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# REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 11 AND 22

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Abstract. In this article, we find bases for the spaces of modular forms $M_2(\Gamma_0(88), (\frac{d}{\cdot}))$ for $d = 1, 8, 44$ and 88. We then derive formulas for the number of representations of a positive integer by the diagonal quaternary quadratic forms with coefficients $1, 2, 11$ and 22.

## 1. Introduction

The determination of positive integers represented by a given quadratic form is an old and interesting problem that has been studied extensively since Gauss. It is also interesting to determine the number of different ways to represent a positive integer $n$ by this quadratic form. The problem of finding an explicit exact formula for the number of representations of $n$ by a quadratic form is a classical problem in number theory. In this paper, we focus on the formulae for the representation numbers of diagonal quaternary quadratic forms with coefficients 1, 2, 11 and 22.

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{C}$ denote the set of positive integers, integers, rational numbers and complex numbers respectively and let $\mathbb{N}_{\mathbf{0}} = \mathbb{N} \cup \{\mathbf{0}\}$. For $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and $n \in \mathbb{N}_{\mathbf{0}}$ let $N(a_1, a_2, a_3, a_4; n)$ denote the number of representations of $n$ by the diagonal quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$. Clearly $N(a_1, a_2, a_3, a_4; 0) = 1$ and for $n \notin \mathbb{N}_{\mathbf{0}}$ we set $N(a_1, a_2, a_3, a_4; n) = 0$. Without loss of generality we may suppose that

$$a_1 \leq a_2 \leq a_3 \leq a_4 \text{ and } \gcd(a_1, a_2, a_3, a_4) = 1. \tag{1.1}$$

The formula for the representation numbers of a positive integer $n$ by the sum of four squares is a classical result due to Jacobi [9]. It simply asserts that the number of representations is 8 times the sum of the positive divisors of $n$ that are not multiples of 4. This can be given using the sum of divisors function as:

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4).$$

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See [3] and [21, Theorem 9.5] for the proofs. Between 1859 and 1866 Liouville made about 90 conjectures on formulas for $N(a_1, a_2, a_3, a_4; n)$ in a series of papers, many of which were later proven. The following formula for $N(1, 1, 2, 2; n)$ is one of them:

$$N(1, 1, 2, 2; n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8).$$

See [3], [14] and [21, Theorem 18.1] for the details. The formulas for $N(1, 1, 1, 2; n)$ and $N(1, 2, 2, 2; n)$ are the other two formulas studied by Liouville in this series of articles, see [15] and [21, Theorem 18.2]. The proofs of these formulae can be found in [4, 6]. The methods used in the articles to derive the representation number formulas differ from study to study. For additional resources where, formulas for some quaternary, sextenary or octonary quadratic forms are derived, one can see [1, 2, 5–7, 12].

Let $k, N \in \mathbb{N}$ and $\chi$ be a Dirichlet character of modulus dividing $N$. We use $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight $k$ with character $\chi$ for the Hecke congruence subgroup $\Gamma_0(N)$. The space $M_k(\Gamma_0(N), \chi)$ decomposes as follows:

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi), \tag{1.2}$$

where $E_k(\Gamma_0(N), \chi)$ denotes the space of Eisenstein forms and $S_k(\Gamma_0(N), \chi)$ the space of cusp forms (see for example [18, p. 83]).

For $n \in \mathbb{N}$ we define $\sigma_{k,\chi,\psi}$ by

$$\sigma_{k,\chi,\psi}(n) := \sum_{1 \le d | n} \psi(d)\chi(n/d)d^k. \tag{1.3}$$

If $n \notin \mathbb{N}$ we set $\sigma_{k,\chi,\psi}(n) = 0$. If $k = 1$, then we simply write $\sigma_{\chi,\psi}(n)$ and if $\chi$ and $\psi$ are trivial characters so that $\chi(n) = \psi(n) = 1$ for all $n \in \mathbb{Z}$, then we have the sum of divisors function $\sigma(n)$. For each quadratic discriminant $t$, we put $\chi_t(n) = \left(\frac{t}{n}\right)$, where $\left(\frac{t}{n}\right)$ is the Kronecker symbol. Let $\chi$ and $\psi$ be primitive Dirichlet characters with conductors $L, R \in \mathbb{N}$, respectively, such that $\chi(-1)\psi(-1) = (-1)^k$. The weight 2 Eisenstein series are defined by

$$E_{2,\chi,\psi} = c_0 + \sum_{n \ge 1} \sigma_{\chi,\psi}(n)q^n, \text{ where } c_0 = \begin{cases} -\frac{B_{2,\chi}}{4} & \text{if } L = 1, \\ 0 & \text{if } L > 1, \end{cases} \tag{1.4}$$

where the generalized Bernoulli numbers $B_{2,\chi}$ attached to $\chi$ are defined by the following formula:

$$B_{2,\chi} = 2[x^2] \sum_{a=1}^{L} \frac{\chi(a)xe^{ax}}{e^{Lx} - 1}. \tag{1.5}$$

We refer to the [18, p. 88] for details. In our case we get the following Bernoulli numbers using (1.5)

$$B_{2,\chi_8} = \frac{-1}{2}, \ B_{2,\chi_{44}} = -7, \ B_{2,\chi_{88}} = -23. \tag{1.6}$$

We give a list of Eisenstein series in Section 2 and use them in Section 3 to build a bases for $E_2(\Gamma_0(88),\chi)$.

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{C}=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$ by the product formula

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\ \text{where } q=q(z):=e^{2\pi iz} \text{ with } z\in\mathbb{C}. \tag{1.7}$$

By an eta quotient we mean any finite product of the form

$$f(z)=\prod_{\delta\in S}\eta^{r_\delta}(\delta z), \tag{1.8}$$

where $S$ is a finite subset of $\mathbb{N}$, and $r_\delta\in\mathbb{Z}\backslash\{0\}$. Since the product is finite, the least common multiple $N=\mathrm{lcm}\{\delta\}_{\delta\in S}$ exist. We may write the eta quotient (1.8) in the form

$$f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z). \tag{1.9}$$

To see the connection between $M_2(\Gamma_0(88),\left(\frac{d}{\cdot}\right))$ and $N(a_1,a_2,a_3,a_4;n)$ we have to recall the generating function for the sum of squares. The Ramanujan's theta function $\varphi(q)$ defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2},$$

has the following representation [8, p. 10] in terms of an eta quotient

$$\varphi(q)=\frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}, \tag{1.10}$$

and it plays a crucial role in getting formula for $N(a_1,a_2,a_3,a_4;n)$ using the theory of modular forms. We have

$$\sum_{n=0}^{\infty}N(a_1,a_2,a_3,a_4;n)q^n=\prod_{i=1}^{4}\varphi(q^{a_i}). \tag{1.11}$$

So, using (1.10) and (1.11) we can rewrite the generating function in terms of eta quotients as follows:

$$\sum_{n=0}^{\infty}N(a_1,a_2,a_3,a_4;n)q^n=\prod_{i=1}^{4}\frac{\eta^5(2a_iz)}{\eta^2(a_iz)\eta^2(4a_iz)}.$$

It will be convenient to express eta-quotients using the following shorthand notation (see [11, p. 31])

$$[\delta_1^{r_{\delta_1}},\delta_2^{r_{\delta_2}},\dots,\delta_{d-1}^{r_{\delta_{d-1}}},\delta_d^{r_{\delta_d}}]=\prod_{i=1}^{d}\eta^{r_{\delta_i}}(\delta_iz),$$

where $\delta_1=1$, $\delta_2,\dots,\delta_{d-1},\delta_d=N$ are the divisors of $N$, in an ascending order.

In this article, we find basis for each of the spaces of modular forms of weight 2 on $\Gamma_0(88)$ with character $\chi_d = \left(\frac{d}{\cdot}\right)$ for $d = 1,\ 8,\ 44,\ 88$. We then use these bases to obtain formulas for $N(a_1, a_2, a_3, a_4; n)$ with $a_i$'s are in $\{1, 2, 11, 22\}$. The total number of diagonal quaternary quadratic forms with these coefficients is 35, however, this can be reduced to 26 under the simplification conditions (1.1). Out of these 26 formulae, 4 of them appeared in the literature and they all agree with our results. We also investigate the elliptic curves with conductor $N = 88$ over $\mathbb{Q}$. We find one curve and express the modular form attached to this curve as a linear combination of the cusp forms in $S_2(\Gamma_0(88), \chi_1)$.

This work is organized as follows. In Section 2, we give some preliminary results. In Section 3, we establish the bases of the spaces of modular forms. In Section 4, we give a lemma containing the required theta function identities for the proof of the main theorem. In Section 5, we give the statement and the proof of the main theorem and in Section 6 we give a concluding remark.

## 2. Preliminary results

We use the following theorem which is referred to as Ligozat's Criteria, see [10, Theorem 5.7, p. 99], [11, Corollary 2.3, p. 37] to check if a particular eta quotient is in $M_k(\Gamma_0(N), \chi)$. By using this theorem and dimension formulas, we construct the spaces of cusp forms and then modular forms for $\Gamma_0(N)$. M. Newman [16,17] used the eta functions to construct modular forms for $\Gamma_0(N)$. The order of vanishing of an eta function at the cusps of $\Gamma_0(N)$ was determined by G. Ligozat [13].

**Theorem 1.** *Let $N$ be a positive integer and let $f(z) = \prod_{1\le\delta|N} \eta^{r_\delta}(\delta z)$ be an eta quotient. Let $s = \prod_{1\le\delta|N} \eta^{|r_\delta|}$. Suppose that the following conditions hold*:

(i) $\sum_{1\le\delta|N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,
(ii) $\sum_{1\le\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,
(iii) *the weight $k = \frac{1}{2}\sum_{1\le\delta|N} r_\delta$ is an even integer,*
(iv) *for each $d \mid N$, $\sum_{1\le\delta|N} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta} \ge 0$.*

*Then, $f(z)$ is in $M_k(\Gamma_0(N), \chi)$, where the character $\chi$ is defined by $\chi(m) = \left(\frac{(-1)^k s}{m}\right)$. In addition to the above conditions, if the inequalities in* (iv) *are all strict, then $f(z)$ is in $S_k(\Gamma_0(N), \chi)$.*

As mentioned before, to construct the basis of the space $E_k(\Gamma_0(N), \chi)$ we use the method described in [18, p. 88]. We first define the following six Dirichlet characters,

$$\begin{aligned}\chi_{-4}(n) &= \left(\frac{-4}{n}\right),\ \chi_{-8}(n) = \left(\frac{-8}{n}\right),\ \chi_{8}(n) = \left(\frac{8}{n}\right),\\ \chi_{-11}(n) &= \left(\frac{-11}{n}\right),\ \chi_{44}(n) = \left(\frac{44}{n}\right),\ \chi_{88}(n) = \left(\frac{88}{n}\right).\end{aligned} \tag{2.1}$$

Using (1.4)-(1.6) weight 2 level 88 Eisenstein series can be written as follows:

$$L(q) := E_{\chi_1,\chi_1}(q) = \frac{-1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{2.2}$$

$$E_{\chi_1,\chi_{44}}(q) = -7 + \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_{44}}(n) q^n, \ E_{\chi_{44},\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{44},\chi_1}(n) q^n, \tag{2.3}$$

$$E_{\chi_{-4},\chi_{-11}}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{-4},\chi_{-11}}(n) q^n, \ E_{\chi_{-11},\chi_{-4}}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{-11},\chi_{-4}}(n) q^n, \tag{2.4}$$

$$E_{\chi_1,\chi_8}(q) = \frac{-1}{2} + \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_8}(n) q^n, \ E_{\chi_8,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_8,\chi_1}(n) q^n, \tag{2.5}$$

$$E_{\chi_1,\chi_{88}}(q) = -23 + \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_{88}}(n) q^n, \ E_{\chi_{88},\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{88},\chi_1}(n) q^n, \tag{2.6}$$

$$E_{\chi_{-8},\chi_{-11}}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{-8},\chi_{-11}}(n) q^n, \ E_{\chi_{-11},\chi_{-8}}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_{-11},\chi_{-8}}(n) q^n. \tag{2.7}$$

The shorted list of 26 diagonal quaternary quadratic forms $(a_1, a_2, a_3, a_4)$ can be grouped according to the modular spaces $M_2(\Gamma_0(88), \chi)$ to which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ belong. See Table 1 for a complete list.

**Table 1**

| $M_2(\Gamma_0(88), \chi_1)$ | $M_2(\Gamma_0(88), \chi_{44})$ | $M_2(\Gamma_0(88), \chi_8)$ | $M_2(\Gamma_0(88), \chi_{88})$ |
|---|---|---|---|
| $(1, 1, 1, 1)*$ | $(1, 1, 1, 11)$ | $(1, 1, 1, 2)*$ | $(1, 1, 1, 22)$ |
| $(1, 1, 2, 2)*$ | $(1, 1, 2, 22)$ | $(1, 1, 11, 22)$ | $(1, 1, 2, 11)$ |
| $(1, 1, 11, 11)$ | $(1, 2, 2, 11)$ | $(1, 2, 2, 2)*$ | $(1, 2, 2, 22)$ |
| $(1, 1, 22, 22)$ | $(1, 11, 11, 11)$ | $(1, 2, 11, 11)$ | $(1, 11, 11, 22)$ |
| $(1, 2, 11, 22)$ | $(1, 11, 22, 22)$ | $(1, 2, 22, 22)$ | $(1, 22, 22, 22)$ |
| $(2, 2, 11, 11)$ | $(2, 11, 11, 22)$ | $(2, 2, 11, 22)$ | $(2, 2, 2, 11)$ |
| | | | $(2, 11, 11, 11)$ |
| | | | $(2, 11, 22, 22)$ |

$*$ Previously known results

## 3. Bases for the spaces of modular forms

### 3.1. A basis for $M_2(\Gamma_0(88), \chi_1)$

We use the dimension formulae given in [18, p. 98] to calculate the dimension of each of the space. The modular form space $M_2(\Gamma_0(88), \chi_1)$ has dimension 16 with $\dim(E_2(\Gamma_0(88), \chi_1) = 7$ and $\dim(S_2(\Gamma_0(88), \chi_1) = 9$. For the space of cusp forms, we need the following 9 eta-quotients. These are selected among many others such that all satisfy the conditions of Theorem 1 and they are linearly independent.

$$A_1(q) = [2, 4, 22, 44], \tag{3.1}$$

$$A_2(q) = [1^2, 11^2], \tag{3.2}$$

$$A_3(q) = [2^2, 22^2], \tag{3.3}$$

$$A_4(q) = [4^2, 44^2], \tag{3.4}$$

$$A_5(q) = [8^2, 88^2], \tag{3.5}$$

$$A_6(q) = [2^3, 4^{-1}, 22^3, 44^{-1}], \tag{3.6}$$

$$A_7(q) = [2^{-1}, 4^3, 22^{-1}, 44^3], \tag{3.7}$$

$$A_8(q) = [1^2, 2^{-1}, 4, 11^2, 22^{-1}, 44], \tag{3.8}$$

$$A_9(q) = [2, 4^{-1}, 8^2, 22, 44^{-1}, 88^2]. \tag{3.9}$$

We define the integers $a_r(n)$ $(n \in \mathbb{N})$ by:

$$A_r(q) = \sum_{n=1}^{\infty} a_r(n) q^n, \ \ 1 \leq r \leq 9. \tag{3.10}$$

**Theorem 2.** $\{L(q) - tL(q^t) : t = 2, 4, 8, 11, 22, 44, 88\} \cup \{A_r(q) : r = 1, 2, \ldots, 9\}$ *forms a basis of* $M_2(\Gamma_0(88), \chi_1)$.

*Proof.* It follows from [18, Theorem 5.9, p. 88] with $\chi = \psi = \chi_1$ that $\{L(q) - tL(q^t) : t = 2, 4, 8, 11, 22, 44, 88\}$ is a basis for $E_2(\Gamma_0(88), \chi_1)$. It can be easily seen that the eta quotients $A_r(q)$ $(1 \leq r \leq 9)$ are linearly independent over $\mathbb{C}$. It follows from (3.1)-(3.9), (1.7)-(1.9) and Theorem 1 that for each $r \in \{1, 2, \ldots, 9\}$, $A_r(q)$ is in $S_2(\Gamma_0(88), \chi_1)$. Since the dimension of $S_2(\Gamma_0(88), \chi_1)$ is 9, $\{A_r(q) : r = 1, 2, \ldots, 9\}$ forms a basis of $S_2(\Gamma_0(88), \chi_1)$.

Thus $\{L(q) - tL(q^t) : t = 2, 4, 8, 11, 22, 44, 88\}$ and $\{A_r(q) : r = 1, 2, \ldots, 9\}$ together forms a basis of $M_2(\Gamma_0(88), \chi_1)$. □

### 3.2. A basis for $M_2(\Gamma_0(88), \chi_{44})$

The modular form space $M_2(\Gamma_0(88), \chi_{44})$ has dimension 16 with

$$\dim(E_2(\Gamma_0(88), \chi_{44})) = 8 \text{ and } \dim(S_2(\Gamma_0(88), \chi_{44})) = 8.$$

For the space of cusp forms we need the following 8 eta-quotients:

$$B_1(q) = [1, 2^{-2}, 4^4, 11], \tag{3.11}$$

$$B_2(q) = [1^4, 2^{-1}, 22], \tag{3.12}$$

$$B_3(q) = [2, 11^4, 22^{-1}], \tag{3.13}$$

$$B_4(q) = [1^4, 2^{-2}, 4, 44], \tag{3.14}$$

$$B_5(q) = [4, 11^4, 22^{-2}, 44], \tag{3.15}$$

$$B_6(q) = [2, 4^{-2}, 8^4, 22], \tag{3.16}$$

$$B_7(q) = [2^4, 4^{-2}, 8, 88], \tag{3.17}$$

$$B_8(q) = [1^{-1}, 2^3, 4^{-1}, 11^{-1}, 22, 44^3]. \tag{3.18}$$

We define the integers $b_r(n)$ $(n \in \mathbb{N})$ by:

$$B_r(q) = \sum_{n=1}^{\infty} b_r(n)q^n, \ \ 1 \le r \le 8. \tag{3.19}$$

**Theorem 3.** $\{E_{\chi_1,\chi_{44}}(q^t), E_{\chi_{44},\chi_1}(q^t), E_{\chi_{-4},\chi_{-11}}(q^t), E_{\chi_{-11},\chi_{-4}}(q^t) : t = 1, 2\}$ $\cup\{B_r(q) : r = 1, 2, \dots, 8\}$ *forms a basis of* $M_2(\Gamma_0(88), \chi_{44})$.

*Proof.* Appealing to [18, Theorem 5.9, p. 88] with $\epsilon = \chi_{44}$ and $\psi, \chi \in \{\chi_1, \chi_{44}, \chi_{-4}, \chi_{-11}\}$, we see that $\{E_{\chi_1,\chi_{44}}(q^t), E_{\chi_{44},\chi_1}(q^t), E_{\chi_{-4},\chi_{-11}}(q^t), E_{\chi_{-11},\chi_{-4}}(q^t) : t = 1, 2\}$ is a basis for $E_2(\Gamma_0(88), \chi_{44})$. The eta quotients $B_r(q)(1 \le r \le 8)$ are linearly independent over $\mathbb{C}$. From (3.11)-(3.18), (1.7)-(1.9) and Theorem 1 we see that all the eta quotients $B_r(q)(1 \le r \le 8)$ are contained in $S_2(\Gamma_0(88), \chi_{44})$. Since $\dim(S_2(\Gamma_0(88), \chi_{44}))$ is 8, $\{B_r(q) : r = 1, 2, \dots, 8\}$ forms a basis of $S_2(\Gamma_0(88), \chi_{44})$.

Thus $\{E_{\chi_1,\chi_{44}}(q^t)$, $E_{\chi_{44},\chi_1}(q^t)$, $E_{\chi_{-4},\chi_{-11}}(q^t)$, $E_{\chi_{-11},\chi_{-4}}(q^t) : t = 1, 2\}$ and $\{B_r(q) : r = 1, 2, \dots, 8\}$ together forms a basis of $M_2(\Gamma_0(88), \chi_{44})$. □

### 3.3. A basis for $M_2(\Gamma_0(88), \chi_8)$

The modular form space $M_2(\Gamma_0(88), \chi_8)$ has dimension 14 with

$$\dim(E_2(\Gamma_0(88), \chi_8)) = 4 \text{ and } \dim(S_2(\Gamma_0(88), \chi_8)) = 10.$$

For the space of cusp forms, we need the following 10 eta-quotients:

$$C_1(q) = [1^2, 4^{-1}, 8, 22^{-1}, 44^4, 88^{-1}], \tag{3.20}$$

$$C_2(q) = [1^2, 2^{-1}, 8, 22^{-2}, 44^5, 88^{-1}], \tag{3.21}$$

$$C_3(q) = [1, 4^{-1}, 8^2, 11^{-1}, 22^5, 44^{-2}], \tag{3.22}$$

$$C_4(q) = [1, 2^{-1}, 8^2, 11^{-1}, 22^4, 44^{-1}], \tag{3.23}$$

$$C_5(q) = [2^{-1}, 4^5, 8^{-2}, 11^{-2}, 22^6, 44^{-2}], \tag{3.24}$$

$$C_6(q) = [2^{-1}, 4^4, 8^{-1}, 11^2, 44^{-1}, 88], \tag{3.25}$$

$$C_7(q) = [2^{-2}, 4^6, 8^{-2}, 11^{-2}, 22^5, 44^{-1}], \tag{3.26}$$

$$C_8(q) = [2^{-2}, 4^5, 8^{-1}, 11^2, 22^{-1}, 88], \tag{3.27}$$

$$C_9(q) = [1^{-1}, 2^5, 4^{-2}, 11, 44^{-1}, 88^2], \tag{3.28}$$

$$C_{10}(q) = [1^{-1}, 2^4, 4^{-1}, 11, 22^{-1}, 88^2]. \tag{3.29}$$

We define the integers $c_r(n)$ $(n \in \mathbb{N})$ by:

$$C_r(q) = \sum_{n=1}^{\infty} c_r(n)q^n, \ \ 1 \le r \le 10. \tag{3.30}$$

**Theorem 4.** $\{E_{\chi_1,\chi_8}(q^t),\ E_{\chi_8,\chi_1}(q^t) : t = 1, 11\} \cup \{C_r(q) : r = 1, 2, \ldots, 10\}$ *forms a basis of* $M_2(\Gamma_0(88), \chi_8)$.

*Proof.* Appealing to [18, Theorem 5.9, p. 88] with $\epsilon = \chi_8$, and $\psi, \chi \in \{\chi_1, \chi_8\}$, we see that $\{E_{\chi_1,\chi_8}(q^t),\ E_{\chi_8,\chi_1}(q^t) : t = 1, 11\}$ is a basis for $E_2(\Gamma_0(88), \chi_8)$.

From (3.20)-(3.29), (1.7)-(1.9) and Theorem 1 we see that all the eta quotients $C_r(q)$ $(1 \leq r \leq 10)$ are contained in $S_2(\Gamma_0(88), \chi_8)$. It can be easily seen that the eta quotients $C_r(q)$ $(1 \leq r \leq 10)$ are linearly independent over $\mathbb{C}$. Since $\dim(S_2(\Gamma_0(88), \chi_8))$ is 10, $\{C_r(q) : r = 1, 2, \ldots, 10\}$ forms a basis of $S_2(\Gamma_0(88), \chi_8)$.

Thus, $\{E_{\chi_1,\chi_8}(q^t),\ E_{\chi_8,\chi_1}(q^t) : t = 1, 11\}$ and $\{C_r(q) : r = 1, 2, \ldots, 10\}$ together forms a basis of $M_2(\Gamma_0(88), \chi_8)$. □

### 3.4. A basis for $M_2(\Gamma_0(88), \chi_{88})$

The modular form space $M_2(\Gamma_0(88), \chi_{88})$ has dimension 14 with

$$\dim(E_2(\Gamma_0(88), \chi_{88})) = 4 \text{ and } \dim(S_2(\Gamma_0(88), \chi_{88})) = 10.$$

For the space of cusp forms, we need the following 10 eta-quotients:

$$D_1(q) = [1^{-1}, 2^5, 4^{-3}, 8^2, 11], \tag{3.31}$$

$$D_2(q) = [1^{-2}, 2^4, 4^3, 8^{-2}, 22], \tag{3.32}$$

$$D_3(q) = [1^{-2}, 2^3, 4^4, 8^{-2}, 44], \tag{3.33}$$

$$D_4(q) = [1^2, 2^{-2}, 4^4, 8^{-1}, 22, 44^{-1}, 88], \tag{3.34}$$

$$D_5(q) = [2, 4^{-1}, 8, 11^2, 22^{-2}, 44^4, 88^{-1}], \tag{3.35}$$

$$D_6(q) = [1^2, 2^{-3}, 4^5, 8^{-1}, 88], \tag{3.36}$$

$$D_7(q) = [1, 11^{-1}, 22^5, 44^{-3}, 88^2], \tag{3.37}$$

$$D_8(q) = [4, 11^{-2}, 22^3, 44^4, 88^{-2}], \tag{3.38}$$

$$D_9(q) = [8, 11^2, 22^{-3}, 44^5, 88^{-1}], \tag{3.39}$$

$$D_{10}(q) = [1^{-1}, 2^4, 4^{-2}, 8^2, 11, 22^{-1}, 44]. \tag{3.40}$$

We define the integers $d_r(n)$ $(n \in \mathbb{N})$ by:

$$D_r(q) = \sum_{n=1}^{\infty} d_r(n) q^n,\ 1 \leq r \leq 10. \tag{3.41}$$

**Theorem 5.** $\{E_{\chi_1,\chi_{88}}(q),\ E_{\chi_{88},\chi_1}(q),\ E_{\chi_{-8},\chi_{-11}}(q),\ E_{\chi_{-11},\chi_{-8}}(q)\} \cup \{D_r(q) : r = 1, 2, \ldots, 10\}$ *forms a basis of* $M_2(\Gamma_0(88), \chi_{88})$.

*Proof.* Appealing to [18, Theorem 5.9, p. 88] with $\epsilon = \chi_{88}$ and $\psi, \chi \in \{\chi_1, \chi_{88}, \chi_{-8}, \chi_{-11}\}$ we see that $\{E_{\chi_1,\chi_{88}}(q),\ E_{\chi_{88},\chi_1}(q),\ E_{\chi_{-8},\chi_{-11}}(q),\ E_{\chi_{-11},\chi_{-8}}(q)\}$ is a basis for $E_2(\Gamma_0(88), \chi_{88})$.

From (3.31)-(3.40), (1.7)-(1.9) and Theorem 1 we see that all the eta quotients $D_r(q)(1 \leq r \leq 10)$ are contained in $S_2(\Gamma_0(88), \chi_{88})$. The eta quotients $D_r(q)(1 \leq r \leq 10)$ are linearly independent over $\mathbb{C}$. Since dimension of $S_2(\Gamma_0(88), \chi_{88})$ is 10, $\{D_r(q) : r = 1, 2, \ldots, 10\}$ forms a basis of $S_2(\Gamma_0(88), \chi_{88})$.

Thus, $\{E_{\chi_1,\chi_{88}}(q)$, $E_{\chi_{88},\chi_1}(q)$, $E_{\chi_{-8},\chi_{-11}}(q)$, $E_{\chi_{-11},\chi_{-8}}(q)\}$ and $\{D_r(q) : r = 1, 2, \ldots, 10\}$ together forms a basis of $M_2(\Gamma_0(88), \chi_{88})$. □

## 4. Theta function identities

**Theorem 6.** *The following identities hold*:

(a) $\varphi^4(q) = 8L(q) - 32L(q^4)$,

(b) $\varphi^2(q)\varphi^2(q^2) = L(q) - 4L(q^2) + 8L(q^4) - 32L(q^8)$,

(c) $$\varphi^2(q)\varphi^2(q^{11}) = \frac{4}{5}L(q) - \frac{8}{5}L(q^2) + \frac{16}{5}L(q^4) - \frac{44}{5}L(q^{11}) + \frac{88}{5}L(q^{22}) - \frac{176}{5}L(q^{44}) + \frac{16}{5}A_2(q) + \frac{48}{5}A_3(q) + \frac{64}{5}A_4(q),$$

(d) $$\varphi^2(q)\varphi^2(q^{22}) = \frac{2}{5}L(q) - \frac{2}{5}L(q^2) - \frac{4}{5}L(q^4) + \frac{16}{5}L(q^8) - \frac{22}{5}L(q^{11}) + \frac{22}{5}L(q^{22}) + \frac{44}{5}L(q^{44}) - \frac{176}{5}L(q^{88}) + 6A_1(q) + \frac{8}{5}A_2(q) + \frac{32}{5}A_3(q) + \frac{64}{5}A_4(q) + \frac{64}{5}A_5(q) + 2A_6(q) + 8A_7(q),$$

(e) $$\varphi(q)\varphi(q^2)\varphi(q^{11})\varphi(q^{22}) = \frac{1}{3}L(q) - \frac{1}{3}L(q^2) + \frac{2}{3}L(q^4) - \frac{8}{3}L(q^8) + \frac{11}{3}L(q^{11}) - \frac{11}{3}L(q^{22}) + \frac{22}{3}L(q^{44}) - \frac{88}{3}L(q^{88}) + \frac{31}{3}A_1(q) + \frac{5}{3}A_6(q) + \frac{20}{3}A_7(q) + \frac{4}{3}A_8(q) + \frac{16}{3}A_9(q),$$

(f) $$\varphi^2(q^2)\varphi^2(q^{11}) = \frac{2}{5}L(q) - \frac{2}{5}L(q^2) - \frac{4}{5}L(q^4) + \frac{16}{5}L(q^8) - \frac{22}{5}L(q^{11}) + \frac{22}{5}L(q^{22}) + \frac{44}{5}L(q^{44}) - \frac{176}{5}L(q^{88}) - 6A_1(q) + \frac{8}{5}A_2(q) + \frac{32}{5}A_3(q) + \frac{64}{5}A_4(q) + \frac{64}{5}A_5(q) - 2A_6(q) - 8A_7(q).$$

*Proof.* Let $(a_1, a_2, a_3, a_4)$ be any of the diagonal quaternary quadratic forms listed in the first column of Table 1. Using Theorem 1, it can be easily shown that for each quadruple $(a_1, a_2, a_3, a_4)$, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ is in $M_2(\Gamma_0(88), \chi_1)$. Since $\{L(q) - tL(q^t) : t = 2, 4, 8, 11, 22, 44, 88\} \cup \{A_r(q) : r = 1, 2, \ldots, 9\}$ forms a basis of $M_2(\Gamma_0(88), \chi_1)$ by Theorem 2, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})$ $\varphi(q^{a_4})$ can be written as a linear combination of $L(q) - tL(q^t)$ $(t = 2, 4, 8, 11, 22,$

44, 88) and $A_r(q)$ ($r \in \{1, 2, \ldots, 9\}$). We have

$$\begin{aligned}(4.1)\quad &\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})\\ &= x_1(L(q) - 2L(q^2)) + x_2(L(q) - 4L(q^4)) + x_3(L(q) - 8L(q^8))\\ &\quad + x_4(L(q) - 11L(q^{11})) + x_5(L(q) - 22L(q^{22})) + x_6(L(q) - 44L(q^{44}))\\ &\quad + x_7(L(q) - 88L(q^{88})) + y_1A_1(q) + y_2A_2(q) + y_3A_3(q) + y_4A_4(q)\\ &\quad + y_5A_5(q) + y_6A_6(q) + y_7A_7(q) + y_8A_8(q) + y_9A_9(q)\end{aligned}$$

for some $x_i$ ($1 \le i \le 7$) and $y_j$ ($1 \le j \le 9$). Now, we focus only the case (c) in the theorem as the rest can be proven in a similar way. Let $(a_1, a_2, a_3, a_4) = (1, 1, 11, 11)$. Using the formula given in [10, Theorem 3.13], [19] we determine the Sturm bound for $M_2(\Gamma_0(88), \chi_1)$ as 24. So, equating the coefficients of $q^n$ for $1 \le n \le 24$ on both sides of (4.1) we obtain a system of linear equations with the 16 unknowns $x_i$ ($1 \le i \le 7$) and $y_j$ ($1 \le j \le 9$). Solving this system of equations we obtain

$$\begin{aligned}&x_1 = \frac{4}{5},\ x_2 = -\frac{4}{5},\ x_3 = 0,\ x_4 = \frac{4}{5},\ x_5 = -\frac{4}{5},\ x_6 = \frac{4}{5},\ x_7 = 0,\\ &y_1 = 0,\ y_2 = \frac{16}{5},\ y_3 = \frac{48}{5},\ y_4 = \frac{64}{5},\ y_5 = 0,\ y_6 = 0,\ y_7 = 0,\ y_8 = 0, y_9 = 0.\end{aligned}$$

Substituting these values into (4.1) and making the necessary simplifications we obtain the desired result. □

As a consequence of Theorem 1 and Theorem 3 we can give Theorem 7.

**Theorem 7.** *The following identities hold*:

(a) $$\begin{aligned}\varphi^3(q)\varphi(q^{11}) = &-\frac{1}{7}E_{\chi_1,\chi_{44}}(q) + \frac{22}{7}E_{\chi_{44},\chi_1}(q) - \frac{2}{7}E_{\chi_{-4},\chi_{-11}}(q)\\ &+ \frac{11}{7}E_{\chi_{-11},\chi_{-4}}(q) + \frac{12}{7}B_2(q) + \frac{96}{7}B_6(q),\end{aligned}$$

(b) $$\begin{aligned}\varphi^2(q)\varphi(q^2)\varphi(q^{22}) = &-\frac{1}{7}E_{\chi_1,\chi_{44}}(q^2) + \frac{11}{7}E_{\chi_{44},\chi_1}(q) + \frac{1}{7}E_{\chi_{-4},\chi_{-11}}(q)\\ &+ \frac{11}{7}E_{\chi_{-11},\chi_{-4}}(q^2) - 8B_1(q) + 4B_2(q)\\ &+ \frac{44}{7}B_3(q) - \frac{12}{7}B_4(q) - \frac{176}{7}B_5(q)\\ &+ \frac{256}{7}B_6(q) + \frac{128}{7}B_7(q) + \frac{264}{7}B_8(q),\end{aligned}$$

(c) $$\begin{aligned}\varphi(q)\varphi^2(q^2)\varphi(q^{11}) = &-\frac{1}{7}E_{\chi_1,\chi_{44}}(q^2) + \frac{11}{7}E_{\chi_{44},\chi_1}(q) - \frac{1}{7}E_{\chi_{-4},\chi_{-11}}(q)\\ &- \frac{11}{7}E_{\chi_{-11},\chi_{-4}}(q^2) - \frac{8}{7}B_1(q) + \frac{12}{7}B_2(q)\\ &- \frac{2}{7}B_4(q) - \frac{22}{7}B_5(q) + \frac{80}{7}B_6(q) + \frac{16}{7}B_7(q),\end{aligned}$$

$$\begin{aligned}
\text{(d)}\ \varphi(q)\varphi^3(q^{11}) = & -\frac{1}{7}E_{\chi_1,\chi_{44}}(q) + \frac{2}{7}E_{\chi_{44},\chi_1}(q) + \frac{2}{7}E_{\chi_{-4},\chi_{-11}}(q) \\
& -\frac{1}{7}E_{\chi_{-11},\chi_{-4}}(q) + \frac{144}{77}B_1(q) - \frac{12}{77}B_2(q) \\
& +\frac{60}{77}B_4(q) + \frac{12}{7}B_5(q) - \frac{96}{77}B_6(q),
\end{aligned}$$

$$\begin{aligned}
\text{(e)}\ \varphi(q)\varphi(q^{11})\varphi^2(q^{22}) = & -\frac{1}{7}E_{\chi_1,\chi_{44}}(q^2) + \frac{1}{7}E_{\chi_{44},\chi_1}(q) + \frac{1}{7}E_{\chi_{-4},\chi_{-11}}(q) \\
& +\frac{1}{7}E_{\chi_{-11},\chi_{-4}}(q^2) - \frac{16}{77}B_1(q) + \frac{38}{77}B_2(q) \\
& +\frac{10}{7}B_3(q) + \frac{8}{77}B_4(q) - \frac{16}{7}B_5(q) \\
& +\frac{304}{77}B_6(q) + \frac{16}{7}B_7(q) + \frac{32}{7}B_8(q),
\end{aligned}$$

$$\begin{aligned}
\text{(f)}\ \varphi(q^2)\varphi^2(q^{11})\varphi(q^{22}) = & -\frac{1}{7}E_{\chi_1,\chi_{44}}(q^2) + \frac{1}{7}E_{\chi_{44},\chi_1}(q) - \frac{1}{7}E_{\chi_{-4},\chi_{-11}}(q) \\
& -\frac{1}{7}E_{\chi_{-11},\chi_{-4}}(q^2) - \frac{96}{77}B_1(q) + \frac{30}{77}B_2(q) \\
& +\frac{6}{7}B_3(q) - \frac{18}{77}B_4(q) - \frac{22}{7}B_5(q) + \frac{416}{77}B_6(q) \\
& +\frac{16}{7}B_7(q) + \frac{8}{7}B_8(q).
\end{aligned}$$

As a consequence of Theorem 1 and Theorem 4 we can give Theorem 8.

**Theorem 8.** *The following identities hold*:

$$\text{(a)}\ \varphi^3(q)\varphi(q^2) = -2E_{\chi_1,\chi_8}(q) + 8E_{\chi_8,\chi_1}(q),$$

$$\begin{aligned}
\text{(b)}\ \varphi^2(q)\varphi(q^{11})\varphi(q^{22}) = & -\frac{12}{61}E_{\chi_1,\chi_8}(q) - \frac{110}{61}E_{\chi_1,\chi_8}(q^{11}) + \frac{48}{61}E_{\chi_8,\chi_1}(q) \\
& -\frac{440}{61}E_{\chi_8,\chi_1}(q^{11}) + \frac{48}{61}C_1(q) - \frac{1632}{427}C_2(q) \\
& +\frac{968}{427}C_3(q) + \frac{624}{427}C_4(q) + \frac{8}{7}C_5(q) - \frac{72}{427}C_6(q) \\
& +\frac{24}{7}C_7(q) - \frac{176}{427}C_8(q) - \frac{992}{427}C_9(q) + \frac{104}{61}C_{10}(q),
\end{aligned}$$

$$\text{(c)}\ \varphi(q)\varphi^3(q^2) = -2E_{\chi_1,\chi_8}(q) + 4E_{\chi_8,\chi_1}(q),$$

$$\begin{aligned}
\text{(d)}\ \varphi(q)\varphi(q^2)\varphi^2(q^{11}) = & \frac{10}{61}E_{\chi_1,\chi_8}(q) - \frac{132}{61}E_{\chi_1,\chi_8}(q^{11}) + \frac{40}{61}E_{\chi_8,\chi_1}(q) \\
& +\frac{528}{61}E_{\chi_8,\chi_1}(q^{11}) - \frac{80}{61}C_1(q) - \frac{800}{427}C_2(q) \\
& -\frac{472}{427}C_3(q) + \frac{240}{427}C_4(q) + \frac{16}{7}C_5(q) + \frac{640}{427}C_6(q) \\
& -\frac{8}{7}C_7(q) + \frac{320}{427}C_8(q) + \frac{480}{427}C_9(q) + \frac{120}{61}C_{10}(q),
\end{aligned}$$

$$\text{(e)}\ \varphi(q)\varphi(q^2)\varphi^2(q^{22}) = \frac{10}{61}E_{\chi_1,\chi_8}(q) - \frac{132}{61}E_{\chi_1,\chi_8}(q^{11}) + \frac{20}{61}E_{\chi_8,\chi_1}(q) + \frac{264}{61}E_{\chi_8,\chi_1}(q^{11}) + \frac{52}{61}C_1(q) - \frac{992}{427}C_2(q) - \frac{88}{427}C_3(q) - \frac{72}{427}C_4(q) + \frac{12}{7}C_5(q) + \frac{312}{427}C_6(q) + \frac{8}{7}C_7(q) + \frac{968}{427}C_8(q) - \frac{816}{427}C_9(q) + \frac{48}{61}C_{10}(q),$$

$$\text{(f)}\ \varphi^2(q^2)\varphi(q^{11})\varphi(q^{22}) = -\frac{12}{61}E_{\chi_1,\chi_8}(q) - \frac{110}{61}E_{\chi_1,\chi_8}(q^{11}) + \frac{24}{61}E_{\chi_8,\chi_1}(q) - \frac{220}{61}E_{\chi_8,\chi_1}(q^{11}) + \frac{60}{61}C_1(q) + \frac{480}{427}C_2(q) + \frac{160}{427}C_3(q) + \frac{640}{427}C_4(q) - \frac{4}{7}C_5(q) + \frac{120}{427}C_6(q) + \frac{16}{7}C_7(q) - \frac{472}{427}C_8(q) - \frac{400}{427}C_9(q) - \frac{80}{61}C_{10}(q).$$

As a consequence of Theorem 1 and Theorem 5 we can give Theorem 9.

**Theorem 9.** *The following identities hold*:

$$\text{(a)}\ \varphi^3(q)\varphi(q^{22}) = -\frac{1}{23}E_{\chi_1,\chi_{88}}(q) + \frac{44}{23}E_{\chi_{88},\chi_1}(q) - \frac{4}{23}E_{\chi_{-8},\chi_{-11}}(q) + \frac{11}{23}E_{\chi_{-11},\chi_{-8}}(q) + \frac{496}{115}D_1(q) - \frac{56}{115}D_2(q) + \frac{272}{115}D_3(q) - \frac{224}{115}D_6(q) - \frac{616}{115}D_7(q) - \frac{616}{115}D_8(q) + \frac{2464}{115}D_9(q) + 8D_{10}(q),$$

$$\text{(b)}\ \varphi^2(q)\varphi(q^2)\varphi(q^{11}) = -\frac{1}{23}E_{\chi_1,\chi_{88}}(q) + \frac{44}{23}E_{\chi_{88},\chi_1}(q) + \frac{4}{23}E_{\chi_{-8},\chi_{-11}}(q) - \frac{11}{23}E_{\chi_{-11},\chi_{-8}}(q) + \frac{96}{115}D_1(q) + \frac{8}{5}D_2(q) + \frac{32}{115}D_3(q) - \frac{24}{23}D_4(q) + \frac{88}{23}D_5(q) + \frac{16}{115}D_6(q) - \frac{176}{115}D_7(q) - \frac{176}{115}D_8(q) - \frac{176}{115}D_9(q) - \frac{16}{23}D_{10}(q),$$

$$\text{(c)}\ \varphi(q)\varphi^2(q^2)\varphi(q^{22}) = -\frac{1}{23}E_{\chi_1,\chi_{88}}(q) + \frac{22}{23}E_{\chi_{88},\chi_1}(q) - \frac{2}{23}E_{\chi_{-8},\chi_{-11}}(q) + \frac{11}{23}E_{\chi_{-11},\chi_{-8}}(q) + \frac{112}{115}D_1(q) - \frac{32}{115}D_2(q) + \frac{224}{115}D_3(q) - \frac{24}{23}D_4(q) + \frac{88}{23}D_5(q) + \frac{32}{115}D_6(q) - \frac{352}{115}D_7(q) + \frac{88}{115}D_8(q) - \frac{352}{115}D_9(q) - \frac{16}{23}D_{10}(q),$$

(d) $$\begin{aligned}\varphi(q)\varphi^2(q^{11})\varphi(q^{22}) = & -\frac{1}{23}E_{\chi_1,\chi_{88}}(q)+\frac{4}{23}E_{\chi_{88},\chi_1}(q)+\frac{4}{23}E_{\chi_{-8},\chi_{-11}}(q)\\ & -\frac{1}{23}E_{\chi_{-11},\chi_{-8}}(q)+\frac{2736}{1265}D_1(q)-\frac{536}{1265}D_2(q)\\ & +\frac{592}{1265}D_3(q)-\frac{624}{253}D_4(q)+\frac{208}{23}D_5(q)\\ & +\frac{416}{1265}D_6(q)-\frac{816}{115}D_7(q)-\frac{16}{115}D_8(q)\\ & -\frac{416}{115}D_9(q)-\frac{416}{253}D_{10}(q),\end{aligned}$$

(e) $$\begin{aligned}\varphi(q)\varphi^3(q^{22}) = & -\frac{1}{23}E_{\chi_1,\chi_{88}}(q)+\frac{2}{23}E_{\chi_{88},\chi_1}(q)+\frac{2}{23}E_{\chi_{-8},\chi_{-11}}(q)\\ & -\frac{1}{23}E_{\chi_{-11},\chi_{-8}}(q)+\frac{3136}{1265}D_1(q)-\frac{716}{1265}D_2(q)\\ & +\frac{272}{1265}D_3(q)-\frac{408}{253}D_4(q)+\frac{228}{23}D_5(q)-\frac{344}{1265}D_6(q)\\ & -\frac{656}{115}D_7(q)-\frac{56}{115}D_8(q)+\frac{224}{115}D_9(q)-\frac{272}{253}D_{10}(q),\end{aligned}$$

(f) $$\begin{aligned}\varphi^3(q^2)\varphi(q^{11}) = & -\frac{1}{23}E_{\chi_1,\chi_{88}}(q)+\frac{22}{23}E_{\chi_{88},\chi_1}(q)+\frac{2}{23}E_{\chi_{-8},\chi_{-11}}(q)\\ & -\frac{11}{23}E_{\chi_{-11},\chi_{-8}}(q)-\frac{56}{23}D_1(q)+\frac{44}{23}D_2(q)-\frac{8}{23}D_4(q)\\ & -\frac{308}{23}D_5(q)+\frac{88}{23}D_6(q)+\frac{56}{23}D_{10}(q),\end{aligned}$$

(g) $$\begin{aligned}\varphi(q^2)\varphi^3(q^{11}) = & -\frac{1}{23}E_{\chi_1,\chi_{88}}(q)+\frac{4}{23}E_{\chi_{88},\chi_1}(q)-\frac{4}{23}E_{\chi_{-8},\chi_{-11}}(q)\\ & +\frac{1}{23}E_{\chi_{-11},\chi_{-8}}(q)-\frac{16}{1265}D_1(q)+\frac{16}{1265}D_2(q)\\ & +\frac{48}{1265}D_3(q)+\frac{16}{253}D_4(q)+\frac{56}{23}D_5(q)+\frac{224}{1265}D_6(q)\\ & +\frac{456}{115}D_7(q)+\frac{456}{115}D_8(q)-\frac{624}{115}D_9(q)-\frac{664}{253}D_{10}(q),\end{aligned}$$

(h) $$\begin{aligned}\varphi(q^2)\varphi(q^{11})\varphi^2(q^{22}) = & -\frac{1}{23}E_{\chi_1,\chi_{88}}(q)+\frac{2}{23}E_{\chi_{88},\chi_1}(q)-\frac{2}{23}E_{\chi_{-8},\chi_{-11}}(q)\\ & +\frac{1}{23}E_{\chi_{-11},\chi_{-8}}(q)+\frac{112}{1265}D_1(q)-\frac{112}{1265}D_2(q)\\ & +\frac{104}{1265}D_3(q)-\frac{24}{253}D_4(q)+\frac{8}{23}D_5(q)+\frac{192}{1265}D_6(q)\\ & -\frac{32}{115}D_7(q)+\frac{208}{115}D_8(q)-\frac{32}{115}D_9(q)-\frac{16}{253}D_{10}(q).\end{aligned}$$

## 5. Statement of the main theorem

**Theorem 10.** *Let* $n \in \mathbb{N}$ *and* $a_r(n)$ $(1 \le r \le 9)$ *be as in* (3.10). *Then*

(a) $N(1,1,1,1;n) = 8\sigma(n) - 32\sigma(n/4)$,

(b) $N(1,1,2,2;n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8)$,

(c) $$N(1,1,11,11;n) = \frac{4}{5}\sigma(n) - \frac{8}{5}\sigma(\frac{n}{2}) + \frac{16}{5}\sigma(\frac{n}{4}) - \frac{44}{5}\sigma(\frac{n}{11}) + \frac{88}{5}\sigma(\frac{n}{22}) - \frac{176}{5}\sigma(\frac{n}{44}) + \frac{16}{5}a_2(n) + \frac{48}{5}a_3(n) + \frac{64}{5}a_4(n),$$

(d) $$N(1,1,22,22;n) = \frac{2}{5}\sigma(n) - \frac{2}{5}\sigma(\frac{n}{2}) - \frac{4}{5}\sigma(\frac{n}{4}) + \frac{16}{5}\sigma(\frac{n}{8}) - \frac{22}{5}\sigma(\frac{n}{11}) + \frac{22}{5}\sigma(\frac{n}{22}) + \frac{44}{5}\sigma(\frac{n}{44}) - \frac{176}{5}\sigma(\frac{n}{88}) + 6a_1(n) + \frac{8}{5}a_2(n) + \frac{32}{5}a_3(n) + \frac{64}{5}a_4(n) + \frac{64}{5}a_5(n) + 2a_6(n) + 8a_7(n),$$

(e) $$N(1,2,11,22;n) = \frac{1}{3}\sigma(n) - \frac{1}{3}\sigma(\frac{n}{2}) + \frac{2}{3}\sigma(\frac{n}{4}) - \frac{8}{3}\sigma(\frac{n}{8}) + \frac{11}{3}\sigma(\frac{n}{11}) - \frac{11}{3}\sigma(\frac{n}{22}) + \frac{22}{3}\sigma(\frac{n}{44}) - \frac{88}{3}\sigma(\frac{n}{88}) + \frac{31}{3}a_1(n) + \frac{5}{3}a_6(n) + \frac{20}{3}a_7(n) + \frac{4}{3}a_8(n) + \frac{16}{3}a_9(n),$$

(f) $$N(2,2,11,11;n) = \frac{2}{5}\sigma(n) - \frac{2}{5}\sigma(\frac{n}{2}) - \frac{4}{5}\sigma(\frac{n}{4}) + \frac{16}{5}\sigma(\frac{n}{8}) - \frac{22}{5}\sigma(\frac{n}{11}) + \frac{22}{5}\sigma(\frac{n}{22}) + \frac{44}{5}\sigma(\frac{n}{44}) - \frac{176}{5}\sigma(\frac{n}{88}) - 6a_1(n) + \frac{8}{5}a_2(n) + \frac{32}{5}a_3(n) + \frac{64}{5}a_4(n) + \frac{64}{5}a_5(n) - 2a_6(n) - 8a_7(n).$$

*Proof.* We just give the proof of part (c) as the rest can be proven in a similar way. From (1.11) and Theorem 6(c) we have

$$\sum_{n=0}^{\infty} N(1,1,11,11;n)q^n = \varphi^2(q)\varphi^2(q^{11}) = \frac{4}{5}L(q) - \frac{8}{5}L(q^2) + \frac{16}{5}L(q^4) - \frac{44}{5}L(q^{11}) + \frac{88}{5}L(q^{22}) - \frac{176}{5}L(q^{44}) + \frac{16}{5}A_2(q) + \frac{48}{5}A_3(q) + \frac{64}{5}A_4(q).$$

Using (2.2) and (3.10) we obtain the desired result. □

**Theorem 11.** *Let* $n \in \mathbb{N}$. *Let* $\sigma_{k,\chi_i,\chi_j}(n)$ *be as in* (1.3) *for* $i,j \in \{1,-4,-11,44\}$ *and* $b_r(n)$ $(1 \le r \le 8)$ *as in* (3.19). *Then*

(a) $$N(1,1,1,11;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(n) + \frac{22}{7}\sigma_{\chi_{44},\chi_1}(n) - \frac{2}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) + \frac{11}{7}\sigma_{\chi_{-11},\chi_{-4}}(n) + \frac{12}{7}b_2(n) + \frac{96}{7}b_6(n),$$

(b) $N(1,1,2,22;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(\frac{n}{2}) + \frac{11}{7}\sigma_{\chi_{44},\chi_1}(n) + \frac{1}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) + \frac{11}{7}\sigma_{\chi_{-11},\chi_{-4}}(\frac{n}{2}) - 8b_1(n) + 4b_2(n) + \frac{44}{7}b_3(n) - \frac{12}{7}b_4(n) - \frac{176}{7}b_5(n) + \frac{256}{7}b_6(n) + \frac{128}{7}b_7(n) + \frac{264}{7}b_8(n),$

(c) $N(1,2,2,11;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(\frac{n}{2}) + \frac{11}{7}\sigma_{\chi_{44},\chi_1}(n) - \frac{1}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) - \frac{11}{7}\sigma_{\chi_{-11},\chi_{-4}}(\frac{n}{2}) - \frac{8}{7}b_1(n) + \frac{12}{7}b_2(n) - \frac{2}{7}b_4(n) - \frac{22}{7}b_5(n) + \frac{80}{7}b_6(n) + \frac{16}{7}b_7(n),$

(d) $N(1,11,11,11;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(n) + \frac{2}{7}\sigma_{\chi_{44},\chi_1}(n) + \frac{2}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) - \frac{1}{7}\sigma_{\chi_{-11},\chi_{-4}}(n) + \frac{144}{77}b_1(n) - \frac{12}{77}b_2(n) + \frac{60}{77}b_4(n) + \frac{12}{7}b_5(n) - \frac{96}{77}b_6(n),$

(e) $N(1,11,22,22;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(\frac{n}{2}) + \frac{1}{7}\sigma_{\chi_{44},\chi_1}(n) + \frac{1}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) + \frac{1}{7}\sigma_{\chi_{-11},\chi_{-4}}(\frac{n}{2}) - \frac{16}{77}b_1(n) + \frac{38}{77}b_2(n) + \frac{10}{7}b_3(n) + \frac{8}{77}b_4(n) - \frac{16}{7}b_5(n) + \frac{304}{77}b_6(n) + \frac{16}{7}b_7(n) + \frac{32}{7}b_8(n),$

(f) $N(2,11,11,22;n) = -\frac{1}{7}\sigma_{\chi_1,\chi_{44}}(\frac{n}{2}) + \frac{1}{7}\sigma_{\chi_{44},\chi_1}(n) - \frac{1}{7}\sigma_{\chi_{-4},\chi_{-11}}(n) - \frac{1}{7}\sigma_{\chi_{-11},\chi_{-4}}(\frac{n}{2}) - \frac{96}{77}b_1(n) + \frac{30}{77}b_2(n) + \frac{6}{7}b_3(n) - \frac{18}{77}b_4(n) - \frac{22}{7}b_5(n) + \frac{416}{77}b_6(n) + \frac{16}{7}b_7(n) + \frac{8}{7}b_8(n).$

*Proof.* The assertion follows from (2.3), (2.4) and Theorem 7. □

**Theorem 12.** *Let* $n \in \mathbb{N}$. *Let* $\sigma_{k,\chi_i,\chi_j}(n)$ *be as in* (1.3) *for* $i,j \in \{1,8\}$ *and* $c_r(n)$ $(1 \le r \le 10)$ *as in* (3.30). *Then*

(a) $N(1,1,1,2;n) = -2\sigma_{\chi_1,\chi_8}(n) + 8\sigma_{\chi_8,\chi_1}(n),$

(b) $N(1,1,11,22;n) = -\frac{12}{61}\sigma_{\chi_1,\chi_8}(n) - \frac{110}{61}\sigma_{\chi_1,\chi_8}(\frac{n}{11}) + \frac{48}{61}\sigma_{\chi_8,\chi_1}(n) - \frac{440}{61}\sigma_{\chi_8,\chi_1}(\frac{n}{11}) + \frac{48}{61}c_1(n) - \frac{1632}{427}c_2(n) + \frac{968}{427}c_3(n) + \frac{624}{427}c_4(n) + \frac{8}{7}c_5(n) - \frac{72}{427}c_6(n) + \frac{24}{7}c_7(n) - \frac{176}{427}c_8(n) - \frac{992}{427}c_9(n) + \frac{104}{61}c_{10}(n),$

(c) $N(1,2,2,2;n) = -2\sigma_{\chi_1,\chi_8}(n) + 4\sigma_{\chi_8,\chi_1}(n),$

(d) $N(1,2,11,11;n) = \frac{10}{61}\sigma_{\chi_1,\chi_8}(n) - \frac{132}{61}\sigma_{\chi_1,\chi_8}(\frac{n}{11}) + \frac{40}{61}\sigma_{\chi_8,\chi_1}(n) + \frac{528}{61}\sigma_{\chi_8,\chi_1}(\frac{n}{11}) - \frac{80}{61}c_1(n) - \frac{800}{427}c_2(n) - \frac{472}{427}c_3(n) + \frac{240}{427}c_4(n) + \frac{16}{7}c_5(n) + \frac{640}{427}c_6(n) - \frac{8}{7}c_7(n) + \frac{320}{427}c_8(n) + \frac{480}{427}c_9(n) + \frac{120}{61}c_{10}(n),$

(e) $N(1,2,22,22;n) = \frac{10}{61}\sigma_{\chi_1,\chi_8}(n) - \frac{132}{61}\sigma_{\chi_1,\chi_8}(\frac{n}{11}) + \frac{20}{61}\sigma_{\chi_8,\chi_1}(n) + \frac{264}{61}\sigma_{\chi_8,\chi_1}(\frac{n}{11}) + \frac{52}{61}c_1(n) - \frac{992}{427}c_2(n) - \frac{88}{427}c_3(n) - \frac{72}{427}c_4(n) + \frac{12}{7}c_5(n) + \frac{312}{427}c_6(n) + \frac{8}{7}c_7(n) + \frac{968}{427}c_8(n) - \frac{816}{427}c_9(n) + \frac{48}{61}c_{10}(n),$

(f) $N(2,2,11,22;n) = -\frac{12}{61}\sigma_{\chi_1,\chi_8}(n) - \frac{110}{61}\sigma_{\chi_1,\chi_8}(\frac{n}{11}) + \frac{24}{61}\sigma_{\chi_8,\chi_1}(n) - \frac{220}{61}\sigma_{\chi_8,\chi_1}(\frac{n}{11}) + \frac{60}{61}c_1(n) + \frac{480}{427}c_2(n) + \frac{160}{427}c_3(n) + \frac{640}{427}c_4(n) - \frac{4}{7}c_5(n) + \frac{120}{427}c_6(n) + \frac{16}{7}c_7(n) - \frac{472}{427}c_8(n) - \frac{400}{427}c_9(n) - \frac{80}{61}c_{10}(n).$

*Proof.* The assertion follows from (2.5) and Theorem 8. □

**Theorem 13.** *Let* $n\in\mathbb{N}$. *Let* $\sigma_{k,\chi_i,\chi_j}(n)$ *be as in* (1.3) *for* $i,j\in\{1,88,-8,-11\}$ *and* $d_r(n)$, $(1 \leq r \leq 10)$ *as in* (3.41). *Then*

(a) $N(1,1,1,22;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{44}{23}\sigma_{\chi_{88},\chi_1}(n) - \frac{4}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) + \frac{11}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) + \frac{496}{115}d_1(n) - \frac{56}{115}d_2(n) + \frac{272}{115}d_3(n) - \frac{224}{115}d_6(n) - \frac{616}{115}d_7(n) - \frac{616}{115}d_8(n) + \frac{2464}{115}d_9(n) + 8d_{10}(n),$

(b) $$N(1,1,2,11;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{44}{23}\sigma_{\chi_{88},\chi_1}(n) + \frac{4}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) - \frac{11}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) + \frac{96}{115}d_1(n) + \frac{8}{5}d_2(n) + \frac{32}{115}d_3(n) - \frac{24}{23}d_4(n) + \frac{88}{23}d_5(n) + \frac{16}{115}d_6(n) - \frac{176}{115}d_7(n) - \frac{176}{115}d_8(n) - \frac{176}{115}d_9(n) - \frac{16}{23}d_{10}(n),$$

(c) $$N(1,2,2,22;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{22}{23}\sigma_{\chi_{88},\chi_1}(n) - \frac{2}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) + \frac{11}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) + \frac{112}{115}d_1(n) - \frac{32}{115}d_2(n) + \frac{224}{115}d_3(n) - \frac{24}{23}d_4(n) + \frac{88}{23}d_5(n) + \frac{32}{115}d_6(n) - \frac{352}{115}d_7(n) + \frac{88}{115}d_8(n) - \frac{352}{115}d_9(n) - \frac{16}{23}d_{10}(n),$$

(d) $$N(1,11,11,22;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{4}{23}\sigma_{\chi_{88},\chi_1}(n) + \frac{4}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) - \frac{1}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) + \frac{2736}{1265}d_1(n) - \frac{536}{1265}d_2(n) + \frac{592}{1265}d_3(n) - \frac{624}{253}d_4(n) + \frac{208}{23}d_5(n) + \frac{416}{1265}d_6(n) - \frac{816}{115}d_7(n) - \frac{16}{115}d_8(n) - \frac{416}{115}d_9(n) - \frac{416}{253}d_{10}(n),$$

(e) $$N(1,22,22,22;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{2}{23}\sigma_{\chi_{88},\chi_1}(n) + \frac{2}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) - \frac{1}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) + \frac{3136}{1265}d_1(n) - \frac{716}{1265}d_2(n) + \frac{272}{1265}d_3(n) - \frac{408}{253}d_4(n) + \frac{228}{23}d_5(n) - \frac{344}{1265}d_6(n) - \frac{656}{115}d_7(n) - \frac{56}{115}d_8(n) + \frac{224}{115}d_9(n) - \frac{272}{253}d_{10}(n),$$

(f) $$N(2,2,2,11;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n) + \frac{22}{23}\sigma_{\chi_{88},\chi_1}(n) + \frac{2}{23}\sigma_{\chi_{-8},\chi_{-11}}(n) - \frac{11}{23}\sigma_{\chi_{-11},\chi_{-8}}(n) - \frac{56}{23}d_1(n) + \frac{44}{23}d_2(n) - \frac{8}{23}d_4(n) - \frac{308}{23}d_5(n) + \frac{88}{23}d_6(n) + \frac{56}{23}d_{10}(n),$$

(g) $N(2,11,11,11;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n)+\frac{4}{23}\sigma_{\chi_{88},\chi_1}(n)-\frac{4}{23}\sigma_{\chi_{-8},\chi_{-11}}(n)$
$+\frac{1}{23}\sigma_{\chi_{-11},\chi_{-8}}(n)-\frac{16}{1265}d_1(n)+\frac{16}{1265}d_2(n)$
$+\frac{48}{1265}d_3(n)+\frac{16}{253}d_4(n)+\frac{56}{23}d_5(n)+\frac{224}{1265}d_6(n)$
$+\frac{456}{115}d_7(n)+\frac{456}{115}d_8(n)-\frac{624}{115}d_9(n)-\frac{664}{253}d_{10}(n),$

(h) $N(2,11,22,22;n) = -\frac{1}{23}\sigma_{\chi_1,\chi_{88}}(n)+\frac{2}{23}\sigma_{\chi_{88},\chi_1}(n)-\frac{2}{23}\sigma_{\chi_{-8},\chi_{-11}}(n)$
$+\frac{1}{23}\sigma_{\chi_{-11},\chi_{-8}}(n)+\frac{112}{1265}d_1(n)-\frac{112}{1265}d_2(n)$
$+\frac{104}{1265}d_3(n)-\frac{24}{253}d_4(n)+\frac{8}{23}d_5(n)+\frac{192}{1265}d_6(n)$
$-\frac{32}{115}d_7(n)+\frac{208}{115}d_8(n)-\frac{32}{115}d_9(n)-\frac{16}{253}d_{10}(n).$

*Proof.* The assertion follows from (2.6), (2.7) and Theorem 9. □

## 6. Concluding remark

By the modularity theorem, behind any modular curve there is a weight 2 cusp form. By using the L-functions and modular forms database (LMFDB), we find the elliptic curve $y^2 = x^3 - 4x + 4$ (labeled as 88.a) with conductor 88 over $\mathbb{Q}$. The modular form attached to this curve has a Fourier expansion

$$f(z) = q - 3q^3 - 3q^5 - 2q^7 + 6q^9 - q^{11} + 9q^{15} - 6q^{17} + 4q^{19} + 6q^{21} + q^{23} + O(q^{25}). \tag{6.1}$$

As we construct a basis of the space $S_2(\Gamma_0(88),\chi_1)$, we can easily express $f(z)$ as a combination of eta-quotients. We obtain

$$f(z) = [2^3,4^{-1},22^3,44^{-1}] - 4[2^{-1},4^3,22^{-1},44^3]. \tag{6.2}$$

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## References

[1] A. Alaca, *Representations by quaternary quadratic forms with coefficients* 1*,* 3*,* 5 *or* 15, Integers **18** (2018), Paper No. A12, 14 pp.
[2] A. Alaca, Ş. Alaca, and Z. S. Aygin, *Theta products and eta quotients of level* 24 *and weight* 2, Funct. Approx. Comment. Math. **57** (2017), no. 2, 205–234. https://doi.org/10.7169/facm/1628
[3] A. Alaca, Ş. Alaca, M. F. Lemire, and K. S. Williams, *Nineteen quaternary quadratic forms*, Acta Arith. **130** (2007), no. 3, 277–310. https://doi.org/10.4064/aa130-3-5

[4] A. Alaca, Ş. Alaca, M. F. Lemire, and K. S. Williams, *The number of representations of a positive integer by certain quaternary quadratic forms*, Int. J. Number Theory **5** (2009), no. 1, 13–40. https://doi.org/10.1142/S1793042109001943

[5] A. Alaca and J. Alanazi, *Representations by quaternary quadratic forms with coefficients* 1*,* 2*,* 7 *or* 14, Integers **16** (2016), Paper No. A55, 16 pp.

[6] A. Alaca and M. Altiary, *Representations by quaternary quadratic forms with coefficients* 1*,* 2*,* 5 *or* 10, Commun. Korean Math. Soc. **34** (2019), no. 1, 27–41. https://doi.org/10.4134/CKMS.c170468

[7] Z. S. Aygin, *Representations by sextenary quadratic forms with coefficients* 1*,* 2*,* 3 *and* 6 *and on newforms in* $S_3(\Gamma_0(24), \chi)$, J. Number Theory **185** (2018), 434–448. https://doi.org/10.1016/j.jnt.2017.09.012

[8] B. C. Berndt, *Number theory in the spirit of Ramanujan*, Student Mathematical Library, 34, American Mathematical Society, Providence, RI, 2006. https://doi.org/10.1090/stml/034

[9] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, in Gesammelte Werke, Erster Band (Chelsea Publishing Co., New York, 1969), pp. 49–239.

[10] L. J. P. Kilford, *Modular Forms*, Imperial College Press, London, 2008. https://doi.org/10.1142/p564

[11] G. Köhler, *Eta products and theta series identities*, Springer Monographs in Mathematics, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-16152-0

[12] B. Köklüce, *Representations by certain octonary quadratic forms with coefficients* 1*,* 2*,* 5 *and* 10, Proc. Math. Sci. **132** (2022), 55. https://doi.org/10.1007/s12044-022-00700-8

[13] G. Ligozat, *Courbes modulaires de genre* 1, Supplément au Bull. Soc. Math. France, Tome 103, no. 3, Société Mathématique de France, Paris, 1975.

[14] J. Liouville, *Sur les deux formes* $x^2 + y^2 + 2(z^2 + t^2)$, J. Math. Pures Appl. **5** (1860), 269–272.

[15] J. Liouville, *Sur les deux formes* $x^2 + y^2 + z^2 + 2t^2, x^2 + 2(y^2 + z^2 + t^2)$, J. Math. Pures Appl. **6** (1861), 225–230.

[16] M. Newman, *Construction and application of a class of modular functions*, Proc. Lond. Math. Soc. (3) **7** (1957), 334–350. https://doi.org/10.1112/plms/s3-7.1.334

[17] M. Newman, *Construction and application of a class of modular functions. II*, Proc. Lond. Math. Soc. (3) **9** (1959), 373–387. https://doi.org/10.1112/plms/s3-9.3.373

[18] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, 79, American Mathematical Society, Providence, RI, 2007. https://doi.org/10.1090/gsm/079

[19] J. Sturm, *On the congruence of modular forms*, in Number theory (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin, 1987. https://doi.org/10.1007/BFb0072985

[20] The LMFDB Collaboration, *The L-functions and modular forms database*, [Online; accessed 10 March 2022]. https://www.lmfdb.org.

[21] K. S. Williams, *Number theory in the spirit of Liouville*, London Mathematical Society Student Texts, 76, Cambridge University Press, Cambridge, 2011.

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