw|^2\,dz\\ &\leqslant \sum_{i=1}^{\infty}\int_{Q_i^{(3)}\cap\{|Dw|>2c_{Lip}\lambda\}} 2|Dw - Dh_{\lambda,i}|^2\,dz\\ &\leqslant 2\varepsilon\lambda^2\sum_{i=1}^{\infty}|Q_i^{(3)}| = 2^{3n+7}\varepsilon\lambda^2\sum_{i=1}^{\infty}|Q_i^{(0)}|.\end{aligned}$$

Hence (15) allows to obtain

$$\begin{aligned}&\int_{E(r_1,2c_{Lip}\lambda)} |Dw|^2\,dz\\ &\quad\leqslant c\varepsilon\sum_{i=1}^{\infty}\int_{Q_i^{(0)}\cap\{|Dw|^2>\frac{\lambda^2}{4}\}} |Dw|^2\,dz + \frac{c\varepsilon}{\delta}\sum_{i=1}^{\infty}\int_{Q_i^{(0)}\cap\{\Phi^2>\frac{\delta\lambda^2}{4}\}} \Phi^2\,dz,\end{aligned}$$

where the constant $c > 0$ depends only on n, ν, and L. Since $Q_i^{(0)} \subset Q_{r_2}$, $i = 1, 2, \ldots$, are mutually disjoint, it turns out that for any $\lambda \geqslant \lambda_1$

$$\int_{E(r_1,2c_{Lip}\lambda)} |Dw|^2\,dz \leqslant c\varepsilon\int_{Q_{r_2}\cap\{|Dw|^2>\frac{\lambda^2}{4}\}} |Dw|^2\,dz + \frac{c\varepsilon}{\delta}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda^2}{4}\}} \Phi^2\,dz. \tag{18}$$

Step 3. (Gradient estimates)

Since the L^q boundedness of Dw cannot be ensured, we employ a truncation argument. We set

$$|Dw|_k := \min\{|Dw|, k\} \quad \text{for } k \geqslant 0.$$

For $k > \lambda$, the inequality $|Dw|_k > \lambda$ holds if and only if the inequality $|Dw| > \lambda$ holds. Then from (18) we obtain

$$\begin{aligned}&\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda\}} |Dw|^2\,dz\\ &\quad\leqslant c\varepsilon\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda^2}{4}\}} |Dw|^2\,dz + \frac{c\varepsilon}{\delta}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda^2}{4}\}} \Phi^2\,dz.\end{aligned}$$

We multiply both sides of the above inequality by λ^{q-3} and integrate with respect to λ over (λ_1,∞) to discover

$$
\begin{aligned}
&\int_{\lambda_1}^{\infty}\lambda^{q-3}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda\}}|Dw|^2\,dzd\lambda\\
&\quad\leqslant c\varepsilon\bigg(\int_{\lambda_1}^{\infty}\lambda^{q-3}\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda^2}{4}\}}|Dw|^2\,dzd\lambda\\
&\qquad+\frac{1}{\delta}\int_{\lambda_1}^{\infty}\lambda^{q-3}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda^2}{4}\}}\Phi^2\,dzd\lambda\bigg)\\
&\quad=:c\varepsilon(I+II).
\end{aligned}
\tag{19}
$$

Here, by using Fubini's theorem, we derive that

$$
\begin{aligned}
&\int_{\lambda_1}^{\infty}\lambda^{q-3}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda\}}|Dw|^2\,dzd\lambda\\
&\quad=\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}}|Dw|^2\int_{\lambda_1}^{\frac{|Dw|_k}{2c_{Lip}}}\lambda^{q-3}\,d\lambda dz\\
&\quad=\frac{1}{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}}|Dw|^2\bigg(\frac{|Dw|_k^{q-2}}{(2c_{Lip})^{q-2}}-\lambda_1^{q-2}\bigg)\,dz\\
&\quad=\frac{(2c_{Lip})^{2-q}}{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}}|Dw|^2|Dw|_k^{q-2}\,dz\\
&\qquad-\frac{\lambda_1^{q-2}}{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}}|Dw|^2\,dz.
\end{aligned}
$$

Similarly, we also use Fubini's theorem to obtain

$$
\begin{aligned}
I&=\int_{Q_{r_1}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}}|Dw|^2\int_{\lambda_1}^{2|Dw|_k}\lambda^{q-3}\,d\lambda dz\\
&=\frac{1}{q-2}\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}}|Dw|^2\Big(2^{q-2}|Dw|_k^{q-2}-\lambda_1^{q-2}\Big)\,d\lambda dz\\
&\leqslant\frac{2^{q-2}}{q-2}\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}}|Dw|^2|Dw|_k^{q-2}\,dz
\end{aligned}
$$

and

$$
II=\frac{1}{\delta}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda_1^2}{4}\}}\Phi^2\int_{\lambda_1}^{\frac{2\Phi}{\sqrt{\delta}}}\lambda^{q-3}\,d\lambda dz\leqslant\frac{2^{q-2}\delta^{-\frac{q}{2}}}{q-2}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda_1^2}{4}\}}\Phi^q\,dz.
$$

Therefore we insert the above estimates into (19) to obtain

$$\begin{aligned}
&\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}} |Dw|^2|Dw|_k^{q-2}\,dz \\
&\leqslant c\varepsilon 4^{q-2}c_{Lip}^{q-2}\bigg(\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}} |Dw|^2|Dw|_k^{q-2}\,dz + \delta^{-\frac{q}{2}}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda_1^2}{4}\}} \Phi^q\,dz\bigg) \\
&\quad + (2c_{Lip}\lambda_1)^{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}} |Dw|^2\,dz \\
&\leqslant \frac{1}{2}\bigg(\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}} |Dw|^2|Dw|_k^{q-2}\,dz + \delta^{-\frac{q}{2}}\int_{Q_{r_2}\cap\{\Phi^2>\frac{\delta\lambda_1^2}{4}\}} \Phi^q\,dz\bigg) \\
&\quad + (2c_{Lip}\lambda_1)^{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}} |Dw|^2\,dz
\end{aligned} \tag{20}$$

by choosing $\varepsilon > 0$ so small that $c\varepsilon 4^{q-2}c_{Lip}^{q-2} \leqslant \frac{1}{2}$. We remark that δ is also determined in this step.

Recalling the definition of λ_1 in (14), we note that

$$(2c_{Lip}\lambda_1)^{q-2} = \left[2c_{Lip}\left(\frac{64r}{r_2-r_1}\right)^{\frac{n+2}{2}}\lambda_0\right]^{q-2} \leqslant c\left(\frac{r}{r_2-r_1}\right)^{\frac{(n+2)(q-2)}{2}}\lambda_0^{q-2}$$

and we conclude

$$\begin{aligned}
&\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}} |Dw|^2|Dw|_k^{q-2}\,dz \\
&\leqslant \frac{1}{2}\int_{Q_{r_2}\cap\{|Dw|_k^2>\frac{\lambda_1^2}{4}\}} |Dw|^2|Dw|_k^{q-2}\,dz + c\int_{Q_{r_2}} \Phi^q\,dz \\
&\quad + c\left(\frac{r}{r_2-r_1}\right)^{\frac{(n+2)(q-2)}{2}}\lambda_0^{q-2}\int_{Q_{r_1}\cap\{|Dw|_k>2c_{Lip}\lambda_1\}} |Dw|^2\,dz
\end{aligned}$$

for any $r \leqslant r_1 < r_2 \leqslant 2r$, where a constant $c > 0$ depends on n, ν, L, and q. Applying Lemma 3.1, we have

$$\int_{Q_r} |Dw|^2|Dw|_k^{q-2}\,dz \leqslant c\lambda_0^{q-2}\int_{Q_{2r}} |Dw|^2\,dz + c\int_{Q_{2r}} \Phi^q\,dz.$$

By virtue of Fatou's lemma, letting $k \to \infty$, we obtain

$$\int_{Q_r} |Dw|^q\,dz \leqslant c\lambda_0^{q-2}\int_{Q_{2r}} |Dw|^2\,dz + c\int_{Q_{2r}} \Phi^q\,dz.$$

Since

$$\lambda_0^{q-2} = \left(\fint_{Q_{2r}} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz\right)^{\frac{q-2}{2}},$$

we conclude that

$$\begin{aligned}\fint_{Q_r} |Dw|^q\,dz &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz\right)^{\frac{q-2}{2}} \fint_{Q_{2r}} |Dw|^2\,dz + c\fint_{Q_{2r}} \Phi^q\,dz \\ &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2 + \frac{1}{\delta}\Phi^2\,dz\right)^{\frac{q}{2}} + c\left(\fint_{Q_{2r}} |Dw|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} \Phi^q\,dz \\ &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} \Phi^q\,dz\end{aligned}$$

by using Young's inequality and Hölder's inequality.

Recalling the definition of Φ, we therefore obtain

$$\begin{aligned}\fint_{Q_r} |Dw|^q\,dz &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} (|F|^2 + |Dv|^2)^{\frac{q}{2}}\,dz \\ &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} |F|^q + |Dv|^q\,dz,\end{aligned}$$

which yields that

$$\begin{aligned}\fint_{Q_r} |Du|^q\,dz &\leqslant c\Big(\fint_{Q_r} |Dw|^q\,dz + \fint_{Q_r} |Dv|^q\,dz\Big) \\ &\leqslant c\left(\fint_{Q_{2r}} |Dw|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} |F|^q + |Dv|^q\,dz + c\fint_{Q_r} |Dv|^q\,dz \\ &\leqslant c\left(\fint_{Q_{2r}} |Du|^2\,dz\right)^{\frac{q}{2}} + c\fint_{Q_{2r}} |Dv|^q + |F|^q\,dz \\ &\leqslant c\left(\fint_{Q_{2r}} |Du|^2\,dz\right)^{\frac{q}{2}} + c\left(\fint_{Q_{2r}} |rg|^{q_*}\,dz\right)^{\frac{q}{q_*}} + c\fint_{Q_{2r}} |F|^q\,dz,\end{aligned}$$

where we applied (7) in the last inequality.

We finally obtain that

$$\begin{aligned}&\left(\fint_{Q_r} |Du|^q\,dz\right)^{\frac{1}{q}} \\ &\leqslant c\Big[\Big(\fint_{Q_{2r}} |Du|^2\,dz\Big)^{\frac{1}{2}} + r\Big(\fint_{Q_{2r}} |g|^{q_*}\Big)^{\frac{1}{q_*}} + \Big(\fint_{Q_{2r}} |F|^q\,dz\Big)^{\frac{1}{q}}\Big]\end{aligned}$$

where a constant $c > 0$ depends on n, ν, L, and q.

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