

## NEW FAMILIES OF HYPERBOLIC TWISTED TORUS KNOTS WITH GENERALIZED TORSION

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**ABSTRACT.** A generalized torsion element is an obstruction for a group to admit a bi-ordering. Only a few classes of hyperbolic knots are known to admit such an element in their knot groups. Among twisted torus knots, the known ones have their extra twists on two adjacent strands of torus knots. In this paper, we give several new families of hyperbolic twisted torus knots whose knot groups have generalized torsion. They have extra twists on arbitrarily large numbers of adjacent strands of torus knots.

### 1. Introduction

It is well known that any knot group is torsion free, but it may contain a generalized torsion element. In a group  $G$ , a *generalized torsion element* is a nontrivial element  $g$  which yields the identity as a nonempty finite product of its conjugates. That is, the equation

$$g^{a_1} g^{a_2} \cdots g^{a_n} = 1$$

holds for some  $a_1, a_2, \dots, a_n \in G$ , where  $g^{a_i}$  denotes the conjugate  $a_i^{-1} g a_i$ . The existence of a generalized torsion element is an obstruction for  $G$  to admit a bi-ordering, which is a strict total ordering invariant under the left and right multiplications.

Among knot groups, those of torus knots are well known to admit generalized torsion (see [17]). Also, it is easy to give satellite knots whose knot groups have generalized torsion. However, the case of hyperbolic knot groups is hard. It is Naylor and Rolfsen [17] who give the first example of hyperbolic knot, which is the  $(-2)$ -twist knot ( $5_2$  in the knot table), that enjoys the property. This hyperbolic knot is extended to all negative twist knots in [24]. Since then, only few classes of hyperbolic knots are verified to have generalized torsion in their knot groups [16]. In particular, we treated two families of twisted torus knots,

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$T(pm + p + 1, pm + 1; 2, 1)$  with  $p \geq 2, m \geq 1$ , and  $T(5, 3; 2, s)$  with  $s \geq 1$  in [16]. These realized a hyperbolic knot with arbitrarily high genus whose knot group admits a generalized torsion element.

A twisted torus knot  $T(p, q; r, s)$  is obtained from a torus knot  $T(p, q)$  by adding  $s$ -full (right handed) twists on  $r$  adjacent strings. We may assume that  $p > q > 1$ , since  $T(p, q; r, s)$  and  $T(q, p; r, s)$  are equivalent [11]. Although it is possible to take  $2 \leq r \leq p + q$  and  $s < 0$ , in general, we consider only the case where  $r < q$  and  $s \geq 1$ . See [6, 12, 13] for basic facts of twisted torus knots.

The purpose of this paper is to give several new families of hyperbolic twisted torus knots having generalized torsion in their knot groups. In particular, we can choose the third parameter  $r$  to be arbitrarily large.

**Theorem 1.1.** *For any integer  $r \geq 3$ , each twisted torus knot of the following families is hyperbolic, and its knot group admits a generalized torsion element.*

- (1)  $T(r + 2, r + 1; r, s)$  for  $s \geq 2$ ,
- (2)  $T(m(r + 1) + 1, r + 1; r, 2)$  for  $m \geq 2$ ,
- (3)  $T(2r + 1, r + 1; r, s)$  for  $s \geq 1$ ,
- (4)  $T(m(r + 1) - 1, r + 1; r, 1)$  for  $m \geq 3$ .

**Corollary 1.2.** *For any integer  $b \geq 4$ , there exist infinitely many hyperbolic twisted torus knots, each of whose knot group admits a generalized torsion element and bridge number is equal to  $b$ .*

*Proof.* Set  $r = b - 1$  for (1) in Theorem 1.1. If  $s > 18(r + 1)$ , then the bridge number of  $T(r + 2, r + 1; r, s)$  is equal to  $r + 1 = b$  by [3].

Also,  $T(r + 2, r + 1; r, s)$  is expressed as the closure of a positive braid, so it is fibered [23]. In particular, its genus is  $(r^2 + r + s(r^2 - r))/2$ . (See the next paragraph after Lemma 3.3.) By varying  $s$ , we can give distinct knots.  $\square$

For  $b = 2$  and  $3$ , we have given such hyperbolic knots in [16, 24]. Also, the examples given in [16] are expected to realize arbitrarily high bridge number, but we could not evaluate it.

In Theorem 1.1, we treat only the case where the first parameter of a twisted torus knot is congruent to  $\pm 1$  modulo the second parameter. We can give more families beyond such a constraint as follows.

**Theorem 1.3.** *Each twisted torus knot of the following families is hyperbolic, and its knot group admits a generalized torsion element.*

- (1)  $T(5m + 2, 5; 3, 1)$  for  $m \geq 1$ ,
- (2)  $T(5m - 2, 5; 3, 1)$  for  $m \geq 2$ ,
- (3)  $T(5m + 2, 5; 4, 1)$  for  $m \geq 1$ ,
- (4)  $T(5m - 2, 5; 4, 1)$  for  $m \geq 2$ .

In general, knot types of twisted torus knots are most determined. Roughly speaking,  $T(p, q; r, s)$  is hyperbolic if  $s \geq 2$  by Lee [13]. However, the case where  $s = 1$  remains to be open. Recently, Paiva [19, Corollary 1.4] shows that  $T(kq + n, q; r, 1)$  is hyperbolic for  $k \geq 2$  and  $q > n \geq r$  with  $\gcd(n, q) = 1$ .

However, these results do not cover all of our cases, so we need a proof of hyperbolicity.

As mentioned before, any bi-orderable group does not admit generalized torsion.

**Corollary 1.4.** *For each twisted torus knot listed in Theorems 1.1, 1.3, its knot group is not bi-orderable.*

Many of our knots are L-space knots, so their knot groups are known to be non-bi-orderable [5]. However, the lack of bi-ordering does not imply the existence of generalized torsion.

In Section 2, we give an algorithmic way to get a presentation for the knot group of a twisted torus knot. Section 3 shows the existence of generalized torsion elements in their knot groups of our twisted torus knots. Finally, we confirm that all of our twisted torus knots are hyperbolic in Sections 4 and 5 by applying some results on Dehn surgery.

### 2. Knot groups

We prepare an algorithmic way to calculate knot groups of twisted torus knots as similar to [16].

Let  $\Sigma$  be the standard genus two closed orientable surface in the 3-sphere  $S^3$ . This gives a genus two Heegaard splitting  $U \cup_{\Sigma} V$  of  $S^3$ , where  $U$  and  $V$  are genus two handlebodies. Let us take simple closed curves  $K_0, C_1, C_2, C_3, C_4$  and  $C_5$  on  $\Sigma$  as shown in Figure 1. Moreover, take simple closed curves  $a, b, c, d$  on  $\Sigma$  as shown in Figure 2, so that  $\{a, b\}$  generates  $\pi_1(U)$  and  $\{c, d\}$  generates  $\pi_1(V)$ , where  $U$  is the inside of  $\Sigma$  and  $V$  is the outside. (We use the same symbol for the homotopy class of a loop.) Finally, we need three loops  $G_0, R_0$  and  $Y_0$  as in Figure 3, where  $\Sigma - K_0$  contracts to the bouquet  $G_0 \vee R_0 \vee Y_0$ .

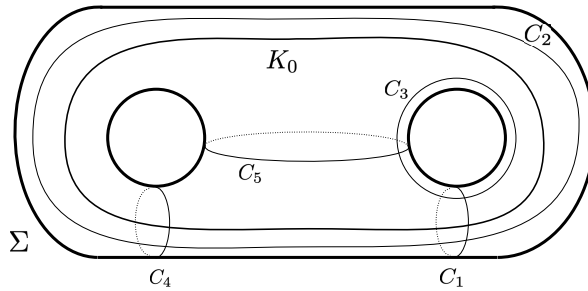


FIGURE 1. The knot  $K_0$  and 5 curves  $C_1, C_2, \dots, C_5$  for Dehn twists on the standard genus two surface  $\Sigma$  in  $S^3$ .

We consider the (right handed) Dehn twists  $D_i$  about  $C_i$  for  $1 \leq i \leq 5$ . (The curve  $C_5$  and the Dehn twist  $D_5$  will be used in Sections 4 and 5.)

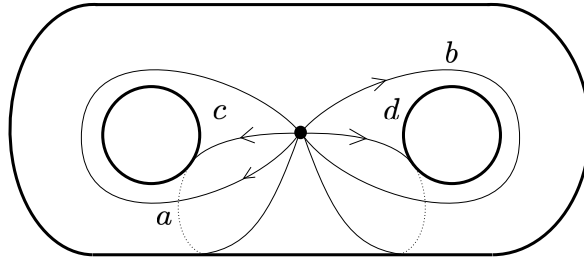


FIGURE 2. Generators  $a, b, c, d$  for  $\pi_1(U)$  and  $\pi_1(V)$ .

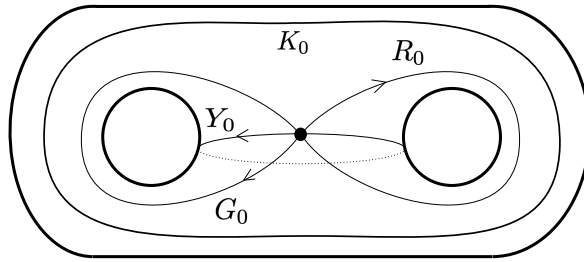


FIGURE 3. Spine  $G_0 \vee R_0 \vee Y_0$  of  $\Sigma - K_0$ .

TABLE 1. Compositions of Dehn twists and the images of  $K_0$ .

	Composition	$K_0 \mapsto$
$\rho_1$	$D_4^s \circ D_1^m \circ D_3^{-1} \circ D_2^{-(r-1)} \circ D_1$	$T(m(r+1) + 1, r+1; r, s)$
$\rho_2$	$D_4^s \circ D_1^m \circ D_3 \circ D_2^{r-1} \circ D_1^{-1}$	$T(m(r+1) - 1, r+1; r, s)$
$\rho_3$	$D_4^s \circ D_1^m \circ D_3^{-1} \circ D_2^{-1} \circ D_1^2$	$T(5m + 2, 5; 3, s)$
$\rho_4$	$D_4^s \circ D_1^m \circ D_3 \circ D_2 \circ D_1^{-2}$	$T(5m - 2, 5; 3, s)$
$\rho_5$	$D_4^s \circ D_1^m \circ D_2^{-2} \circ D_1 \circ D_3^{-1} \circ D_2 \circ D_1$	$T(5m + 2, 5; 4, s)$
$\rho_6$	$D_4^s \circ D_1^m \circ D_2^2 \circ D_1^{-1} \circ D_3 \circ D_2 \circ D_1$	$T(5m - 2, 5; 4, s)$

**Lemma 2.1.** *The compositions of Dehn twists given in Table 1 map the knot  $K_0$  to the indicated twisted torus knot.*

*Proof.* This is straightforward to verify. □

To chase the images of  $G_0, R_0$  and  $Y_0$  under these compositions, it is enough to examine the effects on the generators  $a, b, c, d$  under each Dehn twist  $D_i$  for

TABLE 2. The effects of  $D_i$  on the generators.

	$a$	$b$	$c$	$d$
$D_1^{\pm 1}$	$a$	$bd^{\pm 1}$	$c$	$d$
$D_2^{\pm 1}$	$a$	$b$	$c(ab)^{\mp 1}$	$d(ab)^{\mp 1}$
$D_3^{\pm 1}$	$a$	$b$	$c$	$b^{\mp 1}d$
$D_4^s$	$c^s a$	$b$	$c$	$d$

$1 \leq i \leq 4$ . Table 2 summarizes them. For example,  $D_1^m$  maps  $b$  to  $bd^m$  for any integer  $m$ , and  $D_2^2$  maps  $c$  and  $d$  to  $c(ab)^{-2}$  and  $d(ab)^{-2}$ , respectively.

First, let us examine  $\rho_1 = D_4^s \circ D_1^m \circ D_3^{-1} \circ D_2^{-(r-1)} \circ D_1$  in Table 1. Let  $G, R$  and  $Y$  be the images of  $G_0, R_0$  and  $Y_0$  under  $\rho_1$ .

- For  $G_0$ , we have the transition

$$G_0 = a \mapsto a \mapsto a \mapsto a \mapsto a \mapsto c^s a.$$

Thus  $G = c^s a$  on  $\Sigma$ . Hence, by putting  $c = d = 1$  for  $\pi_1(U)$  or  $a = b = 1$  for  $\pi_1(V)$ , we obtain

$$G = \begin{cases} a & \text{in } \pi_1(U), \\ c^s & \text{in } \pi_1(V). \end{cases}$$

- For  $R_0$ ,

$$\begin{aligned} R_0 = b &\mapsto bd \mapsto bd(ab)^{r-1} \mapsto b^2 d(ab)^{r-1} \\ &\mapsto (bd^m)^2 d(abd^m)^{r-1} \mapsto (bd^m)^2 d(c^s abd^m)^{r-1}. \end{aligned}$$

Hence,

$$R = \begin{cases} b^2 (ab)^{r-1} & \text{in } \pi_1(U), \\ d^{2m+1} (c^s d^m)^{r-1} & \text{in } \pi_1(V). \end{cases}$$

- For  $Y_0$ ,

$$Y_0 = cd^{-1} \mapsto cd^{-1} \mapsto cd^{-1} \mapsto cd^{-1} b^{-1} \mapsto cd^{-m-1} b^{-1} \mapsto cd^{-m-1} b^{-1}.$$

Hence,

$$Y = \begin{cases} b^{-1} & \text{in } \pi_1(U), \\ cd^{-m-1} & \text{in } \pi_1(V). \end{cases}$$

For the remaining compositions, we only exhibit the results in Tables 3, 4 and 5.

**Lemma 2.2.** *Let  $r \geq 1$ ,  $m, s \geq 0$ . The knot groups of twisted torus knots  $T(m(r+1) \pm 1, r+1; r, s)$ ,  $T(5m \pm 2, 5; 3, s)$ ,  $T(5m \pm 2, 5; 4, s)$  have the presentations as shown in Table 6.*

TABLE 3. The images  $G, R, Y$  under the compositions  $\rho_1, \rho_2$ .

	$\rho_1$		$\rho_2$	
	$\pi_1(U)$	$\pi_1(V)$	$\pi_1(U)$	$\pi_1(V)$
$G$	$a$	$c^s$	$a$	$c^s$
$R$	$b^2(ab)^{r-1}$	$d^{2m+1}(c^s d^m)^{r-1}$	$b(ab)^{r-1}b$	$d^m(c^s d^m)^{r-1}d^{m-1}$
$Y$	$b^{-1}$	$cd^{-m-1}$	$b$	$cd^{m-1}$

TABLE 4. The images  $G, R, Y$  under the compositions  $\rho_3, \rho_4$ .

	$\rho_3$		$\rho_4$	
	$\pi_1(U)$	$\pi_1(V)$	$\pi_1(U)$	$\pi_1(V)$
$G$	$a$	$c^s$	$a$	$c^s$
$R$	$b(bab)^2$	$d^m(d^{m+1}c^s d^m)^2$	$b(ab^2)^2$	$d^m(c^s d^{2m-1})^2$
$Y$	$b^{-1}$	$cd^{-m-1}$	$b$	$cd^{m-1}$

TABLE 5. The images  $G, R, Y$  under the compositions  $\rho_5, \rho_6$ .

	$\rho_5$		$\rho_6$	
	$\pi_1(U)$	$\pi_1(V)$	$\pi_1(U)$	$\pi_1(V)$
$G$	$a$	$c^s$	$a$	$c^s$
$R$	$b(ab)^2bab$	$d^{m+1}(c^s d^m)^2 d^{m+1}c^s d^m$	$(ab)^{-5}$	$d(c^s d^m)^{-2}d(c^s d^m)^{-3}$
$Y$	$(babab)^{-1}$	$cd^{-1}(c^s d^m)^{-2}d^{-m-1}$	$b(ab)^2$	$cd^{m-1}(c^s d^m)^2 d^{-1}$

*Proof.* We apply the van Kampen theorem to calculate the knot group.

Let  $\rho_1 = D_4^s \circ D_1^m \circ D_3^{-1} \circ D_2^{-(r-1)} \circ D_1$  as in Table 1. Then  $K = \rho_1(K_0)$  is  $T(m(r+1)+1, r+1; r, s)$ . The knot complement is the union of  $U - K$  and  $V - K$  along  $\Sigma - K$ . Note that  $\pi_1(U - K) \cong \pi_1(U)$ ,  $\pi_1(V - K) \cong \pi_1(V)$  and  $\pi_1(\Sigma - K)$  is generated by  $\{G, R, Y\}$ .

By Table 3, we have a presentation of the knot group as

$$\langle a, b, c, d \mid a = c^s, b^2(ab)^{r-1} = d^{2m+1}(c^s d^m)^{r-1}, b^{-1} = cd^{-m-1} \rangle.$$

If we eliminate  $a$  and  $b$  by using the first and last relations, then we have

$$\begin{aligned} & \langle c, d \mid (d^{m+1}c^{-1})^2(c^s d^{m+1}c^{-1})^{r-1} = d^{2m+1}(c^s d^m)^{r-1} \rangle \\ & = \langle c, d \mid c^{-1}d^{m+1}(c^{s-1}d^{m+1})^{r-1}c^{-1} = d^m(c^s d^m)^{r-1} \rangle \end{aligned}$$

TABLE 6. The presentations of knot groups.

	$T(m(r+1)+1, r+1; r, s)$
	$\langle c, d \mid d^{m+1}(c^{s-1}d^{m+1})^{r-1} = cd^m(c^s d^m)^{r-1}c \rangle$
	$T(m(r+1)-1, r+1; r, s)$
	$\langle c, d \mid d^{m-1}(c^{s+1}d^{m-1})^{r-1} = c^{-1}d^m(c^s d^m)^{r-1}c^{-1} \rangle$
Knot	$T(5m+2, 5; 3, s)$
and	$\langle c, d \mid c^{-1}d^{m+1}c^{s-1}d^{m+1}c^{-1}d^{m+1}c^{s-1}d^{m+1}c^{-1} = d^m c^s d^{2m+1} c^s d^m \rangle$
Presentation	$T(5m-2, 5; 3, s)$
of	$\langle c, d \mid cd^{m-1}c^{s+1}d^{m-1}cd^{m-1}c^{s+1}d^{m-1}c = d^m c^s d^{2m-1} c^s d^m \rangle$
knot	$T(5m+2, 5; 4, s)$
group	$\langle c, d \mid (d^{m+1}c^s d^m c^s d^{m+1}(cd^m c^s d^m c)^{-1})^2 = cd^m c^s d^m c^s \rangle$
	$T(5m-2, 5; 4, s)$
	$\langle c, d \mid (d^{m-1}c^s d^m c^s d^{m-1}(c^{-1}d^m c^s d^m c^{-1})^{-1})^2 = c^{-1}d^m c^s d^m c^s \rangle$

$$= \langle c, d \mid d^{m+1}(c^{s-1}d^{m+1})^{r-1} = cd^m(c^s d^m)^{r-1}c \rangle.$$

For other compositions  $\rho_i$  ( $2 \leq i \leq 4$ ), the process is very similar, so we omit it.

For  $\rho_5$ , we have a presentation

$$\begin{aligned} \langle a, b, c, d \mid a = c^s, b(ab)^2bab = d^{m+1}(c^s d^m)^2 d^{m+1} c^s d^m, \\ (babab)^{-1} = cd^{-1}(c^s d^m)^{-2} d^{-m-1} \rangle \end{aligned}$$

from Table 5. Eliminate the generator  $a$  by using the first relation. Then the second and third relations change to

$$(2.1) \quad bc^s bc^s b \cdot bc^s b = d^{m+1}(c^s d^m)^2 d^{m+1} c^s d^m,$$

$$(2.2) \quad bc^s bc^s b = d^{m+1}(c^s d^m)^2 dc^{-1},$$

respectively. By (2.2), (2.1) changes to

$$(2.3) \quad bc^s b = cd^m c^s d^m.$$

Furthermore by (2.3), (2.2) changes to

$$(2.4) \quad cd^m c^s d^m \cdot c^s b = d^{m+1}(c^s d^m)^2 dc^{-1}.$$

By (2.4), we have

$$\begin{aligned} (2.5) \quad b &= c^{-s} \cdot (cd^m c^s d^m)^{-1} \cdot d^{m+1}(c^s d^m)^2 dc^{-1} \\ &= c^{-s+1}(cd^m c^s d^m c)^{-1} \cdot d^{m+1}(c^s d^m)^2 dc^{-1} \\ &= c^{-s+1}(cd^m c^s d^m c)^{-1} \cdot d^{m+1}c^s d^m c^s d^{m+1}c^{-1}. \end{aligned}$$

Thus we can eliminate the generator  $b$  by (2.5), and (2.3) changes to

$$\begin{aligned} & c^{-s+1}(cd^m c^s d^m c)^{-1} \cdot d^{m+1} c^s d^m c^s d^{m+1} c^{-1} \cdot c^s. \\ & c^{-s+1}(cd^m c^s d^m c)^{-1} \cdot d^{m+1} c^s d^m c^s d^{m+1} c^{-1} = cd^m c^s d^m. \end{aligned}$$

Then

$$\begin{aligned} & d^{m+1} c^s d^m c^s d^{m+1} (cd^m c^s d^m c)^{-1} \cdot d^{m+1} c^s d^m c^s d^{m+1} c^{-1} (cd^m c^s d^m)^{-1} \\ & = (cd^m c^s d^m c) c^{s-1}. \end{aligned}$$

Hence we have the presentation

$$\langle c, d \mid (d^{m+1} c^s d^m c^s d^{m+1} (cd^m c^s d^m c)^{-1})^2 = cd^m c^s d^m c^s \rangle.$$

For  $\rho_6$ , the process is similar.  $\square$

### 3. Generalized torsion elements

The commutator of  $x$  and  $y$  is  $[x, y] = x^{-1}y^{-1}xy$ , and recall  $g^z = z^{-1}gz$ .

**Lemma 3.1.** *In a group  $G$ ,  $[x, yz] = [x, z][x, y]^z$ . Moreover, if  $w$  consists of  $x^{\pm 1}$  and  $y$ , then  $[x, w]$  is decomposed as a product of conjugates of  $[x, y]$ .*

*Proof.* The first claim follows from a direct calculation. The second follows from the first one and  $[x, x^{\pm 1}] = 1$ .  $\square$

Our argument uses the next lemma repeatedly.

**Lemma 3.2.** *Let  $G$  be a group generated by two elements  $x$  and  $y$ , and let  $w$  consist of  $x^{\pm 1}$  and  $y$ . Suppose that  $G$  is not abelian. If  $[x, w] = 1$ , then the commutator  $[x, y]$  is a generalized torsion element in  $G$ .*

*Proof.* By Lemma 3.1,  $[x, w]$  is decomposed as a product of conjugates of  $[x, y]$ . If  $[x, y] = 1$ , then  $G$  would be abelian. Hence  $[x, y] \neq 1$ , so it is a generalized torsion element.  $\square$

**Lemma 3.3.** *Each twisted torus knot listed in Theorems 1.1, 1.3 is a nontrivial fibered knot. Therefore, its knot group is not abelian.*

*Proof.* Each twisted torus knot is expressed as a closure of a positive braid, so it is fibered [23]. Its genus is calculated as below.  $\square$

If a knot  $K$  has a form of the closure of a positive braid, then its genus  $g(K)$  is given as

$$g(K) = \frac{1 - b + c}{2},$$

where  $b$  is the number of strands and  $c$  is the number of crossings. We record the genera for some classes for later use in Table 7.

**Proposition 3.4.** *Each twisted torus knot listed in Theorems 1.1, 1.3 admits a generalized torsion element in its knot group.*



TABLE 7. Genera for some families.

Knot	Genus
$T(5m + 2, 5; 3, 1)$	$10m + 5$
$T(5m + 2, 5; 4, 1)$	$10m + 8$
$T(5m - 2, 5; 4, 1)$	$10m$

*Proof.* For each knot, the knot group is not abelian by Lemma 3.3. To apply Lemma 3.2, it is necessary to find a presentation with two generators for the knot group and a word  $w$  which satisfies the condition of Lemma 3.2.

Let  $G_1$  be the knot group of  $T(r + 2, r + 1; r, s)$  given in Lemma 2.2 (set  $m = 1$  for Table 6). The presentation is changed to

$$\begin{aligned}
G_1 &= \langle c, d \mid d^2(c^{s-1}d^2)^{r-1} = cd(c^s d)^{r-1}c \rangle \\
&= \langle c, d, g \mid d^2(c^{s-1}d^2)^{r-1} = cd(c^s d)^{r-1}c, g = c^s d \rangle \\
&= \langle c, g \mid (c^{-s}g)^2(c^{s-1}(c^{-s}g)^2)^{r-1} = c^{1-s}g \cdot g^{r-1}c \rangle \\
&= \langle c, g \mid (c^{-s}g)^2(c^{-1}gc^{-s}g)^{r-1} = c^{1-s}g^r c \rangle \\
&= \langle c, g \mid (c^{-1}gc^{-s}g)^r c^{-1} = g^r \rangle.
\end{aligned}$$

Let  $w(g, c^{-1})$  be the left hand side of the last relation. Thus  $G_1$  has the presentation  $\langle c, g \mid w(g, c^{-1}) = g^r \rangle$ , so  $[g, w(g, c^{-1})] = 1$ . Since each power of the generator  $c$  is negative in the word  $w(g, c^{-1})$ ,  $[g, c^{-1}]$  gives a generalized torsion element in  $G_1$  by Lemma 3.2.

For the remaining cases, the process is the same. So, we only exhibit the transition of the presentation. The word  $w$  is set to be the left or right side of the last relation.

Let  $G_2$  be the knot group of  $T(m(r + 1) + 1, r + 1; r, 2)$ . Then,

$$\begin{aligned}
G_2 &= \langle c, d \mid d^{m+1}(cd^{m+1})^{r-1} = cd^m(c^2 d^m)^{r-1}c \rangle \\
&= \langle c, d, g \mid d^{m+1}(cd^{m+1})^{r-1} = cd^m(c^2 d^m)^{r-1}c, g = cd^{m+1} \rangle \\
&= \langle d, g \mid d^{m+1}g^{r-1} = gd^{-1}((gd^{-m-1})^2 d^m)^{r-1}gd^{-m-1} \rangle \\
&= \langle d, g \mid g^{r-1} = d^{-m-1}gd^{-1}(gd^{-m-1}gd^{-1})^{r-1}gd^{-m-1} \rangle.
\end{aligned}$$

Let  $G_3$  and  $G_4$  be the knot groups of  $T(2r + 1, r + 1; r, s)$  and  $T(m(r + 1) - 1, r + 1; r, 1)$ , respectively. Then,

$$\begin{aligned}
G_3 &= \langle c, d \mid d(c^{s+1}d)^{r-1} = c^{-1}d^2(c^s d^2)^{r-1}c^{-1} \rangle \\
&= \langle c, d, g \mid d(c^{s+1}d)^{r-1} = c^{-1}d^2(c^s d^2)^{r-1}c^{-1}, g = c^{s+1}d \rangle \\
&= \langle c, g \mid c^{-s-1}g \cdot g^{r-1} = c^{-1}(c^{-s-1}g)^2(c^s(c^{-s-1}g)^2)^{r-1}c^{-1} \rangle \\
&= \langle c, g \mid g^r = (c^{-1}gc^{-s-1}g)^r c^{-1} \rangle,
\end{aligned}$$

and

$$\begin{aligned}
G_4 &= \langle c, d \mid d^{m-1}(c^2 d^{m-1})^{r-1} = c^{-1} d^m (cd^m)^{r-1} c^{-1} \rangle \\
&= \langle c, d, g \mid d^{m-1}(c^2 d^{m-1})^{r-1} = c^{-1} d^m (cd^m)^{r-1} c^{-1}, g = cd^m \rangle \\
&= \langle d, g \mid d^{m-1}((gd^{-m})^2 d^{m-1})^{r-1} = d^m g^{-1} d^m g^{r-1} d^m g^{-1} \rangle \\
&= \langle d, g \mid d^{-m} g d^{-1} (g d^{-m} g d^{-1})^{r-1} g d^{-m} = g^{r-1} \rangle \\
&= \langle d, g \mid (g d^{-m} g d^{-1})^r g d^{-m} = g^r \rangle.
\end{aligned}$$

Let  $G_5$  and  $G_6$  be the knot groups of  $T(5m+2, 5; 3, 1)$  and  $T(5m-2, 5; 3, 1)$ , respectively. We have

$$\begin{aligned}
G_5 &= \langle c, d \mid c^{-1} d^{2m+2} c^{-1} d^{2m+2} c^{-1} = d^m c d^{2m+1} c d^m \rangle \\
&= \langle c, d \mid (c^{-1} d^{2m+2})^3 = d^m c d^{2m+1} c d^{3m+2} \rangle \\
&= \langle c, d, g \mid (c^{-1} d^{2m+2})^3 = d^m c d^{2m+1} c d^{3m+2}, g = c^{-1} d^{2m+2} \rangle \\
&= \langle d, g \mid g^3 = d^{3m+2} g^{-1} d^{4m+3} g^{-1} d^{3m+2} \rangle,
\end{aligned}$$

and

$$\begin{aligned}
G_6 &= \langle c, d \mid cd^{m-1} c^2 d^{m-1} cd^{m-1} c^2 d^{m-1} c = d^m c d^{2m-1} c d^m \rangle \\
&= \langle c, d \mid cd^{m-1} c^2 d^{m-1} cd^{m-1} c^2 d^{m-1} c = d(d^{m-1} c d^m)^2 \rangle \\
&= \langle c, d, g \mid cd^{m-1} c^2 d^{m-1} cd^{m-1} c^2 d^{m-1} c = d(d^{m-1} c d^m)^2, g = d^{m-1} c d^m \rangle \\
&= \langle d, g \mid d^{1-m} g d^{-m} g d^{-2m+1} g d^{-m} g d^{-m} g d^{-2m+1} g d^{-m} g d^{-m} = dg^2 \rangle \\
&= \langle d, g \mid d^{-m} g d^{-m} g d^{-2m+1} g d^{-m} g d^{-m} g d^{-2m+1} g d^{-m} g d^{-m} = g^2 \rangle.
\end{aligned}$$

Finally, let  $G_7$  and  $G_8$  be the knot groups of  $T(5m+2, 5; 4, 1)$  and  $T(5m-2, 5; 4, 1)$ , respectively. Then

$$\begin{aligned}
G_7 &= \langle c, d \mid (d^{m+1} c d^m c d^{m+1} (c d^m c d^m c)^{-1})^2 = c d^m c d^m c \rangle \\
&= \langle c, d \mid (d^{m+1} c d^m c d^{m+1} (c d^m c d^m c)^{-1})^2 d^m = (c d^m)^3 \rangle \\
&= \langle c, d, g \mid (d^{m+1} c d^m c d^{m+1} (c d^m c d^m c)^{-1})^2 d^m = (c d^m)^3, g = c d^m \rangle \\
&= \langle d, g \mid (d^{m+1} g^2 d^{m+1} g^{-3})^2 d^m = g^3 \rangle,
\end{aligned}$$

and

$$\begin{aligned}
G_8 &= \langle c, d \mid (d^{m-1} c d^m c d^{m-1} (c^{-1} d^m c d^m c^{-1})^{-1})^2 = c^{-1} d^m c d^m c \rangle \\
&= \langle c, d \mid c(d^{m-1} c d^m c d^{m-1} (c^{-1} d^m c d^m c^{-1})^{-1})^2 = d^m c d^m c \rangle \\
&= \langle c, d, g \mid c(d^{m-1} c d^m c d^{m-1} (c^{-1} d^m c d^m c^{-1})^{-1})^2 = (d^m c)^2, g = d^m c \rangle \\
&= \langle d, g \mid d^{-m} g (d^{-1} g^2 d^{-1} g d^{-m} g^{-1} d^{-m} g)^2 = g^2 \rangle.
\end{aligned}$$

□

#### 4. Hyperbolicity and proof of Theorem 1.1

In this section, we discuss the geometric types of our twisted torus knots, and complete the proof of Theorem 1.1. In general, most of twisted torus knots  $T(p, q; r, s)$  are hyperbolic if  $|s| \geq 2$  by Lee [12]. The problem happens when  $|s| = 1$ . We extract a few facts in a restricted form from Lee's and Paiva's works for our purpose.

**Lemma 4.1** ([13]). *Let  $p > q > r \geq 3$  and  $s \geq 1$ . Then  $T(p, q; r, s)$  is a torus knot if and only if  $(p, q; r, s) = (m(r+1)+1, r+1; r, 1)$  for some integer  $m \geq 1$ .*

**Lemma 4.2** ([12]). *Let  $p > q > r \geq 3$  and  $s \geq 2$ . Then  $T(p, q; r, s)$  is hyperbolic.*

**Lemma 4.3** ([19]).  *$T(kq + n, q; r, 1)$  is hyperbolic for  $k \geq 2$  and  $q > n \geq r$  with  $\gcd(n, q) = 1$ .*

**Proposition 4.4.** *Each twisted torus knot listed in Theorem 1.1 is hyperbolic.*

*Proof.* This follows from Lemmas 4.5, 4.10 and 4.11 below.  $\square$

We divide the argument into subsections. Recall that  $r \geq 3$ .

##### 4.1. Families (1) and (2) of Theorem 1.1

**Lemma 4.5.**  *$T(r+2, r+1; r, s)$  and  $T(m(r+1)+1, r+1; r, 2)$  are hyperbolic for  $s \geq 2, m \geq 2$ .*

*Proof.* This immediately follows from Lemma 4.2.  $\square$

##### 4.2. Families (3) and (4) of Theorem 1.1

We recall the notion of doubly primitive construction [1] and primitive/Seifert-fibered construction [6].

Let  $H$  be a standardly embedded genus two handlebody in  $S^3$ . A knot  $K$  lying on  $\partial H$  is said to be *primitive* with respect to  $H$  if it represents one of two generators of  $\pi_1(H)$ . Then a 2-handle addition to  $H$  along  $K$  yields a solid torus. This is also equivalent to the fact that the quotient group of  $\pi_1(H)$  by the normal closure of the element represented by  $K$  is isomorphic to the infinite cyclic group. In addition, if  $K$  is primitive with respect to  $H'$ , which is the outside of  $H$  in  $S^3$ , then  $K$  is said to be *doubly primitive*. We should remark here that if  $K$  is primitive with respect to  $H$ , then  $K$  has tunnel number one [8].

Also,  $K$  is said to be  $(m, n)$  *Seifert-fibered* with respect to  $H$  if the quotient group of  $\pi_1(H)$  by the normal closure of the element represented by  $K$  is isomorphic to a group with presentation  $\langle a, b \mid a^m = b^n \rangle$  for some integers  $m, n \geq 2$ . Then a 2-handle addition to  $H$  along  $K$  yields a Seifert fibered manifold over the disk with two exceptional fibers of indices  $m$  and  $n$ . If  $K$  is primitive with respect to one side and  $(m, n)$  Seifert-fibered with respect to the other side, then  $K$  is said to be *primitive/Seifert-fibered*.

For  $K$  on  $\partial H$ , the *surface slope* is the slope determined by one of the loops  $\partial N(K) \cap \partial H$ . The surface slope surgery on  $K$  is decomposed as the union of two 2-handle additions  $H \cup (2\text{-handle})$  and  $H' \cup (2\text{-handle})$  along  $K$ . Hence, if  $K$  is doubly primitive (resp. primitive/Seifert-fibered), then the surface slope surgery on  $K$  yields a lens space (resp. a Seifert fibered manifold over the 2-sphere with three exceptional fibers or a connected sum of two lens spaces).

For the family (3) of Theorem 1.1, we use the composition  $\rho_2$  (set  $m = 2$  and  $s = 1$ ) in Table 1 of Section 2.

**Lemma 4.6.** *On  $\Sigma$ ,  $\rho_2(K_0)$  with  $m = 2$ ,  $s = 1$  is doubly primitive. Its surface slope is  $3r^2 + 3r + 1$ .*

*Proof.* By tracing  $\rho_2(K_0)$  on  $\Sigma$ , we can read off that the curve represents  $(ab)^r b \in \pi_1(U)$  and  $(cd^2)^r d \in \pi_1(V)$ . Consider the quotient of  $\pi_1(U)$  by the normal closure of this element  $(ab)^r b$ . We have

$$\begin{aligned} \langle a, b \mid (ab)^r b = 1 \rangle &= \langle a, b \mid (ab)^r = b^{-1} \rangle \\ &= \langle a, b, e \mid (ab)^r = b^{-1}, e = ab \rangle \\ &= \langle a, b, e \mid e^r = b^{-1}, e = ab \rangle \\ &= \langle a, e \mid e = ae^{-r} \rangle \\ &= \langle e \rangle. \end{aligned}$$

This shows that  $\rho_2(K_0)$  is primitive with respect to  $U$ .

Similarly, consider the quotient of  $\pi_1(V)$  by the above element  $(cd^2)^r d$ . Then,

$$\begin{aligned} \langle c, d \mid (cd^2)^r d = 1 \rangle &= \langle c, d \mid (cd^2)^r = d^{-1} \rangle \\ &= \langle c, d, f \mid (cd^2)^r = d^{-1}, f = cd^2 \rangle \\ &= \langle c, d, f \mid f^r = d^{-1}, f = cd^2 \rangle \\ &= \langle c, f \mid f = cf^{-2r} \rangle \\ &= \langle c, f \mid f^{2r+1} = c \rangle \\ &= \langle f \rangle. \end{aligned}$$

Hence  $\rho_2(K_0)$  is also primitive with respect to  $V$ .

It is straightforward to count the surface slope on  $\Sigma$ . The  $(2r+1, r+1)$ -torus knot part contributes  $(2r+1)(r+1)$  to the surface slope, and an extra full twist on  $r$  adjacent strands contributes  $r^2$  to it. These give  $3r^2 + 3r + 1$ .  $\square$

Let  $\rho'_2 = D_5^{-1} \circ \rho_2$ , where  $D_5$  is the Dehn twist along the curve  $C_5$  defined in Section 2. The point is that this extra Dehn twist does not change the knot type, but the position on  $\Sigma$ , and hence, the surface slope changes.

**Lemma 4.7.** *Suppose  $m = 2$  and  $s = 1$ . On  $\Sigma$ ,  $\rho'_2(K_0)$  is primitive with respect to  $U$ , but  $(r-1, 2)$  Seifert-fibered with respect to  $V$ . Its surface slope is  $3r^2 + 3r$ .*

*Proof.* The argument is similar to the proof of Lemma 4.6. We read off  $\rho'_2(K_0)$  on  $\Sigma$ . It represents  $(ab)^r b \in \pi_1(U)$  as the same as  $\rho_2(K_0)$ , but  $(cd^2)^{r-1} cdcd \in \pi_1(V)$ . Hence, this is primitive with respect to  $U$  as before. Also,

$$\begin{aligned} \langle c, d \mid (cd^2)^{r-1} cdcd = 1 \rangle &= \langle c, d, f \mid (cd^2)^{r-1} (cd)^2 = 1, f = cd^2 \rangle \\ &= \langle d, f \mid f^{r-1} (fd^{-1})^2 = 1 \rangle \\ &= \langle d, f, g \mid f^{r-1} (fd^{-1})^2 = 1, g = fd^{-1} \rangle \\ &= \langle f, g \mid f^{r-1} g^2 = 1 \rangle. \end{aligned}$$

This shows that  $\rho'_2(K_0)$  is  $(r - 1, 2)$  Seifert-fibered with respect to  $V$ .

As an effect of  $D_5^{-1}$ , the surface slope decreases by one from that of  $\rho_2(K_0)$ . □

**Lemma 4.8.** *Suppose  $m = 2$  and  $s = 1$ . The surface slope surgery on  $\rho'_2(K_0)$  yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(r - 1, 2, r + 2)$ .*

*Proof.* The only information that we need is the index of the third exceptional fiber after the surface slope surgery. We follow the procedure of [6, 4.2 and 4.3]. Take a non-separating curve  $\ell$  on  $\Sigma$  disjoint from  $\rho'_2(K_0)$  so that  $\ell$  represents  $g^2 (= (cd)^2) \in \pi_1(V)$  as shown in Figure 4.

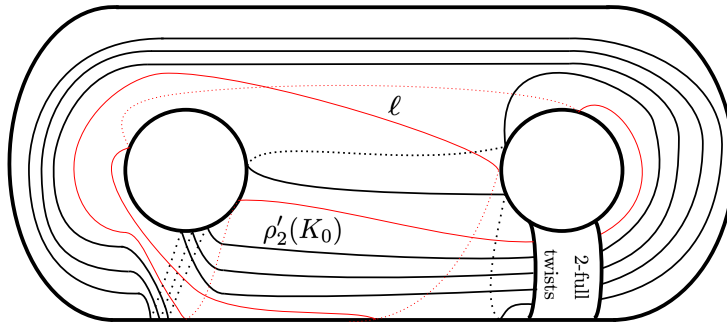


FIGURE 4. The loop  $\ell$  disjoint from  $\rho'_2(K_0)$  ( $r = 3$ ) on  $\Sigma$ .

This implies that  $\ell$  becomes a regular fiber in the Seifert fibration of  $V \cup h$ , where  $h$  is a 2-handle attached to  $V$  along  $\rho'_2(K_0)$ . (Recall that any regular fiber in  $V \cup h$  represents  $g^2$  as shown in the proof of Lemma 4.7.)

Also,  $\ell = a^2 b \in \pi_1(U)$ . Let  $U \cup h'$  be the 2-handle addition to  $U$  along  $\rho_2(K_0)$ . Since  $\pi_1(U \cup h')$  is generated by  $e (= ab)$ ,  $\ell = e^{r+2} \in \pi_1(U \cup h')$ .

Recall that the surface slope surgery on  $\rho'_2(K_0)$  is decomposed as the union of  $V \cup h$  and  $U \cup h'$ . Here,  $V \cup h$  is a Seifert fibered manifold over the disk with two exceptional fibers of indices  $r - 1$  and  $2$ , and  $U \cup h'$  is a solid torus by Lemma 4.7. As shown above, a regular fiber  $\ell$  of  $V \cup h$  runs  $r + 2$  times

along the core of the solid torus  $U \cup h'$ . This means that the Seifert fibration of  $V \cup h$  extends to  $U \cup h'$  so that the core of  $U \cup h'$  becomes an exceptional fiber of index  $r + 2$ . Hence the resulting manifold of the surgery slope surgery is a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(r - 1, 2, r + 2)$ .  $\square$

*Remark 4.9.* In Lemma 4.8, when  $r = 3$ , the resulting manifold is a prism manifold, which also admits a Seifert fibration over the projective plane with at most one exceptional fiber.

**Lemma 4.10.**  $T(2r + 1, r + 1; r, s)$  is hyperbolic for  $s \geq 1$ .

*Proof.* If  $s \geq 2$ , then this is hyperbolic by Lemma 4.2. Hence we show that  $K = T(2r + 1, r + 1; r, 1)$  is hyperbolic. By Lemma 4.6,  $K$  is doubly primitive on  $\Sigma$ . Then the surface slope surgery, which is  $(3r^2 + 3r + 1)$ -surgery, yields a lens space.

By Lemma 4.1,  $K$  is not a torus knot. So, assume that  $K$  is a satellite for a contradiction. By [2, 25], the only satellite knot that admits a cyclic surgery is a  $(2uv + \varepsilon, 2)$ -cable of a  $(u, v)$ -torus knot  $T(u, v)$ , and the surgery slope is  $4uv + \varepsilon$ , where  $\varepsilon = \pm 1$ .

On the other hand, Lemma 4.8 claims that  $(3r^2 + 3r)$ -surgery on  $K$  yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers. Thus an adjacent slope to the cyclic surgery slope on  $K$  yields such an irreducible Seifert fibered manifold. This is impossible. For, if  $K$  is a  $(2uv + 1, 2)$ -cable of  $T(u, v)$ , then the only cyclic surgery slope is  $4uv + 1$  as mentioned. We examine its adjacent slopes  $4uv + 2$  and  $4uv$ . Then,  $(4uv + 2)$ -surgery yields a reducible manifold with lens space summand  $L(2, 1)$ , and  $4uv$ -surgery yields a graph manifold which is the union of two Seifert fibered manifolds over the disk with two exceptional fibers [7]. (The resulting manifold of  $4uv$ -surgery also admits a Seifert fibration over the projective plane with two exceptional fibers of indices  $u$  and  $v$  [14, Example 8.2].) The argument is similar for a  $(2uv - 1, 2)$ -cable of  $T(u, v)$ .  $\square$

**Lemma 4.11.**  $T(m(r + 1) - 1, r + 1; r, 1)$  is hyperbolic for  $m \geq 3$ .

*Proof.* Since  $T(m(r + 1) - 1, r + 1; r, 1) = T((m - 1)(r + 1) + r, r + 1; r, 1)$ , the conclusion immediately follows from Lemma 4.3.  $\square$

*Proof of Theorem 1.1.* This follows from Propositions 3.4 and 4.4.  $\square$

## 5. Proof of Theorem 1.3

**Proposition 5.1.** Each twisted torus knot listed in Theorem 1.3 is hyperbolic.

*Proof.* This follows from Lemmas 5.5, 5.6, 5.8 and 5.9 below.  $\square$

*Remark 5.2.* For the family (2) of Theorem 1.3,  $T(5m - 2, 5; 3, 1) = T(5(m - 1) + 3, 5; 3, 1)$ , so Paiva's result [19] (Lemma 4.3) guarantees the hyperbolicity

if  $m \geq 3$ . However, we also present its proof including the case  $m = 2$ , because our argument is different from [19].

### 5.1. Families (1) and (2) of Theorem 1.3

For two families  $T(5m + 2, 5; 3, 1)$  ( $m \geq 1$ ) and  $T(5m - 2, 5; 3, 1)$  ( $m \geq 2$ ), we use the compositions  $\rho_3$  and  $\rho_4$  (with  $s = 1$ ) in Table 1 of Section 2.

**Lemma 5.3.** *On  $\Sigma$ , we have the following.*

- (1)  $\rho_3(K_0)$  is primitive with respect to  $U$ , and  $(3, m + 1)$  Seifert-fibered with respect to  $V$ . Its surface slope is  $25m + 19$ .
- (2)  $\rho_4(K_0)$  is doubly primitive if  $m = 2$ . If  $m > 2$ , then it is primitive with respect to  $U$ , and  $(3, m - 1)$  Seifert-fibered with respect to  $V$ . Its surface slope is  $25m - 1$ .

*Proof.* The argument is the same as the proofs of Lemmas 4.6 and 4.7.

(1)  $\rho_3(K_0)$  represents  $abab^2ab^2 \in \pi_1(U)$  and  $cd^mcd^{2m+1}cd^{2m+1} \in \pi_1(V)$ . Then,

$$\begin{aligned} \langle a, b \mid abab^2ab^2 = 1 \rangle &= \langle a, b \mid (ab^2)^3 = b \rangle \\ &= \langle a, b, e \mid (ab^2)^3 = b, e = ab^2 \rangle \\ &= \langle a, b, e \mid e^3 = b, e = ab^2 \rangle \\ &= \langle a, e \mid e = ae^6 \rangle \\ &= \langle e \rangle, \end{aligned}$$

$$\begin{aligned} \langle c, d \mid cd^mcd^{2m+1}cd^{2m+1} = 1 \rangle &= \langle c, d \mid (cd^{2m+1})^3 = d^{m+1} \rangle \\ &= \langle c, d, f \mid (cd^{2m+1})^3 = d^{m+1}, f = cd^{2m+1} \rangle \\ &= \langle c, d, f \mid f^3 = d^{m+1}, f = cd^{2m+1} \rangle \\ &= \langle d, f \mid f^3 = d^{m+1} \rangle. \end{aligned}$$

Hence  $\rho_3(K_0)$  is primitive with respect to  $U$ , and  $(3, m + 1)$  Seifert-fibered with respect to  $V$ .

(2) Consider  $\rho_4(K_0)$ . It represents  $abab^2ab^2 \in \pi_1(U)$  and  $cd^mcd^{2m-1}cd^{2m-1} \in \pi_1(V)$ . Thus it is primitive with respect to  $U$  as in (1), and  $(3, m - 1)$  Seifert-fibered with respect to  $V$  when  $m \geq 3$  as follows:

$$\begin{aligned} \langle c, d \mid cd^mcd^{2m-1}cd^{2m-1} = 1 \rangle &= \langle c, d \mid (cd^{2m-1})^3 = d^{m-1} \rangle \\ &= \langle c, d, f \mid (cd^{2m-1})^3 = d^{m-1}, f = cd^{2m-1} \rangle \\ &= \langle c, d, f \mid f^3 = d^{m-1}, f = cd^{2m-1} \rangle \\ &= \langle d, f \mid f^3 = d^{m-1} \rangle. \end{aligned}$$

If  $m = 2$ , this is isomorphic to the infinite cyclic group  $\langle f \rangle$ , so  $\rho_4(K_0)$  is primitive with respect to  $V$ .

For these knots, it is straightforward to calculate their surface slopes on  $\Sigma$ .  $\square$

**Lemma 5.4.** *We have the following.*

- (1) *The surface slope surgery on  $\rho_3(K_0)$  yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(2, 3, m + 1)$ .*
- (2) *The surface slope surgery on  $\rho_4(K_0)$  yields a lens space if  $m = 2$ , or a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(8, 3, m - 1)$  if  $m \geq 3$ .*

*Proof.* The process is the same as the proof of Lemma 4.8.

(1) We use the same notation there. Take a non-separating curve  $\ell$  on  $\Sigma$  disjoint from  $\rho_3(K_0)$  so that  $\ell$  represents  $d^{m+1} \in \pi_1(V)$  as shown in Figure 5. It represents a regular fiber in  $V \cup h$ .

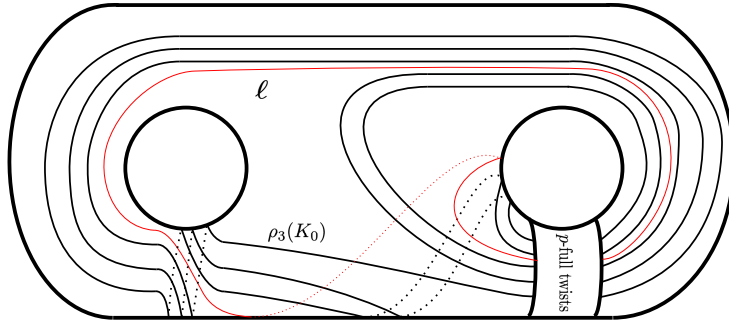


FIGURE 5. The loop  $\ell$  disjoint from  $\rho_3(K_0)$  on  $\Sigma$ .

Here,  $\ell = ab \in \pi_1(U)$ . Then,  $\ell = e^{-2} \in \pi_1(U \cup h')$ . Hence the resulting manifold of the surface slope surgery is a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(2, 3, m + 1)$ .

(2) If  $m = 2$ , then  $\rho_4(K_0)$  is doubly primitive by Lemma 5.3(2), so the surface slope surgery yields a lens space. For  $m \geq 3$ , the argument is similar again. Take a non-separating curve  $\ell$  on  $\Sigma$  disjoint from  $\rho_4(K_0)$  as shown in Figure 6. It represents  $d^{m-1} \in \pi_1(V)$ , which becomes a regular fiber of  $V \cup h$ .

Since  $\ell = a^{-1}b \in \pi_1(U)$ ,  $\ell = e^8 \in \pi_1(U \cup h')$ . This shows that the core of the solid torus  $U \cup h'$  becomes an exceptional fiber of index 8 after the surface slope surgery.  $\square$

**Lemma 5.5.**  *$T(5m + 2, 5; 3, 1)$  is hyperbolic for  $m \geq 1$ .*

*Proof.* Let  $K = T(5m + 2, 5; 3, 1)$ . This is not a torus knot by Lemma 4.1. Assume that  $K$  is a satellite for a contradiction.

First, the primitivity (Lemma 5.3(1)) implies that  $K$  has tunnel number one (see also [10]). In particular,  $K$  is prime [18], so the wrapping number in the pattern is at least two. However, the bridge number of  $K$  is at most 5. Hence, the companion is a 2-bridge knot, and the pattern has wrapping number two by [21] (see also [22]).



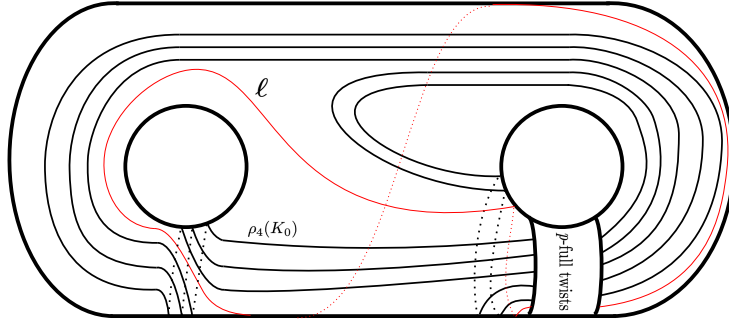


FIGURE 6. The loop  $\ell$  disjoint from  $\rho_4(K_0)$  on  $\Sigma$ .

By Lemma 5.4(1), the surface slope surgery on  $\rho_3(K_0)$  yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers. Thus  $K$  is a  $(4n \pm 1, 2)$ -cable of  $T(2, n)$  by [15, Proposition 2.1].

For a knot of this class, the cabling slope is  $8n \pm 2$ , so  $8n \pm 1, 8n \pm 3$  surgeries are the only integral surgeries that yield a Seifert fibered manifold over the 2-sphere with three exceptional fibers by [7]. Also, in the resulting Seifert fibered manifold, the two exceptional fibers of indices 2 and  $n$  are inherited from the exterior of the companion torus knot  $T(2, n)$ , and the third exceptional fiber corresponds to the core of the solid torus which arose from the pattern solid torus after the surgery. In fact, the index of the third exceptional fiber is given by the distance between the cabling slope  $2n$  of  $T(2, n)$  and  $(8n \pm 1)/4$  or  $(8n \pm 3)/4$  by [7]. Hence it must be 3.

By Lemma 5.4(1) again, the surface slope surgery on  $\rho_3(K_0)$  yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(2, 3, m + 1)$ . Since the Seifert fibration (over the 2-sphere) is unique for this type of Seifert fibered manifolds (see [9]), the two sets of indices  $(2, 3, m + 1)$  and  $(2, 3, n)$  coincide, implying  $n = m + 1$ .

Finally, a  $(4n \pm 1, 2)$ -cable of  $T(2, n)$  has genus  $3n - 1$  or  $3n - 2$  by [20] (see also [4]). On the other hand,  $K$  has genus  $10m + 5 = 10(n - 1) + 5$  by Table 7. Thus their genera do not coincide, a contradiction.  $\square$

**Lemma 5.6.**  $T(5m - 2, 5; 3, 1)$  is hyperbolic for  $m \geq 2$ .

*Proof.* Let  $K = T(5m - 2, 5; 3, 1)$ . This is not a torus knot by Lemma 4.1. We assume that  $K$  is a satellite for a contradiction.

As in the proof of Lemma 5.5,  $K$  is a  $(4n \pm 1, 2)$ -cable of  $T(2, n)$ .

If  $m = 2$ , then the surface slope surgery (for  $\rho_4(K_0)$ ) yields a lens space by Lemma 5.4(2). This surface slope is 49 by Lemma 5.3(2). By [2, 25], the cyclic surgery slope for  $K$  is  $8n \pm 1$ . This is obviously impossible, because  $n$  is odd.

Suppose  $m \geq 3$ . Again, the surface slope surgery yields a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices  $(8, 3, m - 1)$

by Lemma 5.4(2). The rest of argument goes simpler than the proof of Lemma 5.5. Two sets of indices  $(8, 3, m-1)$  and  $(2, 3, n)$  coincide, implying  $m = 3$  and  $n = 8$ . However, this is impossible again, because  $n$  is odd.  $\square$

### 5.2. Families (3) and (4) of Theorem 1.3

For the compositions  $\rho_5$  and  $\rho_6$  with  $s = 1$  in Table 1,  $\rho_5(K_0) = T(5m + 2, 5; 4, 1)$  and  $\rho_6(K_0) = T(5m - 2, 5; 4, 1)$ .

We modify these compositions as  $\rho'_5 = D_5^{-1} \circ \rho_5$  and  $\rho'_6 = D_5^{-1} \circ \rho_6$ .

**Lemma 5.7.** *On  $\Sigma$ , we have the following.*

- (1)  $\rho'_5(K_0)$  is doubly primitive, and its surface slope is  $25m + 25$ .
- (2)  $\rho'_6(K_0)$  is doubly primitive, and its surface slope is  $25m + 5$ .

Hence, the surface slope surgery yields a lens space for each knot.

*Proof.* The argument is similar to that of Lemma 4.6.

- (1) We see that  $\rho'_5(K_0)$  represents  $ababab^2ab \in \pi_1(U)$  and

$$cd^{m+1}cd^mcd^mcd^mcd^m \in \pi_1(V).$$

Then,

$$\begin{aligned} \langle a, b \mid ababab^2ab = 1 \rangle &= \langle a, b \mid (ab^2)^4 = b^{-1} \rangle \\ &= \langle a, b, e \mid (ab^2)^4 = b^{-1}, e = ab^2 \rangle \\ &= \langle a, b, e \mid e^4 = b^{-1}, e = ab^2 \rangle \\ &= \langle a, e \mid e = ae^{-8} \rangle \\ &= \langle e \rangle, \end{aligned}$$

$$\begin{aligned} \langle c, d \mid cd^{m+1}cd^mcd^mcd^mcd^m = 1 \rangle &= \langle c, d \mid (cd^m)^5 = d^{-1} \rangle \\ &= \langle c, d, f \mid (cd^m)^5 = d^{-1}, f = cd^m \rangle \\ &= \langle c, d, f \mid f^5 = d^{-1}, f = cd^m \rangle \\ &= \langle d, f \mid f^5 = d^{-1} \rangle \\ &= \langle f \rangle. \end{aligned}$$

Thus  $\rho'_5(K_0)$  is doubly primitive.

- (2) Similarly,  $\rho'_6(K_0)$  represents  $abab^2abab \in \pi_1(U)$  and

$$cd^mcd^{m-1}cd^{m-1}cd^mcd^{m-1} \in \pi_1(V).$$

It is primitive with respect to  $U$  as above, and

$$\begin{aligned} \langle c, d \mid cd^mcd^{m-1}cd^{m-1}cd^mcd^{m-1} = 1 \rangle &= \langle c, d, f \mid cd^mcd^{m-1}cd^{m-1}cd^mcd^{m-1} = 1, f = cd^{m-1} \rangle \\ &= \langle c, d, f \mid fdf^3df = 1, f = cd^{m-1} \rangle \\ &= \langle d, f \mid f^2df^3d = 1 \rangle \end{aligned}$$

$$\begin{aligned} &= \langle d, f, g \mid f^2df^3d = 1, g = f^2d \rangle \\ &= \langle f, g \mid gfg = 1 \rangle \\ &= \langle g \rangle. \end{aligned}$$

Thus  $\rho'_6(K_0)$  is doubly primitive.

For both knots, the surface slope is calculated easily. □

**Lemma 5.8.**  *$T(5m + 2, 5; 4, 1)$  is hyperbolic for  $m \geq 1$ .*

*Proof.* Let  $K = T(5m + 2, 5; 4, 1)$ . By Lemma 4.1,  $K$  is not a torus knot. Since it is doubly primitive,  $K$  has tunnel number one, so prime. It suffices to show that  $K$  is not a satellite.

If  $K$  is a satellite, then it is a  $(4n + \varepsilon, 2)$ -cable of a  $(2, n)$ -torus knot with  $\varepsilon = \pm 1$ , because it admits a cyclic surgery by Lemma 5.7(1). In particular, the cyclic surgery slope is  $8n + \varepsilon$ .

However, the surface slope for  $\rho'_5(K_0)$  is  $25m + 25$  by Lemma 5.7(1). Thus  $8n + 1 = 25m + 25$  implies  $(m, n) = (8k, 25k + 3)$ , and  $8n - 1 = 25m + 25$  implies  $(8k - 2, 25k - 3)$  for  $k \geq 1$ .

Recall that  $K$  has genus  $10m + 8$  (Table 7). Also, a  $(4n + 1, 2)$ -cable (resp.  $(4n - 1, 2)$ -cable) of a  $(2, n)$ -torus knot has genus  $3n - 1$  (resp.  $3n - 2$ ). If  $(m, n) = (8k, 25k + 3)$ , then we have  $80k + 8 = 3(25k + 3) - 1$ , a contradiction. Otherwise,  $(m, n) = (8k - 2, 25k - 3)$  implies  $10(8k - 2) + 8 = 3(25k - 3) - 2$ , a contradiction again. □

**Lemma 5.9.**  *$T(5m - 2, 5; 4, 1)$  is hyperbolic for  $m \geq 2$ .*

*Proof.* Let  $K = T(5m - 2, 5; 4, 1)$ . By Lemma 4.1,  $K$  is not a torus knot. Again,  $K$  has tunnel number one, so it is prime. Assume that  $K$  is a satellite. Then it is a  $(4n + \varepsilon, 2)$ -cable of a  $(2, n)$ -torus knot with  $\varepsilon = \pm 1$ , because it admits a cyclic surgery by Lemma 5.7(2). In particular, the cyclic surgery slope is  $8n + \varepsilon$ .

The surface slope of  $\rho'_6(K_0)$  is  $25m + 5$  by Lemma 5.7(2). Then  $8n + 1 = 25m + 5$  implies  $(m, n) = (8k - 4, 25k - 12)$  and  $8n - 1 = 25m + 5$  implies  $(8k - 6, 25k - 18)$  for  $k \geq 1$ .

Recall that  $K$  has genus  $10m$  (Table 7). As in the proof of Lemma 5.8, the case where  $(m, n) = (8k - 4, 25k - 12)$  leads to a contradiction, because  $10(8k - 4) = 3(25k - 12) - 1$  does not hold. Finally, the case where  $(8k - 6, 25k - 18)$  is impossible, because  $10(8k - 6) = 3(25k - 18) - 2$  does not hold. □

*Proof of Theorem 1.3.* This follows from Propositions 3.4 and 5.1. □

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