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A NOTE ON THE PROPERTIES OF PSEUDO-WEIGHTED BROWDER SPECTRUM

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Abstract. The goal of this article is to introduce the concept of pseudo-weighted Browder spectrum when the underlying Hilbert space is not necessarily separable. To attain this goal, the notion of α-pseudo-Browder operator has been introduced. The properties and the relation of the weighted spectrum, pseudo-weighted spectrum, weighted Browder spectrum, and pseudo-weighted Browder spectrum have been investigated by extending analogous properties of their corresponding essential pseudo-spectrum and essential pseudo-weighted spectrum. The weighted spectrum, pseudo-weighted spectrum, weighted Browder, and pseudo-weighted Browder spectrum of the sum of two bounded linear operators have been characterized in the case when the Hilbert space (not necessarily separable) is a direct sum of its closed invariant subspaces. This exploration ends with a characterization of the pseudo-weighted Browder spectrum of the sum of two bounded linear operators defined over the arbitrary Hilbert spaces under certain conditions.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators, where $\mathcal{H}$ is a not necessarily separable complex Hilbert space of infinite dimension h, where $\aleph_0 \leq h$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a Fredholm operator [2,3,7,10] if $n(T) < \infty$, $\beta(T) < \infty$ and range of T (i.e., $R(T)$) is closed, where $n(T)$ is the nullity of T and $\beta(T)$ is codimension of $R(T)$. $\Psi(\mathcal{H})$ shall denote the class of all Fredholm operators. An important class of operators in Fredholm theory is the class of semi-Browder operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a Browder (resp. an upper semi Browder, a lower semi Browder) operator, i.e., $T \in \Psi_B(\mathcal{H})$ (resp. $\Psi_B^+(\mathcal{H})$, $\Psi_B^-(\mathcal{H})$) [9] if T is Fredholm (resp. upper semi Fredholm, lower semi Fredholm) and $p(T) = q(T) < \infty$ (resp. $p(T) < \infty$, $q(T) < \infty$), where $p(T)$ is ascent [9] and $q(T)$ is descent [9] of T. In [2], for $T \in \mathcal{L}(\mathcal{H})$ and

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a nonnegative integer k let us denote by $T_{[k]}$ the restriction of T to $T^k(\mathcal{H})$ viewed as a map from the space $T^k(\mathcal{H})$ into itself defined by $T_{[k]} = T^{k+1}$. The idea of pseudospectrum was introduced by Varah [14] for linear operators and later on extended by Hinrichsen, Landau, Trefethen, Davies and Pritchard. Trefethen developed this concept for operators and matrices. For $\epsilon > 0$, the pseudospectrum [1] of $T \in \mathcal{L}(\mathcal{H})$ is defined as:

$$\sigma_\epsilon(T) = \sigma(T) \cup \{\lambda \in \mathbb{C} : \left\|(\lambda - T)^{-1}\right\| > 1/\epsilon\},$$

where $\sigma(T)$ denotes the spectrum of T and $\left\|(\lambda - T)^{-1}\right\| = \infty$ if λ is in the spectrum $\sigma(T)$. Let $T \in \mathcal{L}(\mathcal{H})$ and $\epsilon > 0$. Then the pseudo-Browder spectrum [1] of T is defined as:

$$\sigma_{\epsilon B}(T) = \bigcup_{\substack{TD=DT \\ \|D\|<\epsilon}} \sigma_B(T + D).$$

Let h be the dimension of $\mathcal{H}$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called an α-Fredholm (resp. an upper semi α-Fredholm, a lower semi α-Fredholm) operator [13], i.e., $T \in \Psi_\alpha(\mathcal{H})$ (resp. $\Psi_\alpha^+(\mathcal{H})$, $\Psi_\alpha^-(\mathcal{H})$), if $R(T)$ is α-closed and $\max\{n(T), n(T^*)\} < \alpha$ (resp. $n(T) < \alpha$, $n(T^*) < \alpha$), where T^* is the adjoint of T. For a cardinal number α, $i_\alpha(T)$ [8] is defined as

$$i_\alpha(T) = \begin{cases} i(T), & \text{if } \alpha = \aleph_0 \text{ and } \max(n(T), n(T^*)) \geq \alpha, \\ 0, & \text{otherwise} \end{cases}$$

and for cardinal numbers $\aleph_0 \leq \alpha_2 \leq \alpha_1$, sum and product operations extended in [6, 8] as:

(1) $\alpha_1 + \alpha_2 = \alpha_1$,
(2) $\alpha_1 \cdot \alpha_2 = \alpha_1$.

The class of α-Weyl operators [8] is defined as:

$$\Psi_\alpha^0(\mathcal{H}) = \{T \in \Psi_\alpha(\mathcal{H}) : i_\gamma(T) = 0 \text{ for all cardinal numbers } \gamma \text{ such that } \aleph_0 \leq \gamma < \alpha\}.$$

Corresponding to these, the weighted spectrum $\sigma_\alpha(T)$ and the weighted Weyl spectrum $\sigma_\alpha^0(T)$ with weight α are defined as:

$$\sigma_\alpha(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - T \notin \Psi_\alpha(\mathcal{H})\}$$

and

$$\sigma_\alpha^0(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - T \notin \Psi_\alpha^0(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an α-Browder operator (resp. an upper semi α-Browder operator, a lower semi α-Browder operator) [12], if T is an α-Fredholm operator (resp. an upper semi α-Fredholm operator, a lower semi α-Fredholm operator) and $p(T) = q(T) < \infty$ (resp. $p(T) < \infty$, $q(T) < \infty$), denoted by $\Psi_{\alpha B}(\mathcal{H})$ (resp. $\Psi_{\alpha B}^+(\mathcal{H})$, $\Psi_{\alpha B}^-(\mathcal{H})$). For a bounded linear operator T, the upper semi weighted-Browder spectrum and lower semi weighted-Browder spectrum [12] of T are, respectively, defined by

$$\sigma_{U\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{+}_{\alpha B}(\mathcal{H})\},$$

and

$$\sigma_{L\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{-}_{\alpha B}(\mathcal{H})\}.$$

The weighted-Browder spectrum of T is then determined as

$$\sigma_{\alpha B}(T) = \sigma_{U\alpha B}(T) \cup \sigma_{L\alpha B}(T)$$

and is characterised as

$$\sigma_{\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi_{\alpha B}(\mathcal{H})\}.$$

In [4], authors introduced pseudo-Fredholm operator. In this direction, Athmouni, Baloudi, Jeribi and Kacem in [6] introduced the concepts of α-pseudo-Fredholm and α-pseudo-Weyl operators. A linear operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an α-pseudo-Fredholm operator [6] if $T + D \in \Psi_{\alpha}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an α-pseudo-Weyl operator [6] if $T + D \in \Psi^{0}_{\alpha}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$. The classes of these operators are denoted by $\Psi^{\epsilon}_{\alpha}(\mathcal{H})$ and $\Psi^{0,\epsilon}_{\alpha}(\mathcal{H})$, respectively. For a bounded linear operator T, $\sigma^{\epsilon}_{\alpha}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{\epsilon}_{\alpha}(\mathcal{H})\}$ denotes the pseudo-weighted spectrum, and $\sigma^{0,\epsilon}_{\alpha}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{0,\epsilon}_{\alpha}(\mathcal{H})\}$ represents the pseudo-weighted Weyl spectrum with weight α. Motivated by the concepts of α-pseudo-Fredholm and α-pseudo-Weyl operators, we introduce α-pseudo-Browder operator, and investigate its properties. We also examine the pseudo-weighted Browder spectrum of sum of two operators.

The manuscript has been organized as follows. In Section 2, some preliminary definitions have been introduced, properties and the relation of defined operators and corresponding spectra have been established. The next section aims to deduce the main results of the paper which are about the pseudo-weighted Browder spectrum (with weight α) of the sum of two operators and property of operators defined on the direct sum of two arbitrary Hilbert spaces. The last section concludes the manuscript.

2. Preliminaries

This section establishes some preliminary definitions which are required in the remaining sections. Recall that, a bounded linear operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a pseudo-Browder operator [1] if $T + D \in \Psi_B(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. This set is denoted by $\Psi^{\epsilon}_{B}(\mathcal{H})$. This inspires the introduction of α-pseudo-Browder operators in this section which sets the foundation for the remaining article. We demonstrate the basic properties of these operators and their corresponding spectra. In [6], authors introduced and studied pseudo-weighted spectrum and pseudo-weighted Weyl spectrum. In the present manuscript, we extend these notions to pseudo-weighted Browder spectrum.

Definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is an α-pseudo-Browder (an upper semi α-pseudo-Browder, a lower semi α-pseudo-Browder) operator if for a given $\epsilon > 0$, $T + D \in \Psi_{\alpha B}(\mathcal{H})$ (resp. $\Psi^{+}_{\alpha B}(\mathcal{H})$, $\Psi^{-}_{\alpha B}(\mathcal{H})$) for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| \leq \epsilon$. We denote the classes of upper semi α-pseudo-Browders, lower semi α-pseudo-Browders and α-pseudo-Browder operators by $\Psi^{+\epsilon}_{\alpha B}(\mathcal{H})$, $\Psi^{-\epsilon}_{\alpha B}(\mathcal{H})$ and $\Psi^{\epsilon}_{\alpha B}(\mathcal{H})$, respectively.

The corresponding spectra of these operators are the following:
the upper semi pseudo-weighted Browder spectrum;

$$\sigma^{\epsilon}_{U\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{+\epsilon}_{\alpha B}(\mathcal{H})\},$$

the lower semi pseudo-weighted Browder spectrum;

$$\sigma^{\epsilon}_{L\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{-\epsilon}_{\alpha B}(\mathcal{H})\},$$

and the pseudo-weighted Browder spectrum;

$$\sigma^{\epsilon}_{\alpha B}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Psi^{\epsilon}_{\alpha B}(\mathcal{H})\} = \sigma^{\epsilon}_{U\alpha B}(T) \cup \sigma^{\epsilon}_{L\alpha B}(T).$$

The following example shows that the class of α-pseudo-Browder operators defined over not necessarily separable complex Hilbert space of infinite dimension h, where $\aleph_0 \leq h$ is not void.

Example 2.1. Let $X = \oplus_{\alpha\in\Delta} l^2$, where Δ is an uncountable index set. By definition each element of space X is $(x_\alpha)_{\alpha\in\Delta}$ such that $x_\alpha = 0$ for all but finitely many α. Define $\|\bar{x}\| = (\Sigma_{\alpha\in\Delta}\|x_\alpha\|^2)^{1/2} < \infty$. Then the space X with this norm is a complete norm space. Now, let $\{\bar{e}_\alpha\}_{\alpha\in\Delta}$ be a set in X, where $\bar{e}_\alpha = (e_{\alpha\beta})_{\beta\in\Delta}$ such that

$$e_{\alpha\beta} = \begin{cases} e, & \alpha = \beta, \\ 0, & \text{otherwise}, \end{cases}$$

where $e \in l^2$ such that $\|e\| = 1$. Then $\{\bar{e}_\alpha\}_{\alpha\in\Delta}$ is an uncountable set in X such that $\|\bar{e}_\alpha\| = 1$ for all $\alpha \in \Delta$. Therefore X is a non separable Hilbert space. Let $\dim(X)$ be h and choose $\epsilon > 0$. Now define an operator $T_0 : l^2 \to l^2$ as

$$T_0(x) = T_0(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, 0, 0, 0, \ldots)$$

for all $x = (\xi_1, \xi_2, \xi_3, \ldots)$ in l^2 for which $(\Sigma_{i=1}^{n}|\xi_i|^2)^{1/2} < \epsilon/2$, where n is a fixed finite positive integer. Define $D_1 : l^2 \to l^2$ as

$$D_1(x) = D_1(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_2 - \xi_1, \xi_3 - \xi_2, \xi_4 - \xi_3, \ldots, \xi_n - \xi_{n-1}, 0, 0, 0, \ldots)$$

for all $x \in l^2$, where $\|D_1\| = (\Sigma_{i=2}^{n}|\xi_i - \xi_{i-1}|^2)^{1/2} < \epsilon$. Clearly,

$$(T_0 D_1)(x) = (D_1 T_0)(x) \text{ for all } x \in l^2.$$

Now

$$(T_0 + D_1)(x) = (\xi_2, \xi_3, \ldots, \xi_{n-1}, \xi_n, \xi_n, 0, 0, \ldots) \text{ for all } x \in l^2$$

and

$$(T_0 + D_1)^*(x) = (0, \xi_1, \xi_2, \ldots, \xi_{n-2}, \xi_{n-1} + \xi_n, 0, 0, \ldots) \text{ for all } x \in l^2.$$

Since

$$(T_0 + D_1)^{n-1}(x) = (\xi_n, \xi_n, \xi_n, \ldots, \xi_n, 0, 0, 0, 0, \ldots) \text{ for all } x \in l^2.$$

Therefore,

$$(T_0 + D_1)^{n-1}(x) = (T_0 + D_1)^n(x) \text{ for all } x \in l^2$$

and

$$ker((T_0 + D_1)^{n-1}) = ker((T_0 + D_1)^n) = (\xi_1, \xi_2, \xi_2, \ldots, \xi_{n-1}, 0, \xi_{n+1}, \xi_{n+2}, \ldots)$$

for all $x \in l^2$. So, $p(T_0 + D_1) = n - 1 = q(T_0 + D_1) < \infty$ and $n(T_0 + D_1) = \aleph_0 = n((T_0 + D_1)^*)$. Now consider $D_2 : l^2 \to l^2$ defined as

$$D_2(x) = D_2(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_2 - \xi_1, \xi_3 - \xi_2, \xi_4 - \xi_3, \ldots, \xi_{n-1} - \xi_{n-2}, 0, 0, 0, \ldots)$$

for all $x \in l^2$, where $\|D_2\| = (\Sigma_{i=2}^{n-1} \|\xi_i - \xi_{i-1}\|^2)^{1/2} < \epsilon$. It follows that,

$$(T_0 D_2)(x) = (D_2 T_0)(x) \text{ for all } x \in l^2, \text{ and}$$

$$\begin{aligned} ker((T_0 + D_2)^{n-2}) &= ker((T_0 + D_2)^{n-1}) \\ &= (\xi_1, \xi_2, \xi_2, \ldots, \xi_{n-2}, 0, 0, \xi_{n+1}, \xi_{n+2}, \ldots) \end{aligned}$$

for all $x \in l^2$. So, $p(T_0 + D_2) = n - 2 = q(T_0 + D_2) < \infty$ and $n(T_0 + D_2) = \aleph_0 = n((T_0 + D_2)^*)$. By a similar process, we get a finite number of D_n such that $T_0 + D_n$ is an α-Browder operator, where $T_0 D_n = D_n T_0$ and $\|D_n\| < \epsilon$. So, T_0 is a pseudo-Browder operator on the separable space l^2. Now let $T \in \mathcal{L}(X)$, defined as

$$T = \oplus_{\alpha \in \Delta} T_\alpha,$$

where

$$T_\alpha = \begin{cases} T_0, & \alpha = \alpha_0, \\ I, & \text{otherwise} \end{cases}$$

and $S_n \in \mathcal{L}(X)$, defined as

$$S_n = \oplus_{\alpha \in \Delta} S_\alpha,$$

where

$$S_\alpha = \begin{cases} D_n, & \alpha = \alpha_0, \\ I, & \text{otherwise.} \end{cases}$$

Then

$$T + S_n = \oplus_{\alpha \in \Delta} (T_\alpha + S_\alpha),$$

where

$$T_\alpha + S_\alpha = \begin{cases} T_0 + D_n, & \alpha = \alpha_0, \\ I, & \text{otherwise.} \end{cases}$$

By the same argument as in [12, Example 2.1], we get that $T + S_n$ is an α-Browder operator implying thereby that T is an α-pseudo-Browder operator.

We begin this section with propositions aiming at establishing the relations among weighted, pseudo weighted, weighted Weyl, pseudo-weighted Weyl, and pseudo-weighted Browder spectra.

Proposition 2.2. *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ and $\epsilon > 0$. Then,*

(1) $\sigma_\alpha^\epsilon(T) \subseteq \sigma_{\alpha B}^\epsilon(T)$.
(2) $\sigma_{\beta B}^\epsilon(T) \subseteq \sigma_{\alpha B}^\epsilon(T)$, *where* $\aleph_0 \leq \alpha < \beta \leq h$.
(3) $\sigma_{\alpha B}^{\epsilon_1}(T) \subseteq \sigma_{\alpha B}^{\epsilon_2}(T)$, *where* $0 < \epsilon_1 < \epsilon_2$.

Proof. (1) Let $\lambda \notin \sigma_{\alpha B}^\epsilon(T)$. Then since $\lambda I - T + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$ the conclusion follows.

(2) Let $\lambda \notin \sigma_{\alpha B}^\epsilon(T)$. By definition, it follows that $\lambda I - T + D \in \Psi_\alpha(\mathcal{H})$ and $p(\lambda I - T + D) = q(\lambda I - T + D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Therefore $\lambda I - T + D \in \Psi_{\beta B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Thus we get $\lambda I - T \in \Psi_{\beta B}^\epsilon(\mathcal{H})$. Consequently, $\lambda \notin \sigma_{\beta B}^\epsilon(T)$.

(3) Let $\lambda \notin \sigma_{\alpha B}^{\epsilon_2}(T)$. It follows that, $\lambda I - T + D \in \Psi_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon_2$. Since $< \epsilon_1 < \epsilon_2$, $\lambda I - T \in \Psi_{\alpha B}^{\epsilon_1}(\mathcal{H})$ leading to the conclusion. ☐

Proposition 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$, $\epsilon > 0$ and $a, b \in \mathbb{C}$, where $b \neq 0$. Then,*

(1) $\sigma_\alpha(aI + bT) = a + \sigma_\alpha(T)b$.
(2) $\sigma_\alpha^\epsilon(aI + bT) = a + \sigma_\alpha^{\epsilon/|b|}(T)b$.
(3) $\sigma_\alpha^0(aI + bT) = a + \sigma_\alpha^0(T)b$.
(4) $\sigma_\alpha^{0,\epsilon}(aI + bT) = a + \sigma_\alpha^{0,\epsilon/|b|}(T)b$.

Proof. (1) Let $\lambda \notin \sigma_\alpha(aI + bT)$, which means $b^{-1}(\lambda - a)I - T \in \Psi_\alpha(\mathcal{H})$. Equivalently, we have $(\lambda - a) \notin \sigma_\alpha(T)b$ and therefore $\lambda \notin a + \sigma_\alpha(T)b$. Similarly, whenever $\lambda \notin a + \sigma_\alpha(T)b$ then $\lambda \notin \sigma_\alpha(aI + bT)$.

(2) If $\lambda \notin \sigma_\alpha^\epsilon(aI + bT)$, then $\lambda I - (aI + bT) + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$ or equivalently $[b^{-1}(\lambda - a)I - T + b^{-1}D)] \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\left\|b^{-1}D\right\| < \epsilon/|b|$. Therefore, $b^{-1}(\lambda - a)I - T \in \Psi_\alpha^{\epsilon/b}(\mathcal{H})$ which is same as $\lambda \notin a + \sigma_\alpha^{\epsilon/|b|}(T)b$.

Conversely, letting $\lambda \notin a + \sigma_\alpha^{\epsilon/|b|}(T)b$ and using the definition we get $b^{-1}(\lambda - a)I - T \in \Psi_\alpha^{\epsilon/|b|}(\mathcal{H})$. Therefore, $\lambda I - (aI + bT) + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$. Hence, $\lambda \notin \sigma_\alpha^\epsilon(aI + bT)$.

(3) Given $\lambda \notin \sigma_\alpha^0(aI + bT)$, using the definition of $\sigma_\alpha^0(aI + bT)$, we get $b^{-1}(\lambda - a)I - T \in \Psi_\alpha(\mathcal{H})$ and $i_\beta(b^{-1}(\lambda - a)I - T)) = 0$ for all $\aleph_0 \leq \beta < \alpha$. Therefore $b^{-1}(\lambda - a) \notin \sigma_\alpha^0(T)$. Hence, $\lambda \notin a + \sigma_\alpha^0(T)b$.

Conversely, if we let $\lambda \notin a + \sigma_\alpha^0(T)b$, we get $\lambda I - (aI + bT) \in \Psi_\alpha(\mathcal{H})$ and $i_\beta(\lambda I - (aI + bT)) = 0$ which implies $\lambda \notin \sigma_\alpha^0(aI + bT)$.

(4) Let $\lambda \notin \sigma_\alpha^{0,\epsilon}(aI + bT)$. By definition we get, $\lambda I - (aI + bT) + D \in \Psi_\alpha(\mathcal{H})$ and $i_\beta(\lambda I - (aI + bT) + D) = 0$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$. This follows that, $[b^{-1}(\lambda - a)I - T + b^{-1}D] \in \Psi_\alpha(\mathcal{H})$ and $i_\beta(b^{-1}\lambda I - (aI + bT) + b^{-1}D)) = 0$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\left\|b^{-1}D\right\| < \epsilon/|b|$. Hence, $\lambda \notin a + \sigma_\alpha^{0,\epsilon/|b|}(T)b$.

Conversely, if we let $\lambda \notin a+\sigma_\alpha^{0,\epsilon/|b|}(T)b$, then $b^{-1}(\lambda-a)I-T+b^{-1}D \in \Psi_\alpha^0(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon/|b|$. Equivalently $(\lambda-a)I-T+D' \in \Psi_\alpha^0(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $\|D'\| < \epsilon$ implying the conclusion. $\square$

Proposition 2.4. *Let $T \in \mathcal{L}(\mathcal{H})$ and $a, b \in \mathbb{C}$, where $b \neq 0$. Then*

$$\sigma_{\alpha B}^{\epsilon}(aI + bT) = a + \sigma_{\alpha B}^{\epsilon/|b|}(T)b.$$

Proof. Let $\lambda \notin \sigma_{\alpha B}^{\epsilon}(aI + bT)$. Then $\lambda I - (aI + bT) + D \in \Psi_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$ or equivalently, $[b^{-1}(\lambda - a)I - T + b^{-1}D] \in \Psi_{\alpha B}(\mathcal{H})$ for all $b^{-1}D \in \mathcal{L}(\mathcal{H})$ such that $\left\|b^{-1}D\right\| < \epsilon/|b|$ and $TD = DT$. It follows that $\lambda \notin \sigma_{\alpha B}^{\epsilon/|b|}(T)b + a$.

For the reverse inclusion, if $\lambda \notin \sigma_{\alpha B}^{\epsilon/|b|}(T)b + a$, then $b^{-1}(\lambda - a)I - T \in \Psi_{\alpha B}^{\epsilon/|b|}(\mathcal{H})$ which implies, $\lambda I - (aI + bT) \in \Psi_{\alpha B}^{\epsilon/|b|}(\mathcal{H})$. Thus $\lambda I - (aI + bT) + D' \in \Psi_{\alpha B}(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $TD' = D'T$ and $\|D'\| < \epsilon/|b|$. Equivalently, $\lambda I - (aI + bT) + D \in \Psi_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Hence, $\lambda \notin \sigma_{\alpha B}^{\epsilon}(aI + bT)$. $\square$

Proposition 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$, $0 \neq c \in \mathbb{C}$ and $\epsilon > 0$. Then,*

(1) $\sigma_\alpha^{|c|\epsilon}(cT) = c\sigma_\alpha^{\epsilon}(T)$.
(2) $\sigma_{\alpha B}^{|c|\epsilon}(cT) = c\sigma_{\alpha B}^{\epsilon}(T)$.

Proof. (1) For $\lambda \notin \sigma_\alpha^{|c|\epsilon}(cT)$, we have $\lambda I - cT + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < |c|\,\epsilon$. It then follows that, $c^{-1}\lambda I - T + c^{-1}D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\left\|c^{-1}D\right\| < \epsilon$. Hence, $\lambda \notin \sigma_\alpha^{\epsilon}(T)c$. The reverse inclusion follows similarly.

(2) Let $\lambda \notin \sigma_{\alpha B}^{|c|\epsilon}(cT)$. The definition then implies that $\lambda I - cT + D \in \Psi_\alpha(\mathcal{H})$ and $p(\lambda I - cT + D) = q(\lambda I - cT + D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < |c|\,\epsilon$. It follows that $c^{-1}\lambda I - T + c^{-1}D \in \Psi_\alpha(\mathcal{H})$ and $p(c^{-1}\lambda I - T + c^{-1}D) = q(c^{-1}\lambda I - T + c^{-1}D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\left\|c^{-1}D\right\| < \epsilon$. Hence, $\lambda \notin \sigma_{\alpha B}^{\epsilon}(T)c$.

Conversely, let $\lambda \notin \sigma_{\alpha B}^{\epsilon}(T)c$. Then $c^{-1}\lambda I - T + D \in \Psi_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Therefore, $\lambda I - cT + D' \in \Psi_{\alpha B}(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $TD' = D'T$ and $\|D'\| < |c|\,\epsilon$. Hence, $\lambda \notin \sigma_{\alpha B}^{|c|\epsilon}(cT)$. $\square$

Remark 2.6. Let $T \in \mathcal{L}(\mathcal{H})$ and $0 < \delta < \epsilon$. Then,

$$\sigma_{\alpha B}^{\delta}(T) \subseteq \sigma_{\alpha B}^{\epsilon}(T).$$

Clearly by definition of pseudo-Browder spectrum converse of Remark 2.6 does not hold.

Now if $\lambda \notin \sigma_\alpha^{\epsilon+\|S\|}(T)$, then $\lambda I - T + D - S \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon + \|S\| < 2\epsilon$ implying $\lambda \notin \sigma_\alpha^{2\epsilon}(T + S)$.

On the other hand if $\lambda \notin \sigma_\alpha^{2\epsilon}(T + S)$ by using definition, we get $\lambda I - (T + S) + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < 2\epsilon$. It then follows that

$\lambda I - T + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon - \|S\|$. Therefore, $\lambda \notin \sigma_\alpha^{\epsilon-\|S\|}(T)$. This proves the following:

Lemma 2.7. *Let $\mathcal{H}$ be a Hilbert space, $S \in \mathcal{K}_\alpha(\mathcal{H})$ such that $\|S\| < \epsilon$ and $T \in \mathcal{L}(\mathcal{H})$, where $\epsilon > 0$. Then,*

$$\sigma_\alpha^{\epsilon-\|S\|}(T) \subseteq \sigma_\alpha^{2\epsilon}(T+S) \subseteq \sigma_\alpha^{\epsilon+\|S\|}(T).$$

The following theorem characterizes α-Browder spectrum of an operator in $\mathcal{L}(\mathcal{H})$;

Theorem 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$. Then $\sigma_{\alpha B}(T) = \cap_{\epsilon>0}\sigma_{\alpha B}^\epsilon(T)$.*

Proof. Let $\lambda \notin \cap_{\epsilon>0}\sigma_{\alpha B}^\epsilon(T)$. It then follows that, $\lambda I - T \in \Psi_{\alpha B}^\epsilon(\mathcal{H})$ for some $\epsilon > 0$. Then using definition, $\lambda I - T + D \in \Psi_\alpha(\mathcal{H})$ and $p(\lambda I - T + D) = q(\lambda I - T + D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Choosing $D = 0$, $\lambda I - T \in \Psi_{\alpha B}(\mathcal{H})$ and $\lambda \notin \sigma_{\alpha B}(T)$ leading to the conclusion.

To see the converse, let $\lambda \notin \sigma_{\alpha B}(T)$. Then, by [12, Corollary 3.14], there exists an $\epsilon > 0$ such that $\lambda I - T + D \in \Psi_\alpha(\mathcal{H})$ and $p(\lambda I - T + D) = q(\lambda I - T + D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$, $\|D\| < \epsilon$ and $\dim R(D^n) < \infty$ for some integer $n \geq 1$. This implies that, $\lambda \notin \sigma_{\alpha B}^\epsilon(T)$. Hence $\lambda \notin \cap_{\epsilon>0}\sigma_{\alpha B}^\epsilon(T)$ follows the result. $\square$

Remark 2.9. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. For cardinal numbers $\aleph_0 \leq \alpha_2 \leq \alpha_1$ and for real number $\epsilon > 0$,

(1) $\sigma_{\alpha_1\alpha_2}(T) = \sigma_{\alpha_1}(T)$.
(2) $\sigma_{\alpha_1\alpha_2}^\epsilon(T) = \sigma_{\alpha_1}^\epsilon(T)$.
(3) $\sigma_{(\alpha_1\alpha_2)B}(T) = \sigma_{\alpha_1 B}(T)$.
(4) $\sigma_{(\alpha_1\alpha_2)B}^\epsilon(T) = \sigma_{\alpha_1 B}^\epsilon(T)$.

3. Pseudo-weighted and pseudo-weighted Browder spectrum of bounded operators and their properties

In the current section we proceed to prove the main results of this article by developing the relation between weighted, pseudo-weighted, weighted Browder, pseudo-weighted-Browder spectra of operator T and their corresponding invariant subspaces. We further characterize the pseudo-weighted-Browder spectrum of the sum of two bounded linear operators defined over the arbitrary Hilbert space under certain conditions.

Proposition 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{H}$ be a direct sum of closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are T-invariant. If $T_1 = T|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 = T|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_2$, then for $\epsilon > 0$*

(1) $\sigma_\alpha(T_1 \oplus T_2) = \sigma_\alpha(T_1) \cup \sigma_\alpha(T_2)$.
(2) $\sigma_\alpha^\epsilon(T_1 \oplus T_2) = \sigma_\alpha^\epsilon(T_1) \cup \sigma_\alpha^\epsilon(T_2)$.
(3) $\sigma_{\alpha B}(T_1 \oplus T_2) = \sigma_{\alpha B}(T_1) \cup \sigma_{\alpha B}(T_2)$.
(4) $\sigma_{\alpha B}^\epsilon(T_1 \oplus T_2) = \sigma_{\alpha B}^\epsilon(T_1) \cup \sigma_{\alpha B}^\epsilon(T_2)$.

Proof. (1) Let $\lambda \notin \sigma_\alpha(T_1) \cup \sigma_\alpha(T_2)$. This implies that, $T_1 - \lambda I_1 \in \Psi_\alpha(\mathcal{H}_1)$ and $T_2 - \lambda I_2 \in \Psi_\alpha(\mathcal{H}_2)$. This follows from [6], $(T_1 - \lambda I_1) \oplus (T_2 - \lambda I_2) \in \Psi_\alpha(\mathcal{H})$. Hence, $T - \lambda I \in \Psi_\alpha(\mathcal{H})$ and $\lambda \notin \sigma_\alpha(T_1 \oplus T_2)$.

Conversely, let $\lambda \notin \sigma_\alpha(T)$. By definition, $T - \lambda I \in \Psi_\alpha(\mathcal{H})$. Therefore, $(T_1 \oplus T_2) - \lambda(I_1 \oplus I_2) \in \Psi_\alpha(\mathcal{H})$. Therefore $T_1 - \lambda I_1 \in \Psi_\alpha(\mathcal{H}_1)$ and $T_2 - \lambda I_2 \in \Psi_\alpha(\mathcal{H}_2)$. Consequently, $\lambda \notin \sigma_\alpha(T_1) \cup \sigma_\alpha(T_2)$.

(2) Let $\lambda \notin \sigma_\alpha^\epsilon(T_1) \cup \sigma_\alpha^\epsilon(T_2)$. This follows that, $T_1 - \lambda I_1 + D_1 \in \Psi_\alpha(\mathcal{H}_1)$ and $T_2 - \lambda I_2 + D_2 \in \Psi_\alpha(\mathcal{H}_2)$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$. Therefore $(T_1 - \lambda I_1 + D_1) \oplus (T_2 - \lambda I_2 + D_2) \in \Psi_\alpha(\mathcal{H})$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$. Then $T - \lambda I + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| = \max\{\|D_1\|, \|D_2\|\} < \epsilon$, where $D = D_1 \oplus D_2$ and hence $T \notin \sigma_\alpha^\epsilon(T_1 \oplus T_2)$.

Conversely, let $\lambda \notin \sigma_\alpha^\epsilon(T)$. By definition, $T - \lambda I + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$. Therefore, $(T_1 \oplus T_2) - \lambda(I_1 \oplus I_2) + (D_1 \oplus D_2) \in \Psi_\alpha(\mathcal{H})$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$, where $D = D_1 \oplus D_2$. This follows that, $T_1 - \lambda I_1 + D_1 \in \Psi_\alpha(\mathcal{H}_1)$ and $T_2 - \lambda I_2 + D_2 \in \Psi_\alpha(\mathcal{H}_2)$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$. Consequently, $\lambda \notin \sigma_\alpha^\epsilon(T_1) \cup \sigma_\alpha^\epsilon(T_2)$.

(3) Let $\lambda \notin \sigma_{\alpha B}(T_1) \cup \sigma_{\alpha B}(T_2)$. By definition, $T_1 - \lambda I_1 \in \Psi_{\alpha B}(\mathcal{H}_1)$ and $T_2 - \lambda I_2 \in \Psi_{\alpha B}(\mathcal{H}_2)$. Therefore $(T_1 - \lambda I_1) \oplus (T_2 - \lambda I_2) \in \Psi_{\alpha B}(\mathcal{H})$ by [12, Theorem 4.3]. Consequently, $T - \lambda I \in \Psi_{\alpha B}(\mathcal{H})$ and $T \notin \sigma_\alpha(T_1 \oplus T_2)$. Similarly, if $\lambda \notin \sigma_{\alpha B}(T_1 \oplus T_2)$, then $\lambda \notin \sigma_{\alpha B}(T_1) \cup \sigma_{\alpha B}(T_2)$.

(4) Let $\lambda \notin \sigma_{\alpha B}^\epsilon(T_1) \cup \sigma_{\alpha B}^\epsilon(T_2)$. This implies that, $T_1 - \lambda I_1 + D_1 \in \Psi_{\alpha B}(\mathcal{H}_1)$ and $T_2 - \lambda I_2 + D_2 \in \Psi_{\alpha B}(\mathcal{H}_2)$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$. Therefore $(T_1 - \lambda I_1 + D_1) \oplus (T_2 - \lambda I_2 + D_2) \in \Psi_{\alpha B}(\mathcal{H})$ for all $D_1 \in \mathcal{L}(\mathcal{H}_1)$ such that $\|D_1\| < \epsilon$ and $D_2 \in \mathcal{L}(\mathcal{H}_2)$ such that $\|D_2\| < \epsilon$. Then $T - \lambda I + D \in \Psi_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$ and hence $T \notin \sigma_{\alpha B}^\epsilon(T_1 \oplus T_2)$. Similarly, we get the converse part. □

Theorem 3.2. *Let U be a bounded linear operator on the Hilbert space $\mathcal{H}$ and V be a bounded linear operator such that $0 \in \rho(V)$, where $\rho(V)$ denotes the resolvent set of the operator V. Let $\epsilon_0 = \|V\| \left\|V^{-1}\right\|$ and $W = VUV^{-1}$. Then, for $\epsilon > 0$, we have*

(1) $\sigma_\alpha^0(W) = \sigma_\alpha^0(U)$.
(2) $\sigma_\alpha^{\epsilon\epsilon_0}(U) \subseteq \sigma_\alpha^{\epsilon\epsilon_0^2}(W) \subseteq \sigma_\alpha^{\epsilon\epsilon_0^3}(U)$.
(3) $\sigma_\alpha^{0,\epsilon\epsilon_0}(U) \subseteq \sigma_\alpha^{0,\epsilon\epsilon_0^2}(W) \subseteq \sigma_\alpha^{0,\epsilon\epsilon_0^3}(U)$.

Proof. (1) Let $\lambda \in \mathcal{C}$. Then

$$(\lambda I - W) = V(\lambda I - U)V^{-1}$$

and

$$V^{-1}(\lambda I - W)V = (\lambda I - U).$$

Let $\lambda \notin \sigma_\alpha^0(W)$. Then we get $\lambda I - W \in \Psi_\alpha^0(\mathcal{H})$, or equivalently, we have $V(\lambda I - U)V^{-1} \in \Psi_\alpha^0(\mathcal{H})$ and therefore $\lambda \notin \sigma_\alpha^0(U)$. By similar argument, we get the converse. Hence, $\sigma_\alpha^0(W) = \sigma_\alpha^0(U)$.

(2) Let $\lambda \notin \sigma_\alpha^{\epsilon\epsilon_0^3}(U)$. Then the definition of $\sigma_\alpha^{\epsilon\epsilon_0^3}(U)$ leads to $V^{-1}(\lambda I - W)V + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon\epsilon_0^3$. It follows that $V^{-1}[(\lambda I - W) + VDV^{-1}]V \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon\epsilon_0^3$. Therefore, $\lambda I - W + D' \in \Psi_\alpha(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $\|D'\| < \epsilon\epsilon_0^2$. Hence, $\lambda \notin \sigma_\alpha^{\epsilon\epsilon_0^2}(W)$.

Conversely, $\lambda \notin \sigma_\alpha^{\epsilon\epsilon_0^2}(W)$. Then using definition, we get $\lambda I - W + D \in \Psi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon\epsilon_0^2$. Further this gives $V[(\lambda I - U)]V^{-1} + D \in \Psi_\alpha(\mathcal{H})$ and thus $V[(\lambda I - U) + D']V^{-1} \in \Psi_\alpha(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $\|D'\| < \epsilon\epsilon_0$. Consequently, $\lambda \notin \sigma_\alpha^{\epsilon\epsilon_0}(U)$.

(3) Suppose that $\lambda \notin \sigma_\alpha^{0,\epsilon\epsilon_0^2}(W)$. Then by definition, $\lambda I - W + D \in \Psi_\alpha^0(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon\epsilon_0^2$. This implies that, $V(\lambda I - U)V^{-1} + D \in \Psi_\alpha^0(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon\epsilon_0^2$. Then, $\lambda I - U + D' \in \Psi_\alpha^0(\mathcal{H})$ for all $D' \in \mathcal{L}(\mathcal{H})$ such that $\|D'\| < \epsilon\epsilon_0$. Therefore, $\lambda \notin \sigma_\alpha^{0,\epsilon\epsilon_0}(U)$. Hence, $\sigma_\alpha^{0,\epsilon\epsilon_0}(U) \subseteq \sigma_\alpha^{0,\epsilon\epsilon_0{}^2}(W) \subseteq \sigma_\alpha^{0,\epsilon\epsilon_0^3}(U)$. □

Lemma 3.3. *Let T and S be bounded linear operators on the Hilbert space $\mathcal{H}$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. If for each $0 \neq \lambda \in \mathbb{C}$ there exists an integer $k \geq 0$ such that $(\lambda I - (T + S + D))_{[k]}$ is injective and $T(S + D) \in \mathcal{K}_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$ and $TD = DT$. Then, for $\epsilon > 0$*

$$\sigma_{U\alpha B}^\epsilon(T + S) \setminus \{0\} \subseteq [\sigma_{U\alpha B}(T) \cup \sigma_{U\alpha B}^\epsilon(S)] \setminus \{0\}.$$

Proof. Suppose that $0 \neq \lambda \notin [\sigma_{U\alpha B}(T) \cup \sigma_{U\alpha B}^\epsilon(S)] \setminus \{0\}$. Then $\lambda I - T \in \Psi_{\alpha B}^+(\mathcal{H})$ and $\lambda I - S - D \in \Psi_{\alpha B}^+(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Therefore, $(\lambda I - T)(\lambda I - S - D) \in \Psi_\alpha^+(\mathcal{H})$ and $p(\lambda I - T) < \infty, p(\lambda I - S - D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. We can write

$$(\lambda I - T)(\lambda I - S - D) = T(S + D) + \lambda(\lambda I - T - S - D).$$

Since $T(D + S) \in \mathcal{K}_\alpha(\mathcal{H})$, therefore, we have

$$\lambda I - T - S - D \in \Psi_\alpha^+(\mathcal{H}).$$

Since for each $\lambda \neq 0$, there exists an integer $k \geq 0$ such that $(\lambda I - T - S - D)_{[k]}$ is injective. Let $x \in N((\lambda I - T - S - D)^{k+1})$. Then, $(\lambda I - T - S - D)^k(x) \in N(\lambda I - T - S - D) \cap R((\lambda I - T - S - D)^k) = N((\lambda I - T - S - D)_{[k]}) = \{0\}$. Then, $x \in N((\lambda I - T - S - D)^k)$ and we get

$$N((\lambda I - T - S - D)^{k+1}) \subseteq N((\lambda I - T - S - D)^k).$$

Thus $N((\lambda I - T - S - D)^k) = N((\lambda I - T - S - D)^{k+1})$ and $p(\lambda I - T - S - D) \leq k < \infty$. So, $\lambda I - T - S - D \in \Psi_{\alpha B}^+(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$

and $\|D\| < \epsilon$. Therefore, $\lambda \notin \sigma^{\epsilon}_{U\alpha B}(T+S)$. Hence,

$$\sigma^{\epsilon}_{U\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{U\alpha B}(T) \cup \sigma^{\epsilon}_{U\alpha B}(S)] \setminus \{0\}. \qquad \square$$

Lemma 3.4. *Let T and S be bounded linear operators on the Hilbert space $\mathcal{H}$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. If for each $0 \neq \lambda \in \mathbb{C}$ there exists an integer $k \geq 0$ such that $(\lambda I - (T+S+D))_{[k]}$ is onto and $T(D+S) \in \mathcal{K}_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$ and $TD = DT$. Then, for $\epsilon > 0$*

$$\sigma^{\epsilon}_{L\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{L\alpha B}(T) \cup \sigma^{\epsilon}_{L\alpha B}(S)] \setminus \{0\}.$$

Proof. Suppose that $0 \neq \lambda \notin [\sigma_{L\alpha B}(T) \cup \sigma^{\epsilon}_{L\alpha B}(S)] \setminus \{0\}$. Then $\lambda I - T \in \Psi^{-}_{\alpha B}(\mathcal{H})$ and $\lambda I - S - D \in \Psi^{-}_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Therefore, $(\lambda I - T)(\lambda I - S - D) \in \Psi^{-}_{\alpha}(\mathcal{H})$ and $q(\lambda I - T) < \infty, q(\lambda I - S - D) < \infty$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. We can write,

$$(\lambda I - T)(\lambda I - S - D) = T(S+D) + \lambda(\lambda I - T - S - D).$$

Since $T(D+S) \in \mathcal{K}_\alpha(\mathcal{H})$, therefore, we have

$$\lambda I - T - S - D \in \Psi^{-}_{\alpha}(\mathcal{H}).$$

Since there exists an integer $k \geq 0$ such that $(\lambda I - T - S - D)_{[k]}$ is onto. Then,

$$R((\lambda I - T - S - D)^k) = R((\lambda I - T - S - D)^{k+1}).$$

So,

$$q(\lambda I - T - S - D) \leq k < \infty.$$

Thus $\lambda I - T - S - D \in \Psi^{-}_{\alpha B}(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $TD = DT$ and $\|D\| < \epsilon$. Therefore, $\lambda \notin \sigma^{\epsilon}_{L\alpha B}(T+S)$. Hence,

$$\sigma^{\epsilon}_{L\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{L\alpha B}(T) \cup \sigma^{\epsilon}_{L\alpha B}(S)] \setminus \{0\}. \qquad \square$$

From Lemmas 3.3 and 3.4, we can conclude the following theorem:

Theorem 3.5. *Let T and S be bounded linear operators on the Hilbert space $\mathcal{H}$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. If for each $0 \neq \lambda \in \mathbb{C}$ there exists an integer $k \geq 0$ such that $(\lambda I - (T+S+D))_{[k]}$ is invertible and $T(D+S) \in \mathcal{K}_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \epsilon$ and $TD = DT$. Then, for $\epsilon > 0$*

$$\sigma^{\epsilon}_{\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{\alpha B}(T) \cup \sigma^{\epsilon}_{\alpha B}(S)] \setminus \{0\}.$$

4. Conclusion

While the concept of pseudo-spectra, essential pseudo-spectra, and pseudo-Browder spectra on the Banach space [1,4,5,11], were introduced and explored by different authors, the notions of the weighted spectrum, pseudo weighted spectrum and pseudo weighted Weyl spectrum have been discussed in [6]. Motivated by their works, in the present paper, we have introduced the notions of α-pseudo-Browder operator and corresponding spectrum, that is, pseudo-weighted Browder spectrum. The properties, as well as the relation between these operators and their corresponding spectra, have to lead to many interesting conclusions. Relation among pseudo-weighted and pseudo-weighted Browder spectrum has also been established in case $\aleph_0 \leq \alpha < \beta \leq h$ and $0 < \epsilon_1 < \epsilon_2$. When $a, b, c \in \mathbb{C}$, where b and c are non-zero, stability of weighted spectrum, pseudo-weighted spectrum, weighted Weyl spectrum, pseudo-weighted Weyl spectrum and pseudo-weighted Browder spectrum of bT when it is perturbed by a scalar operator aI have been obtained. Similarly, some more properties of the pseudo-weighted Browder spectrum have been discussed in Section 2. The example constructed in the discussion proves that the class of α-pseudo-Browder operator is not void. With the help of the deduced results and some previous results, we develop the relation between the pseudo-weighted Browder spectrum of operator T and its invariant subspaces. We characterize the pseudo-weighted-Browder spectrum of the sum of two bounded linear operators defined over the arbitrary Hilbert space. It has been established that for $\epsilon > 0$,

$$\sigma^{\epsilon}_{\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{\alpha B}(T) \cup \sigma^{\epsilon}_{\alpha B}(S)] \setminus \{0\}$$

if T and S are bounded linear operators on the Hilbert space $\mathcal{H}$, α ($\aleph_0 \leq \alpha \leq h$) is a cardinal number and if for each $0 \neq \lambda \in \mathbb{C}$ there exists an integer $k \geq 0$ such that $(\lambda I - (T+S+D))_{[k]}$ is invertible and $T(D+S) \in \mathcal{K}_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ with $\|D\| < \epsilon$ and $TD = DT$. But it is yet to be explored whether

$$[\sigma_{\alpha B}(T) \cup \sigma^{\epsilon}_{\alpha B}(S)] \setminus \{0\} \subseteq \sigma^{\epsilon}_{\alpha B}(T+S) \setminus \{0\}$$

holds or not?

References

[1] F. Abdmouleh, A. Ammar, and A. Jeribi, *A characterization of the pseudo-Browder essential spectra of linear operators and application to a transport equation*, J. Comput. Theor. Transp. **44** (2015), no. 3, 141–153. `https://doi.org/10.1080/23324309.2015.1033222`

[2] P. Aiena, M. T. Biondi, and C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2839–2848. `https://doi.org/10.1090/S0002-9939-08-09138-7`

[3] P. Aiena and S. Triolo, *Fredholm spectra and Weyl type theorems for Drazin invertible operators*, Mediterr. J. Math. **13** (2016), no. 6, 4385–4400. `https://doi.org/10.1007/s00009-016-0751-3`

[4] A. Ammar, B. Boukettaya, and A. Jeribi, *A note on the essential pseudospectra and application*, Linear Multilinear Algebra **64** (2016), no. 8, 1474–1483. `https://doi.org/10.1080/03081087.2015.1091436`

[5] A. Ammar, H. Daoud, and A. Jeribi, *The stability of pseudospectra and essential pseudospectra of linear relations*, J. Pseudo-Differ. Oper. Appl. **7** (2016), no. 4, 473–491. `https://doi.org/10.1007/s11868-016-0150-3`

[6] N. Athmouni, H. Baloudi, A. Jeribi, and G. Kacem, *On weighted and pseudo-weighted spectra of bounded operators*, Commun. Korean Math. Soc. **33** (2018), no. 3, 809–821. `https://doi.org/10.4134/CKMS.c170245`

[7] S. K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J. **20** (1970/71), 529–544. `https://doi.org/10.1512/iumj.1970.20.20044`

[8] L. Burlando, *Approximation by semi-Fredholm and semi-α-Fredholm operators in Hilbert spaces of arbitrary dimension*, Acta Sci. Math. (Szeged) **65** (1999), no. 1-2, 217–275.

[9] C. R. Carpintero, O. García, E. R. Rosas, and J. E. Sanabria, *B-Browder spectra and localized SVEP*, Rend. Circ. Mat. Palermo (2) **57** (2008), no. 2, 239–254. https://doi.org/10.1007/s12215-008-0017-4

[10] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288. http://projecteuclid.org/euclid.mmj/1031732778

[11] J. Cui, V. Forstall, C. Li, and V. Yannello, *Properties and preservers of the pseudospectrum*, Linear Algebra Appl. **436** (2012), no. 2, 316–325. https://doi.org/10.1016/j.laa.2011.03.044

[12] P. Dharmarha and S. Kumari, *On weighted Browder spectrum*, Bull. Korean Math. Soc. **59** (2022), no. 1, 1–13. https://doi.org/10.4134/BKMS.b200328

[13] G. Edgar, J. Ernest, and S. G. Lee, *Weighing operator spectra*, Indiana Univ. Math. J. **21** (1971/72), 61–80. https://doi.org/10.1512/iumj.1971.21.21005

[14] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrixoperator*, ProQuest LLC, Ann Arbor, MI, 1967.

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