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ZERO SUMS OF DUAL TOEPLITZ PRODUCTS ON THE ORTHOGONAL COMPLEMENT OF THE DIRICHLET SPACE

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Abstract. We consider dual Toeplitz operators acting on the orthogonal complement of the Dirichlet space on the unit disk. We give a characterization of when a finite sum of products of two dual Toeplitz operators is equal to 0. Our result extends several known results by using a unified way.

1. Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$ and dA denote the normalized Lebesgue measure on D. The Sobolev space $\mathscr{W}^{1,2}$ is the completion of the space of all smooth functions f on D for which

$$\left|\int_D f\,dA\right|^2 + \int_D \left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right) dA < \infty.$$

As is well known, the space $\mathscr{W}^{1,2}$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_D f dA \int_D \bar{g}\, dA + \int_D \left(\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}}\overline{\frac{\partial g}{\partial \bar{z}}}\right) dA.$$

The Dirichlet space $\mathscr{D}$ is the closed subspace of $\mathscr{W}^{1,2}$ consisting of all holomorphic functions $f \in \mathscr{W}^{1,2}$ with $f(0) = 0$. Let P denote the orthogonal projection from $\mathscr{W}^{1,2}$ onto $\mathscr{D}$. Put

$$\mathscr{L}^{1,\infty} = \left\{u \in \mathscr{W}^{1,2} : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in L^\infty\right\},$$

where $L^p = L^p(D, dA)$ denotes the usual Lebesgue space on D and the derivatives are taken in the distribution sense.

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Given $u \in \mathscr{L}^{1,\infty}$, the *dual Toeplitz operator* S_u with symbol u is defined on $\mathscr{D}^\perp$ by

$$S_u f = (I - P)(uf)$$

for functions $f \in \mathscr{D}^\perp$. Then, for $u \in \mathscr{L}^{1,\infty}$, it is easy to check that the dual Toeplitz operator S_u is a bounded linear operator on $\mathscr{D}^\perp$.

Some algebraic properties for dual Toeplitz operators have been well studied on various function spaces. In [8], Stroethoff and Zheng characterized the (semi)-commuting dual Toeplitz operators and obtained a characterization of when a product of two dual Toeplitz operators is another dual Toeplitz operator on the orthogonal complement of the Bergman space on the unit disk. Also, the corresponding results on the Hardy space have been obtained in [1] and [4]. More recently, Kong and Lu [5] recovered several known results concerning the commutativity or product problem by characterizing zero sums of products of two dual Toeplitz operators on the orthogonal complement of the Bergman space or Hardy space.

Also, the corresponding problems have been studied for dual Toeplitz operators acting on the orthogonal complement of the Dirichlet space. Yu and Wu [12] characterized harmonic symbols of (semi-)commuting dual Toeplitz operators. They also obtained a characterization of when a product of two dual Toeplitz operators with harmonic symbols is another dual Toeplitz operator. Later, Yu [10] extended the results of [12] to general symbols by using complete different arguments from those used in [12].

Motivated by these results, in this paper, we consider a more general class of operators that include (semi-)commutators or products of two dual Toeplitz operators. More explicitly, we consider operators which are finite sums of products of two dual Toeplitz operators with general symbols and then obtain a characterization of when such an operator is equal to 0. Our results extend several known results mentioned above by providing a unified way of treating them.

In Section 2, we collect some preliminary results which will be useful in our characterization. In Section 3, we state and prove our main result; see Theorem 3. Also, as applications of our result, we obtain several consequences and recover known results; see Corollaries 4 and 5.

2. Preliminaries

Since each point evaluation is a bounded linear functional on $\mathscr{D}$, there corresponds to every $z \in D$ a unique function $K_z \in \mathscr{D}$ which has the following reproducing property

$$f(z) = \langle f, K_z \rangle$$

for every $f \in \mathscr{D}$. It is known that the function K_z can be given by

$$K_z(w) = \log\left(\frac{1}{1 - \bar{z}w}\right), \quad w \in D.$$

Using the explicit formula above for K_z, one can see that P can be represented by the integral formula

$$P\psi(z) = \int_D \frac{z}{1 - z\bar{w}} \frac{\partial \psi}{\partial w}(w)\, dA(w), \quad z \in D \tag{1}$$

for functions $\psi \in \mathscr{W}^{1,2}$; see [6] or [9] for details and related facts.

It is known that each $f \in \mathscr{W}^{1,2}$ admits the following polar decomposition

$$f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k(r) e^{ik\theta},$$

where $f_k(r) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta}\, d\theta$. Moreover, $\sum_{|k| \le n} f_k(r) e^{ik\theta}$ converges to f in $\mathscr{W}^{1,2}$ as $n \to \infty$. Also, it is known that the radial limit $f(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta})$ exists for almost every θ and $f(e^{i\theta}) \in L^1(\partial D)$. For $f \in \mathscr{W}^{1,2}$ and $k \in \mathbb{Z}$, the set of all integers, we let

$$f_k(1) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta}\, d\theta$$

and put $\mathcal{A} = \mathcal{A}_0 + \mathbb{C}$, where

$$\mathcal{A}_0 = \left\{ \sum_{k \in \mathbb{Z}} [f_k(r) - f_k(1) r^{|k|}] e^{ik\theta} : f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} f_k(r) e^{ik\theta} \in \mathscr{W}^{1,2} \right\}.$$

Then, it is known that $u\mathcal{A}_0 \subset \mathcal{A}_0$ for all $u \in \mathscr{L}^{1,\infty}$ and $P\psi = 0$ for all $\psi \in \mathcal{A}_0$. Furthermore, we have the following decomposition for $\mathscr{W}^{1,2}$:

$$\mathscr{W}^{1,2} = \mathscr{D} \oplus \overline{\mathscr{D}} \oplus \mathcal{A}. \tag{2}$$

See [2] or [11] for details and more information.

3. Main result

Given $u \in \mathscr{L}^{1,\infty}$, the Toeplitz operator T_u, the Hankel operator H_u and dual Hankel operator R_u with symbol u are defined, respectively, by

$$T_u f = P(uf),$$
$$H_u f = (I - P)(uf),$$
$$R_u g = P(ug)$$

for functions $f \in \mathscr{D}$ and $g \in \mathscr{D}^\perp$. Then one can check that the operators $T_u : \mathscr{D} \to \mathscr{D}$, $H_u : \mathscr{D} \to \mathscr{D}^\perp$ and $R_u : \mathscr{D}^\perp \to \mathscr{D}$ are all bounded linear operators. Also, given $u, v \in \mathscr{L}^{1,\infty}$, it is easy to see that the following useful relation holds on $\mathscr{D}^\perp$:

$$S_u S_v = S_{uv} - H_u R_v. \tag{3}$$

Given $u \in \mathscr{L}^{1,\infty}$, it is known that $R_u = 0$ if and only if $u \in \mathscr{D}^\perp$; see Theorem 1 of [10]. Since the set $\{K_a : a \in D\}$ spans a dense subset of $\mathscr{D}$, we check that $R_u^* K_a = 0$ for all $a \in D$ if and only if $u \in \mathscr{D}^\perp$. The following shows that the same is true even though $R_u^* K_a = 0$ for some nonzero a, which will be useful in our proof.

Proposition 1. *Let $u \in \mathscr{L}^{1,\infty}$ and $a \in D$ be nonzero. Then $R_u^* K_a = 0$ if and only if $u \in \mathscr{D}^\perp$.*

Proof. First assume $R_u^* K_a = 0$. Decompose u as $u = f + \bar{g} + \varphi$, where $f, g \in \mathscr{D}$, $\varphi \in \mathcal{A}$ as in (2). Since $R_u^* K_a = 0$, we have $\langle R_u^* K_a, F \rangle = 0$ for all $F \in \mathscr{D}^\perp$. Write $f(z) = \sum_{j=1}^\infty b_j z^j$ for the Taylor series expansions of f. By (1), we note $\overline{z^n} \in \mathscr{D}^\perp$ and

$$P((\bar{g} + \varphi)\overline{z^n}) = 0$$

for all $n = 0, 1, 2, \ldots$. Also, by a simple calculation using (1), we can see that for integers $m, n \geq 0$

$$P(\bar{z}^m z^n)(z) = \begin{cases} z^{n-m} & \text{if } \ n > m, \\ 0 & \text{if } \ n \leq m \end{cases}$$

for $z \in D$. It follows that

$$P(f\overline{z^n})(a) = \sum_{j=1}^\infty b_j P(z^j \overline{z^n})(a) = \sum_{j=n+1}^\infty b_j a^{j-n}$$

for all $n = 0, 1, 2, \ldots$. Now, taking $F = \overline{z^n}$ and using the above, we can check that

$$\begin{aligned} 0 &= \langle R_u^* K_a, \overline{z^n} \rangle \\ &= \langle K_a, P(u\overline{z^n}) \rangle \\ &= \langle K_a, f\overline{z^n} \rangle \\ &= \overline{P(f\overline{z^n})(a)} \\ &= \overline{a^{-n} \sum_{j=n+1}^\infty b_j a^j} \end{aligned}$$

for all $n = 0, 1, 2, \ldots$, which implies $b_j = 0$ for all $j \geq 1$. Thus $f = 0$ and $u = \bar{g} + \varphi \in \mathscr{D}^\perp$, as desired. Conversely, if $u \in \mathscr{D}^\perp$, then $R_u = 0$ and then $R_u^* K_a = 0$ holds. The proof is complete. □

Given two vectors x, y in a Hilbert space $\mathcal{H}$ with an inner product $(\ , \ )$, the rank one operator $x \otimes y$ is defined on $\mathcal{H}$ by

$$[x \otimes y]z = (z, y)x$$

for functions $x \in \mathcal{H}$. We note that the following operator equation

$$L_1(x \otimes y)L_2 = (L_1 x) \otimes (L_2^* y) \tag{4}$$

holds for bounded operators L_1, L_2. Also, given nonzero vectors $x_1, x_2, y_1, y_2 \in \mathcal{H}$, we observe that $x_1 \otimes y_1 = x_2 \otimes y_2$ if and only if there exists a nonzero $\alpha \in \mathbb{C}$ such that $x_1 = \alpha x_2$ and $y_2 = \bar{\alpha} y_1$. More generally, we have the following lemma which is essentially proved in Proposition 4 of [3]. In the following, for a given positive integer N, we let $\mathbb{M}_N$ be the set of all $N \times N$ matrices and $\mathbb{S}_N$ be the set of all permutations on $\{1, \dots, N\}$.

Lemma 2. *Let $x_j, y_j \in \mathscr{D}^\perp$ for $j = 1, \dots, N$. Then*

$$\sum_{j=1}^{N} x_j \otimes y_j = 0 \quad on \quad \mathscr{D}^\perp$$

if and only if there exist $A \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that

$$[A - I] \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(N)} \end{pmatrix} = 0 \quad and \quad A^* \begin{pmatrix} y_{\sigma(1)} \\ \vdots \\ y_{\sigma(N)} \end{pmatrix} = 0.$$

Given a point $a \in D$, we let

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in D$$

denote the usual automorphism of D and ρ_a be the function on D defined by

$$\rho_a(z) = a - \varphi_a(z), \quad z \in D.$$

Then, for each $a \in D$, one can check that

$$\rho_a \otimes K_a = a(I - T_{\varphi_a} T_{\overline{\varphi_a}}) \tag{5}$$

holds on $\mathscr{D}$; see Lemma 7 of [10].

We are now ready to state and prove our main theorem characterizing zero sums of products of two dual Toeplitz operators. In the proof, we will use an argument in [7] where zero sums of products of two *ordinary* Toeplitz operators have been characterized. In the following, the notation $\mathscr{H}$ denotes the set of all holomorphic functions on D. Also, given a set X and an integer $N \geq 1$, X^N denotes the set of all $(x_1, x_2, \dots, x_N)$, where $x_j \in X$.

Theorem 3. *Let $u_j, v_j \in \mathscr{L}^{1,\infty}$ for $j = 1, \dots, N$. Then*

$$\sum_{j=1}^{N} S_{u_j} S_{v_j} = 0 \quad on \quad \mathscr{D}^\perp \tag{6}$$

if and only $\sum_{j=1}^{N} u_j v_j = 0$ on D and one of the following equivalent conditions holds.

(a) $\sum_{j=1}^{N} H_{u_j} R_{v_j} = 0$.

(b) *There exist $A \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that the following conditions hold;*

  (b1) $(A - I)U_\sigma \in \mathscr{H}^N$*, where* $U_\sigma = (u_{\sigma(1)}, \dots, u_{\sigma(N)})^T$.

  (b2) $\bar{A}^* V_\sigma \in (\mathscr{D}^\perp)^N$*, where* $V_\sigma = (v_{\sigma(1)}, \dots, v_{\sigma(N)})^T$.

Proof. By (3), we note that

$$\sum_{j=1}^{N} S_{u_j} S_{v_j} = S_{\sum_{j=1}^{N} u_j v_j} - \sum_{j=1}^{N} H_{u_j} R_{v_j}. \tag{7}$$

It is known that $\sum_{j=1}^{N} S_{u_j} S_{v_j}$ is compact if and only if $\sum_{j=1}^{N} u_j v_j = 0$; see Corollary 2.5 of [11]. Note that (6) implies the compactness of $\sum_{j=1}^{N} S_{u_j} S_{v_j}$. Thus, (7) shows that (6) holds if and only if $\sum_{j=1}^{N} u_j v_j = 0$ and (a) holds. So, in order to prove the theorem, it suffices to prove that (a) is equivalent to (b).

Now, suppose (a) holds and fix a nonzero point a in D. Noting

$$H_{u_j} T_{\varphi_a} = S_{\varphi_a} H_{u_j}, \qquad T_{\overline{\varphi_a}} R_{v_j} = R_{v_j} S_{\overline{\varphi_a}},$$

we can see from (5) that

$$\begin{aligned} [H_{u_j}\rho_a] \otimes [R_{v_j}^* K_a] &= H_{u_j}[\rho_a \otimes K_a] R_{v_j} \\ &= H_{u_j}[a(I - T_{\varphi_a} T_{\overline{\varphi_a}})] R_{v_j} \\ &= a[H_{u_j} R_{v_j} - H_{u_j} T_{\varphi_a} T_{\overline{\varphi_a}} R_{v_j}] \\ &= a[H_{u_j} R_{v_j} - S_{\varphi_a} H_{u_j} R_{v_j} S_{\overline{\varphi_a}}] \end{aligned}$$

for each j. Hence we have

$$\sum_{j=1}^{N} [H_{u_j}\rho_a] \otimes [R_{v_j}^* K_a] = a \sum_{j=1}^{N} H_{u_j} R_{v_j} - a S_{\varphi_a} \left(\sum_{j=1}^{N} H_{u_j} R_{v_j} \right) S_{\overline{\varphi_a}} = 0.$$

By Lemma 2, there exist $A = [a_{ij}] \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that

$$[A - I] \begin{pmatrix} H_{u_{\sigma(1)}}\rho_a \\ \vdots \\ H_{u_{\sigma(N)}}\rho_a \end{pmatrix} = 0 \quad \text{and} \quad A^* \begin{pmatrix} R_{v_{\sigma(1)}}^* K_a \\ \vdots \\ R_{v_{\sigma(N)}}^* K_a \end{pmatrix} = 0. \tag{8}$$

By the first equation of (8), one can see that

$$H_{\sum_{j=1}^{N} a_{ij} u_{\sigma(j)}} \rho_a = \sum_{j=1}^{N} a_{ij} H_{u_{\sigma(j)}} \rho_a = H_{u_{\sigma(i)}} \rho_a$$

and hence

$$(I - P) \left(\rho_a \Big[\sum_{j=1}^{N} a_{ij} u_{\sigma(j)} - u_{\sigma(i)} \Big] \right) = H_{\sum_{j=1}^{N} a_{ij} u_{\sigma(j)} - u_{\sigma(i)}} \rho_a = 0$$

for each i. Hence $\rho_a \Big[\sum_{j=1}^{N} a_{ij} u_{\sigma(j)} - u_{\sigma(i)} \Big] \in \mathscr{H}$ for each i. Note $\rho_a(z) = 0$ if and only if $z = 0$. Since each u_j is bounded, we see that $\sum_{j=1}^{N} a_{ij} u_{\sigma(j)} - u_{\sigma(i)} \in$

$\mathscr{H}$ for each i and then (b1) holds. Note $R^*_{\alpha\varphi} = \bar{\alpha} R^*_\varphi$ for every $\varphi \in \mathscr{L}^{1,\infty}$ and constant α. Thus, by the second equation of (8), one obtains

$$R^*_{\sum_{i=1}^N a_{ij} v_{\sigma(i)}} K_a = \sum_{i=1}^N \overline{a_{ij}} R^*_{v_{\sigma(i)}} K_a = 0,$$

which implies that $\sum_{i=1}^N a_{ij} v_{\sigma(i)} \in \mathscr{D}^\perp$ for every j by Proposition 1. Hence (b2) holds and (b) follows as desired.

Now suppose (b) holds. Letting $A = [a_{ij}]$, we have $\sum_{j=1}^N a_{ij} u_{\sigma(j)} - u_{\sigma(i)} \in \mathscr{H}$ and $\sum_{j=1}^N a_{ji} v_{\sigma(j)} \in \mathscr{D}^\perp$ for each i. For $x \in \mathscr{H} \cap \mathscr{L}^{1,\infty}$, note $H_x = 0$. Hence

$$H_{u_{\sigma(i)}} = H_{\sum_{j=1}^N a_{ij} u_{\sigma(j)}}, \qquad R_{\sum_{j=1}^N a_{ji} v_{\sigma(j)}} = 0$$

for each i. It follows that

$$\begin{aligned}
\sum_{j=1}^N H_{u_j} R_{v_j} &= \sum_{j=1}^N H_{u_{\sigma(j)}} R_{v_{\sigma(j)}} \\
&= \sum_{j=1}^N \left(H_{\sum_{i=1}^N a_{ji} u_{\sigma(i)}} \right) R_{v_{\sigma(j)}} \\
&= \sum_{j=1}^N \left(\sum_{i=1}^N a_{ji} H_{u_{\sigma(i)}} \right) R_{v_{\sigma(j)}} \\
&= \sum_{i=1}^N H_{u_{\sigma(i)}} \left(\sum_{j=1}^N a_{ji} R_{v_{\sigma(j)}} \right) \\
&= \sum_{i=1}^N H_{u_{\sigma(i)}} \left(R_{\sum_{j=1}^N a_{ji} v_{\sigma(j)}} \right) \\
&= 0
\end{aligned}$$

and (a) holds. The proof is complete. □

As immediate consequences, we obtain several applications. First, in the special case when $N = 2$ in Theorem 3, we obtain a more concrete description as shown in the next corollary.

Corollary 4. *Let $u, v, \varphi, \psi \in \mathscr{L}^{1,\infty}$. Then $S_u S_v = S_\varphi S_\psi$ on $\mathscr{D}^\perp$ if and only if $uv = \varphi\psi$ on D and one of the following statements holds*:

(a) $u, \varphi \in \mathscr{H}$.
(b) $v, \psi \in \mathscr{D}^\perp$.
(c) $u \in \mathscr{H}, \psi \in \mathscr{D}^\perp$.
(d) $v \in \mathscr{D}^\perp, \varphi \in \mathscr{H}$.
(e) $u + \beta\varphi \in \mathscr{H}$ *and* $\psi + \beta v \in \mathscr{D}^\perp$ *for some nonzero constant* $\beta \in \mathbb{C}$.

Proof. First suppose $S_uS_v = S_\varphi S_\psi$. By Theorem 3 (with the identity σ without loss of generality), we have $uv = \varphi\psi$ and

$$\begin{aligned} (a-1)u - b\varphi &\in \mathscr{H}, \\ cu - (d-1)\varphi &\in \mathscr{H}, \\ c\psi + av &\in \mathscr{D}^\perp, \\ d\psi + bv &\in \mathscr{D}^\perp \end{aligned} \tag{9}$$

for some constants a, b, c and d. If $u \in \mathscr{H}$ and $b \neq 0$, then the first line above shows $\varphi \in \mathscr{H}$ and then (a) holds. Also, if $u \in \mathscr{H}$, $b = 0$ and $d \neq 0$, then the last line above shows $\psi \in \mathscr{D}^\perp$ and then (c) holds. If $u \in \mathscr{H}$ and $b = d = 0$, then the second line above shows $\varphi \in \mathscr{H}$ and then (a) holds. Hence, if $u \in \mathscr{H}$, then (a) or (c) holds. Similarly, we can see that if $v \in \mathscr{D}^\perp$, then (b) or (d) holds. Also, if $\varphi \in \mathscr{H}$, then (a) or (d) holds. Finally, if $\psi \in \mathscr{D}^\perp$, (b) or (c) holds. Note $S_1 = I$. By the last case what we have done above, we characterize the semi-commuting dual Toeplitz operators by taking $\psi = 1 \in \mathscr{D}^\perp$ together with (3) as follows: $S_uS_v = S_{uv}$ if and only if either $u \in \mathscr{H}$ or $v \in \mathscr{D}^\perp$. This fact will be used in the proof of the last case below.

Now, assume $u, \varphi \notin \mathscr{H}$ and $v, \psi \notin \mathscr{D}^\perp$. If $a - 1 = b = c = d - 1 = 0$, then the last two conditions above show $v, \psi \in \mathscr{D}^\perp$, which is impossible and then one of $a-1, b, c, d-1$ is nonzero. On the other hand, using the first two conditions in (9), we note $a - 1 \neq 0$ if and only if $b \neq 0$, and $c \neq 0$ if and only if $d \neq 1$. Thus we have $u + \epsilon\varphi \in \mathscr{H}$ for some nonzero constant ϵ. Also, if $a = b = c = d = 0$, then the first two conditions in (9) show $u, \varphi \in \mathscr{H}$, which is impossible as before. So, one of a, b, c, d is nonzero and then by the same argument above we see $\psi + \delta v \in \mathscr{D}^\perp$ for some nonzero constant δ. Now we show $\epsilon = \delta$. For $x \in \mathscr{L}^{1,\infty}$, recall that $R_x = 0$ if and only if $x \in \mathscr{D}^\perp$. Also, note that $H_x = 0$ for $x \in \mathscr{H} \cap \mathscr{L}^{1,\infty}$. Since $u + \epsilon\varphi \in \mathscr{H}$ and $\psi + \delta v \in \mathscr{D}^\perp$, (3) shows that

$$\begin{aligned} S_{u+\epsilon\varphi}S_v &= S_{v(u+\epsilon\varphi)} - H_{u+\epsilon\varphi}R_v = S_{v(u+\epsilon\varphi)}, \\ S_\varphi S_{\psi+\delta v} &= S_{\varphi(\psi+\delta v)} - H_\varphi R_{\psi+\delta v} = S_{\varphi(\psi+\delta v)}. \end{aligned} \tag{10}$$

It follows that

$$\begin{aligned} S_uS_v &= S_{uv} + \epsilon S_{\varphi v} - \epsilon S_\varphi S_v, \\ S_\varphi S_\psi &= S_{\varphi\psi} + \delta S_{\varphi v} - \delta S_\varphi S_v. \end{aligned} \tag{11}$$

Since $S_uS_v = S_\varphi S_\psi$ and $uv = \varphi\psi$, we have $(\epsilon - \delta)[S_\varphi S_v - S_{\varphi v}] = 0$. But, since $\varphi \notin \mathscr{H}$ and $v \notin \mathscr{D}^\perp$, we see $S_\varphi S_v \neq S_{\varphi v}$ by the remark mentioned before. Thus $\epsilon = \delta$ and (e) holds, as desired.

Now, suppose $uv = \varphi\psi$ and one of (a)∼(e) holds. If either $x \in \mathscr{H}$ or $y \in \mathscr{D}^\perp$, we recall $H_xR_y = 0$. Hence, if one of (a), (b), (c) and (d) holds, we have $S_uS_v = S_\varphi S_\psi$ by (3). If (e) holds, (10) and (11) with $\beta = \epsilon = \delta$ show that $S_uS_v = S_\varphi S_\psi$. The proof is complete. □

If we take $\varphi = v$ and $\psi = u$ in Corollary 4, we characterize commuting dual Toeplitz operators as in Corollary 5(a) below which recovers Theorem 2 of [10]. Also, taking $\psi = 1$ in Corollary 4, we have Corollary 5(b) below solving the product problem of when a product of two dual Toeplitz operators is another dual Toeplitz operator. Finally, as mentioned before in the proof of Corollary 4, taking $\varphi = uv$ in Corollary 5(b), we characterize semi-commuting dual Toeplitz operators, which recovers Theorem 3 of [10]; see Corollary 5(c) below.

Corollary 5. *Let $u, v, \varphi \in \mathscr{L}^{1,\infty}$. Then the following statements hold.*

(a) $S_u S_v = S_v S_u$ *if and only if* $u, v \in \mathscr{H}$ *or* $u, v \in \mathscr{D}^\perp$ *or a nontrivial linear combination of* u *and* v *is constant on* D*.*

(b) $S_\varphi = S_u S_v$ *if and only if* $\varphi = uv$ *on* D*, and* $u \in \mathscr{H}$ *or* $v \in \mathscr{D}^\perp$*.*

(c) $S_{uv} = S_u S_v$ *if and only if either* $u \in \mathscr{H}$ *or* $v \in \mathscr{D}^\perp$*.*

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