Bull. Korean Math. Soc. **60** (2023), No. 1, pp. 257–279
https://doi.org/10.4134/BKMS.b220116
pISSN: 1015-8634 / eISSN: 2234-3016

STABILITY OF BIFURCATING STATIONARY PERIODIC SOLUTIONS OF THE GENERALIZED SWIFT–HOHENBERG EQUATION

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Abstract. Applying the Lyapunov–Schmidt reduction, we consider spectral stability of small amplitude stationary periodic solutions bifurcating from an equilibrium of the generalized Swift–Hohenberg equation. We follow the mathematical framework developed in [15, 16, 19, 23] to construct such periodic solutions and to determine regions in the parameter space for which they are stable by investigating the movement of the spectrum near zero as parameters vary.

1. Introduction

This paper concerns the existence and stability of bifurcating stationary periodic solutions of the 1D generalized Swift–Hohenberg equation with a quadratic–cubic nonlinearity

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 u + u^2 - 2u^3, \quad t \geq 0,\ u \in \mathbb{R}^1, \tag{1}$$

where $\varepsilon \in \mathbb{R}^1$ is a bifurcation parameter. This model has been used in many physical and biological contexts, in particular, in pattern formation. Various types of solutions to (1) have been extensively studied numerically and analytically. In the present paper we are interested in the stationary periodic solutions bifurcating from a uniform state. For numerous studies of other types of solutions, see, e.g., [1–3, 7, 8, 14, 22] and the references therein.

The equation (1) has the uniform state $u \equiv 0$. The eigenvalue of the linearization of (1) about $u \equiv 0$ takes the form

$$\lambda(k) = -(1-k^2) + \varepsilon^2,$$

so that the instability of $u \equiv 0$ occurs at $\varepsilon = 0$ corresponding to the wave number $k^2 = 1$. Thus, one can expect small amplitude periodic solutions of

Received February 11, 2022; Revised March 24, 2022; Accepted May 4, 2022.
2020 *Mathematics Subject Classification.* Primary 35B35.
Key words and phrases. Generalized Swift–Hohenberg equations, bifurcating periodic solutions, Lyapunov–Schmidt reduction.
This work was financially supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (No. 2022R1F1A1074414).

the form

$$\tilde{u}(x) \approx \varepsilon e^{i(1+\varepsilon\omega)x} + \mathcal{O}(\varepsilon^2) + c.c. \tag{2}$$

bifurcating at $\varepsilon = 0$ from $u \equiv 0$. Here, *c.c.* denotes the complex conjugate and ω, defined in Section 2, is another parameter associated with a wave number k satisfying $k^2 = 1$ at $\omega = 0$. In this work we rigorously construct such solutions (2) and study their spectral stability and instability as the parameters vary.

Our analysis is based entirely on the classical Lyapunov–Schmidt reduction to carry out the characterization of small amplitude periodic solutions $\tilde{u}$ and the spectrum of the linearized operator of (1) about $\tilde{u}$. This method was introduced by Alexander Mielke and Guido Schneider [15,16,19]. In particular, Alexander Mielke developed a general mathematical framework based solely on the Lyapunov–Schmidt reduction, which is more applicable than other methods (e.g., a spatial center manifold) previously used, to study the bifurcation of periodic patterns and their stability. As a first example, they applied the framework to the Swift–Hohenberg equation with a cubic nonlinearity

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 u - u^3, \quad t \geq 0,\ u \in \mathbb{R}^1. \tag{3}$$

According to their results, the equation (3) possess for small $\varepsilon > 0$ and all $4\omega^2 \leq 1$ a two-parametric family of stationary solutions

$$U_{\varepsilon,\omega}(x) = \varepsilon\sqrt{(1-4\omega^2)/3}e^{i(1+\varepsilon\omega)x} + \mathcal{O}(\varepsilon^3) + c.c. \tag{4}$$

bifurcating at $\varepsilon = 0$ from $u \equiv 0$. Moreover, the solutions $U_{\varepsilon,\omega}$ are spectrally stable, i.e., the linearization of (3) about $U_{\varepsilon,\omega}$ has the spectrum on $(-\infty, 0]$ when ω satisfies

$$4\omega^2 < \frac{1}{3}. \tag{5}$$

However, the number $\frac{1}{3}$ is not a coincidence. The stability condition (5) is called Eckhaus criterion that determines the stability of Ginzburg–Landau equation, so that these solutions $U_{\varepsilon,\omega}$ are often called Eckhaus–stable, see [4,6,12,24] for the studies of Eckhaus criterion. Indeed, the existence condition $4\omega^2 \leq 1$ and the stability condition $4\omega^2 < \frac{1}{3}$ can be predicted by an associated formal amplitude equation given by the real Ginzburg–Landau equation [6,15–17,19]

$$\partial_T A = 4\partial_X^2 A + A - 3|A|^2 A, \tag{6}$$

where $(T, X) = (\varepsilon^2 t, \varepsilon x)$ and $A = A(T, X) \in \mathbb{C}$. The equation (6) can be derived by inserting the ansatz $u(x,t) = \varepsilon A(T,X)e^{ix}+c.c.$ into (3) and equating equal powers in ε and e^{ix} (e.g., see [21, Section 10.2]). The solution of (6) is given explicitly by

$$A_\omega(X) = \sqrt{(1-4\omega^2)/3}e^{i\omega X}$$

for all $4\omega^2 \leq 1$, which agrees with (4) at leading order of ε. Moreover, the amplitude A is stable if the parameter satisfies the Eckhaus criterion (5). That is, in [15, 16, 19], their results precisely verified that Ginzburg–Landau formalism provides a valid approximation, for both existence and stability of small bifurcating periodic solutions $U_{\varepsilon,\omega}$.

Another contribution of Guido Schneider is nonlinear stability, called diffusive stability, of such solutions $U_{\varepsilon,\omega}$ [19, 20]. It is well known that by the Floquet theory the spectrum of linear differential operators with spatially periodic coefficients must be entirely essential spectrum (i.e., entirely continuous), and moreover the essential spectrum can be decomposed into an uncountable union of the point spectrum of the Bloch operators, dependent upon the Floquet exponent σ (e.g., see [13, Section 3.3]). Since the Bloch operator at $\sigma = 0$ has a zero eigenvalue and the spectrum has no spectral gap from zero, the movement of the spectral curve bifurcating at $\sigma = 0$ from zero is very important to study nonlinear stability of stationary periodic solutions. In particular, in [19, 20], the characterization of the spectral curve

$$\lambda(\varepsilon, \omega, \sigma) = -C(\varepsilon, \omega)\sigma^2 + h.o.t \tag{7}$$

for some $C > 0$, played a crucial role in the study of nonlinear stability of $U_{\varepsilon,\omega}$. Since then, the diffusive stability in [19, 20] has had a profound effect on the study of nonlinear stability of stationary periodic solutions even for other differential equations (e.g., see [5, 9–11, 18] and the reference therein).

The main purpose of the present paper is to establish similar existence and stability results for the equation (1) by applying the same framework. The difficulty compared to (3) is arising from the quadratic nonlinearity. In our case it is necessary to compute the ε^2–order terms in the reduced equation when applying the Lyapunov–Schmidt reduction, while these terms identically vanish for the case (3). Therefore, we need more effort to calculate many new terms and to ensure that they are still small enough to obtain the desired results from the reduced equation. However, these technical difficulties have also appeared in the recent work [23], in which the Lyapunov–Schmidt reduction approach was applied to the Brusslator model, a system of reaction–diffusion equations with Turing instability. In their analysis, an appropriate scaling between parameters has been used to overcome the computational difficulties and to reframe the stability analysis of the Ginzburg–Landau equation for understanding its connection with the Lyapunov–Schmidt reduction. We follow several computational techniques laid out in [23].

The plan of the paper is as follows. The remaining part of the introduction provides the main results of this paper and future work we are interested in. In the next section, we first outline the Lyapunov–Schmidt reduction which is generally used in our paper. Then we construct a unique branch of stationary periodic solutions $\tilde{u}$ with the accurate ε^2–order terms, and we also compare it to the solution of the associated Ginzberg–Landau equation. In the last section, we study the eigenvalue problem of the Bloch operator for σ sufficiently small by reducing it to a 2×2 eigenvalue problem through the Lyapunov–Schmidt reduction. We then characterize the spectrum near zero to prove that the solution $\tilde{u}$ is Eckhaus–stable.

1.1. Main results

We now state our main theorems. We first make a coordinate change $x \to \xi := kx$, where k is a wave number. Then in the (t, ξ) coordinates, the equation (1) reads

$$\partial_t u = -(1 + k^2\partial_\xi^2)^2 u + \varepsilon^2 u + u^2 - 2u^3, \tag{8}$$

and we look for stationary periodic solutions $\tilde{u}(\xi)$ with period 2π bifurcating from the constant solution $u \equiv 0$. Our first result states the existence of such solutions.

Theorem 1.1 (Existence)**.** *Let ω be a parameter satisfying $k^2 - 1 = 2\varepsilon\omega$. Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ there is a unique (up to translation) stationary 2π–periodic solution $\tilde{u}_{\varepsilon,\omega}(\xi) \in H^4_{per}([0, 2\pi])$ of (8), which is even in ξ and bifurcating from the uniform state $u \equiv 0$. These solutions have the following expansion*

$$\begin{aligned}\tilde{u}_{\varepsilon,\omega}(\xi) = \frac{3}{2}\varepsilon\sqrt{1-4\omega^2}\cos\xi + \varepsilon^2\Big[\frac{9}{8}(1-4\omega^2) - \frac{1}{2}\omega\sqrt{1-4\omega^2}\cos\xi \\ + \frac{1}{8}(1-4\omega^2)\cos 2\xi\Big] + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{9}$$

In particular, if $\omega = \pm\frac{1}{2}$, then $\tilde{u}_{\varepsilon,k} \equiv 0$.

Proof. Given in Section 2. □

Before we state the stability result of $\tilde{u}_{\varepsilon,\omega}$, we briefly discuss the Bloch operators. A detailed analysis is given in Section 3.1. Linearization of (8) about $\tilde{u}_{\varepsilon,\omega}$ yields

$$\mathcal{L}_{\varepsilon,\omega} := -(1 + (1 + 2\varepsilon\omega)\partial_\xi^2)^2 + (\varepsilon^2 + 2\tilde{u}_{\varepsilon,\omega} - 6\tilde{u}^2_{\varepsilon,\omega})$$

acting on $L^2(\mathbb{R})$ with densely defined domain $H^4(\mathbb{R})$. By the standard Floquet theory, the $L^2(\mathbb{R})$–spectrum of $L_{\varepsilon,\omega}$ is purely continuous (i.e., entirely essential) and corresponds with the union of the $L^2_{per}([0, 2\pi])$–eigenvalues of the Bloch operators given by

$$B(\varepsilon, \omega, \sigma) := -(1 + (1 + 2\varepsilon\omega)(\partial_\xi + i\sigma)^2)^2 + (\varepsilon^2 + 2\tilde{u}_{\varepsilon,\omega} - 6\tilde{u}^2_{\varepsilon,\omega}) \ \text{ for } \sigma \in [-\frac{1}{2}, \frac{1}{2})$$

acting on $L^2_{per}([0, 2\pi])$ with densely defined domain $H^4_{per}([0, 2\pi])$. In fact, $B(\varepsilon, \omega, 0)$ has a zero eigenvalue because the original problem is translation invariant. As long as $|\sigma| > \tilde{\sigma}$ for some sufficiently small $\tilde{\sigma} > 0$ the eigenvalues of the Bloch operator have negative upper bound implying the stability of $\tilde{u}_{\varepsilon,\omega}$. Thus, to study the stability of $\tilde{u}_{\varepsilon,\omega}$ it is enough to investigate the eigenvalues, called the critical eigenvalues, of the Bloch operator in a neighborhood of $\sigma = 0$. The following theorem states the stability and instability of the small bifurcating solutions $\tilde{u}_{\varepsilon,\omega}$.

Theorem 1.2 (Stability). *Let ε_0 be taken from Theorem 1.1. Then there exist $\tilde{\varepsilon}_0 \in (0, \varepsilon_0]$ and sufficiently small $\tilde{\sigma} > 0$ such that $\tilde{u}_{\varepsilon,\omega}(\xi)$ is spectrally stable for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$ and all $\omega \in (-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$ with the following two critical eigenvalues*

$$\lambda_1(\varepsilon, \omega, \sigma) \leq -c(\varepsilon^2 + \sigma^2) \quad \text{and} \quad \lambda_2(\varepsilon, \omega, \sigma) \leq -c\sigma^2 \tag{10}$$

for all $|\sigma| < \tilde{\sigma}$ and some constant $c = c(\varepsilon, \omega) > 0$. In particular, $\lambda_2 = 0$ at $\sigma = 0$. Moreover, $\tilde{u}_{\varepsilon,\omega}(\xi)$ is spectrally unstable if $\omega \in [-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}]$.

Proof. Given in Section 3. □

Remark 1.3. From the stability result we see that small periodic solutions with $\omega = 0$, i.e., $k^2 = 1$, are stable.

Remark 1.4. Our results are also established in the generalized Swift–Hohenberg equation with different constants

$$\partial_t u = -(1 + \partial_x^2)^2 u + \varepsilon^2 u + bu^2 - su^3, \quad t \geq 0, \ u \in \mathbb{R}^1,$$

where $s > \frac{38}{27} b^2$. This can be obtained naturally in the process of solving the bifurcation equation in the Lyapunov–Schmidt reduction; see Remark 2.1. For simplicity we set $b = 1$ and $s = 2$ in the present work.

1.2. Discussion

Similarly as in [15, 16, 19], our main theorems rigorously validate the predictions of the formal Ginzburg–Landau approximation regarding existence and stability of periodic solutions bifurcating from a constant solution. The Ginzburg–Landau derivation for our case is given in Section 2.4. However, in Theorem 1.2, the critical eigenvalues (10) are unable to be predicted like (7) because the error terms in the 2×2 reduced eigenvalue equation are not small enough, in particular, where $\frac{\sigma}{\varepsilon}$ is lower bounded away from zero. Indeed, the error terms are the same as that of [23]. Since the Bloch operators of the Swift-Hohenburg equation are self–adjoint, all eigenvalues are real-valued, while self-adjointness was lost in [23]. In that case, the estimates (10) have been replaced by $\Re\lambda_1 \leq -c(\varepsilon^2 + \sigma^2)$ and $\Re\lambda_2 \leq -c\sigma^2$, which are the diffusive stability conditions in general.

In the present work we start the investigation with bifurcating stationary solutions in the 1D generalized Swift–Hohenberg equation. As future work we will study the stability of rolls $u_{rolls}(t, x, y)$, that is, bifurcating stationary solutions which depend only upon one variable in the 2D generalized Swift–Hohenberg equation

$$\partial_t u = -(1 + \Delta)^2 u + \varepsilon^2 u + u^2 - 2u^3, \quad t \geq 0, \ (x, y) \in \mathbb{R}^2, \ u \in \mathbb{R}^1, \tag{11}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator. Indeed, in [15, 16], the stability of roll waves in the Swift–Hohenberg equation with a cubic nonlinear term has been studied. Without loss of generality if we assume the rolls $u_{rolls}(t, x, y)$ are periodic in x and independent in y, then the existence problem of $u_{rolls}(t, x, y)$

in turn becomes the existence of bifurcating stationary solution $\tilde{u}_{\varepsilon,\omega}(x)$ for the 1D case. Therefore, Theorem 1.1 also covers the existence result of such rolls, i.e., $u_{rolls}(t,\xi,y)=\tilde{u}_{\varepsilon,\omega}(\xi)$ in (9) with $\xi=kx$. However, the stability problem of $u_{rolls}(t,x,y)$ is quite different from the 1D case. Linearizing (11) about u_{rolls} gives

$$\mathcal{L}_{\varepsilon,\omega}:=-(1+(1+2\varepsilon\omega)\partial_\xi^2+\partial_y^2)^2+(\varepsilon^2+2\tilde{u}_{\varepsilon,\omega}-6\tilde{u}_{\varepsilon,\omega}^2),$$

and the associated Bloch operators are defined by

$$B(\varepsilon,\omega,\sigma,\gamma):=-(1+(1+2\varepsilon\omega)(\partial_\xi+i\sigma_1)^2-\gamma_2^2)^2+(\varepsilon^2+2\tilde{u}_{\varepsilon,\omega}-6\tilde{u}_{\varepsilon,\omega}^2)$$

for $(\sigma,\gamma)\in[-\frac{1}{2},\frac{1}{2})\times\mathbb{R}$. That is, we need to analyze the spectrum with one more parameter γ, which would cause immense difficulties. After that, the stability of rolls of the Brusselator model [23] would also be interesting direction for future study. As mentioned above, the error terms in the reduced eigenvalue problem in the present 1D case turn out to be the same as that of the Brusselator model. Thus, we expect that the study of rolls of the generalized Swift–Hohenberg equation would provide valid clues for the Brusselator model.

2. Periodic solutions bifurcating from a uniform state

The purpose of this section is to construct a two–parametric family of stationary 2π–periodic solutions $\tilde{u}(\xi)$ of (8) bifurcating from $u\equiv 0$. The problem for $\tilde{u}$ reads

$$0=N(\varepsilon,k,\tilde{u}):=-(1+k^2\partial_\xi^2)^2\tilde{u}+\varepsilon^2\tilde{u}+\tilde{u}^2-2\tilde{u}^3, \tag{12}$$

where $N:\mathbb{R}^2\times H^4_{per}([0,2\pi])\to L^2_{per}([0,2\pi])$ is an analytic mapping. Recalling the instability of u $\equiv 0$ occurs at $\varepsilon=0$ corresponding to the wave number $k^2=1$, we notice that $N(0,\pm1,0)=0$, so that we look for 2π–periodic solutions $\tilde{u}(\xi)$ satisfying the equation (12) in a neighborhood of $(\varepsilon,k,\tilde{u})=(0,\pm1,0)$. Such bifurcating periodic solutions can be found by the Lyapunov–Schmidt reduction. We first provide a sketch of the idea how to apply the Lyapunov–Schmidt reduction.

2.1. Lyapunov–Schmidt reduction for (12)

The main idea of the Lyapunov–Schmidt reduction is to reduce an infinite dimensional problem (12) solving for $\tilde{u}\in H^4_{per}([0,2\pi])$ to an appropriate finite dimensional problem which is equivalent to (12). In order to solve the equation (12) in a neighborhood of $(0,\pm1,0)$, we linearize $N(\varepsilon,k,\tilde{u})$ about $(\varepsilon,k,\tilde{u})=(0,\pm1,0)$ to obtain the linear operator

$$L:=\partial_{\tilde{u}}N(0,\pm1,0)=-(1+\partial_\xi^2)^2,$$

where $L:H^4_{per}([0,2\pi])\subset L^2_{per}([0,2\pi])\to L^2_{per}([0,2\pi])$ is a Fredholm operator. Since the kernel of L, denoted by $ker(L)$, is spanned by

$$U_1(\xi)=\cos\xi \quad\text{and}\quad U_2(\xi)=\sin\xi, \tag{13}$$

the linear operator L is not invertible. This is why the Implicit Function Theorem cannot be used directly. In order to apply the Lyapunov–Schmidt reduction we consider the orthogonal projection of L onto the kernel of L

$$P : L^2_{per}([0, 2\pi]) \to ker(L)$$

defined by

$$Pu := \frac{1}{\pi}\int_0^{2\pi} uU_1 \; d\xi U_1 + \frac{1}{\pi}\int_0^{2\pi} uU_2 \; d\xi U_2,$$

which can also be defined as a vector form

$$\tilde{P}u := \frac{1}{\pi}\Big(\int_0^{2\pi} uU_1 \; d\xi, \int_0^{2\pi} uU_2 \; d\xi\Big)^T \in \mathbb{R}^2.$$

Noting that L is self-adjoint and a Fredholm operator of index zero,

$$PL^2_{per}([0, 2\pi]) = ker(L) \quad \text{and} \quad (I - P)L^2_{per}([0, 2\pi]) = ran(L).$$

Here $ran(L)$ denotes the range of L.

Let us solve the equation (12) by decomposing $\tilde{u} \in H^4_{per}([0, 2\pi])$ into $U + V$, where $U = P\tilde{u} = \alpha_1 U_1 + \alpha_2 U_2$ and $V = (I - P)\tilde{u}$. Then one can rewrite the equation (12) as

$$\begin{aligned} 0 &= PN(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V), \\ 0 &= (I - P)N(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V). \end{aligned} \tag{14}$$

We first focus on the second equation of (14). Upon setting

$$G(\varepsilon, k, U, V) := (I - P)N(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V),$$

we see that $G(0, \pm 1, 0, 0) = (I - P)N(0, \pm 1, 0, 0) = 0$. In particular,

$$\partial_V G(0, \pm 1, 0, 0) = (I - P)\partial_{\tilde{u}} N(0, \pm 1, 0, 0) = (I - P)L$$

which is a bijection between $(I-P)H^4_{per}([0, 2\pi])$ and $ran(L)$. Consequently, by the Implicit Function Theorem there exist a neighborhood of $(0, \pm 1, 0, 0)$ and a unique function $V = V(\varepsilon, k, U) \in \text{ran}(I - P)$ such that $G(\varepsilon, k, U, V(\varepsilon, k, U)) = 0$.

We then substitute $V = V(\varepsilon, k, U)$ into the first equation of (14), and we solve

$$0 = PN(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V(\varepsilon, k, U)). \tag{15}$$

This is frequently referred to as the bifurcation equation (or the reduced equation) for (12), which is an equivalent problem to (12). That is, by solving the bifurcation equation (15) for $U = \alpha_1 U_1 + \alpha_2 U_2$ in two-dimensional space $ker(L)$, one can characterize all small solutions

$$\tilde{u}(\xi) = \alpha_1 U_1 + \alpha_2 U_2 + V(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2) \tag{16}$$

in a neighborhood of $(\varepsilon, k, \tilde{u}) = (0, \pm 1, 0)$.

2.2. The form of the bifurcation equation for (12)

Before we construct stationary periodic solutions (16), we first investigate the form of the bifurcation equation, which is extremely useful in finding $(\alpha_1,\alpha_2)^T$ from (15). Let us set the bifurcation equation (15) in the $\alpha = (\alpha_1,\alpha_2)^T$–vector form (i.e., taking $\tilde{P}$ instead of P in (15)) as

$$\begin{pmatrix}0\\0\end{pmatrix} = f(\varepsilon,k,\alpha) := \begin{pmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix},$$

where $f_{ij} = f_{ij}(\varepsilon,k,\alpha)$ in a neighborhood of $(\varepsilon,k,\tilde{u}) = (0,\pm1,0)$. However, as discussed in [15, 23], the original differential equation (12) is translation invariant ($\xi \mapsto \xi+\eta$) and reflection symmetric ($\xi \mapsto -\xi$). More precisely, by the fact that $(U_1,U_2)(\xi+\eta) = (\cos\eta\, U_1(\xi) - \sin\eta\, U_2(\xi),\ \sin\eta\, U_1(\xi) + \cos\eta\, U_2(\xi))$ and $(U_1,U_2)(-\xi) = (U_1,-U_2)(\xi)$, the bifurcation equation in the α-vector form is also invariant under the symmetries

$$\alpha \mapsto R(\eta)\alpha := \begin{pmatrix}\cos\eta & -\sin\eta\\ \sin\eta & \cos\eta\end{pmatrix}\alpha \quad\text{and}\quad \alpha \mapsto S\alpha := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\alpha,$$

which lead to

$$f(\varepsilon,k,R(\eta)\alpha) = R(\eta)f(\varepsilon,k,\alpha) \quad\text{and}\quad f(\varepsilon,k,S\alpha) = Sf(\varepsilon,k,\alpha).$$

These symmetries give the special form of the bifurcation equation $f(\varepsilon,k,\alpha)$, which is the product of a scalar matrix and the vector α, i.e.,

$$\begin{pmatrix}0\\0\end{pmatrix} = \tilde{f}(\varepsilon,k,\alpha)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$

for a scalar function $\tilde{f}(\varepsilon,k,\alpha)$. Consequently, solving (15) is equivalent to solving

$$0 = \tilde{f}(\varepsilon,k,\alpha)$$

in a neighborhood of $(\varepsilon,k,\alpha) = (0,\pm1,0)$.

2.3. Stationary periodic solutions $\tilde{u}$

We now construct 2π–periodic bifurcating stationary solutions $\tilde{u}_{\varepsilon,k}(\xi)$ of (12). Recalling the decomposition $\tilde{u}_{\varepsilon,k} = \alpha_1U_1 + \alpha_2U_2 + V$ where $PV = 0$, the first step is to solve

$$\begin{aligned} 0 &= (I-P)N(\varepsilon,k,\alpha_1U_1+\alpha_2U_2+V)\\ &= (I-P)(\varepsilon^2 - (1+k^2\partial_\xi^2)^2)(\alpha_1U_1+\alpha_2U_2+V)\\ &\quad + (I-P)(\alpha_1U_1+\alpha_2U_2+V)^2 - 2(I-P)(\alpha_1U_1+\alpha_2U_2+V)^3 \end{aligned} \tag{17}$$

for $V = V(\varepsilon,k,U) \in (I-P)L^2_{per}([0,2\pi])$ locally about (ε,k,U) near $(0,\pm1,0)$. Noting that $(I-P)U_1 = (I-P)U_2 = 0$, $(I-P)V = V$, and $(I-P)L^2_{per}([0,2\pi])$

is invariant under the operator $\varepsilon^2-(1+k^2\partial_\xi^2)^2$, the first term on the right–hand side of (17) can be simplified to

$$\begin{aligned}&(I-P)(\varepsilon^2-(1-k^2)^2)(\alpha_1U_1+\alpha_2U_2)+(I-P)(\varepsilon^2-(1+k^2\partial_\xi^2)^2)V\\ &=(\varepsilon^2-(1+k^2\partial_\xi^2)^2)V.\end{aligned}$$

It follows from (17) that

$$V=V(\varepsilon,k,\alpha_1,\alpha_2)=\mathcal{O}(|\alpha|^2)$$

as $\alpha\to 0$. Indeed, since V is analytic in (α_1,α_2) near $(0,0)$, a direct computation gives $V(\varepsilon,k,0,0)=0$ and $\partial_{\alpha_j}V(\varepsilon,k,0,0)=0$ for $j=1,2$.

Let us find the leading order term of V denoted by $V|_{\mathcal{O}(|\alpha|^2)}$. Collecting the $|\alpha|^2$–order terms in (17) and using $P[U_1^2]=0=P[U_2^2]=P[U_1U_2]$ yield

$$\begin{aligned}&(\varepsilon^2-(1+k^2\partial_\xi^2)^2)V|_{\mathcal{O}(|\alpha|^2)}\\ &=\ -(I-P)(\alpha_1U_1+\alpha_2U_2)^2\\ &=\ -(\alpha_1U_1+\alpha_2U_2)^2\\ &=\ -\frac{1}{2}(\alpha_1^2+\alpha_2^2)-\frac{1}{2}(\alpha_1^2-\alpha_2^2)\cos 2\xi-\alpha_1\alpha_2\sin 2\xi\\ &\in span\{1,\cos 2\xi,\sin 2\xi\}.\end{aligned}\tag{18}$$

Since $span\{1,\cos 2\xi,\sin 2\xi\}$ is an invariant subspace of $(I-P)L^2_{per}([0,2\pi])$ under the operator $\varepsilon^2-(1+k^2\partial_\xi^2)^2$, the leading order term of V has the form

$$V|_{\mathcal{O}(|\alpha|^2)}=c_1+c_2\cos 2\xi+c_3\sin 2\xi$$

for some $c_j=c_j(\varepsilon,k,\alpha_1,\alpha_2)\in\mathbb{R}$, $j=1,2,3$. By inserting this form into the left–hand side of (18), we obtain

$$\begin{aligned}&(\varepsilon^2-1)c_1+(\varepsilon^2-(1-4k^2)^2)(c_2\cos 2\xi+c_3\sin 2\xi)\\ &=\ -\frac{1}{2}(\alpha_1^2+\alpha_2^2)-\frac{1}{2}(\alpha_1^2-\alpha_2^2)\cos 2\xi-\alpha_1\alpha_2\sin 2\xi,\end{aligned}$$

which determines the values c_j for $j=1,2,3$. Then we deduce that

$$V|_{\mathcal{O}(|\alpha|^2)}=\tilde{c}_1(\alpha_1^2+\alpha_2^2)+\tilde{c}_2(\alpha_1^2-\alpha_2^2)\cos 2\xi+2\tilde{c}_2\alpha_1\alpha_2\sin 2\xi,\tag{19}$$

where

$$\tilde{c}_1(\varepsilon,k)=\frac{1}{2(1-\varepsilon^2)}\quad\text{and}\quad\tilde{c}_2(\varepsilon,k)=\frac{1}{2((1-4k^2)^2-\varepsilon^2)}.$$

Next, we look for $V|_{\mathcal{O}(|\alpha|^3)}$ denoting $|\alpha|^3$–order terms of V. After inserting (19) into (17) and collecting the $|\alpha|^3$–order terms in (17), a simple calculation gives

$$\begin{aligned}&(\varepsilon^2-(1+k^2\partial_\xi^2)^2)V|_{\mathcal{O}(|\alpha|^3)}\\ &=\ -(I-P)(\alpha_1U_1+\alpha_2U_2)V|_{\mathcal{O}(|\alpha|^2)}+2(I-P)(\alpha_1U_1+\alpha_2U_2)^3\\ &\in span\{\cos 3\xi,\sin 3\xi\}\end{aligned}$$

which is also an invariant subspace of $(I-P)L^2_{per}([0,2\pi])$ under the operator $\varepsilon^2-(1+k^2\partial_\xi^2)^2$; thus $V|_{\mathcal{O}(|\alpha|^3)}$ has the form

$$V|_{\mathcal{O}(|\alpha|^3)} = c_4\cos 3\xi + c_5\sin 3\xi$$

for some $c_j = c_j(\varepsilon,k,\alpha_1,\alpha_2)$, $j=4,5$. However, we do not require finding the exact expressions of c_4 and c_5. So far, we have shown that the equation of (17) can be solved locally for V:

$$\begin{aligned} V(\varepsilon,k,\alpha_1,\alpha_2) &= \tilde{c}_1(\alpha_1^2+\alpha_2^2)+\tilde{c}_2(\alpha_1^2-\alpha_2^2)\cos 2\xi + 2\tilde{c}_2\alpha_1\alpha_2\sin 2\xi \\ &\quad + c_4\cos 3\xi + c_5\sin 3\xi + \mathcal{O}(|\alpha|^4) \end{aligned} \tag{20}$$

in a neighborhood of $(\varepsilon,k,\alpha_1,\alpha_2)=(0,\pm1,0,0)$.

As the second step we will obtain the reduced bifurcation equation by solving the first equation of (14):

$$\begin{aligned} 0 &= PN(\varepsilon,k,\alpha_1U_1+\alpha_2U_2+V) \\ &= P(\varepsilon^2-(1+k^2\partial_\xi^2)^2)(\alpha_1U_1+\alpha_2U_2+V) + P(\alpha_1U_1+\alpha_2U_2+V)^2 \\ &\quad - 2P(\alpha_1U_1+\alpha_2U_2+V)^3. \end{aligned} \tag{21}$$

Since $PV=0$ and $(I-P)L^2_{per}([0,2\pi])$ is invariant under the operator $\varepsilon^2-(1+k^2\partial_\xi^2)^2$, the first term on the right–hand side of (21) becomes

$$(\varepsilon^2-(1-k^2)^2)(\alpha_1U_1+\alpha_2U_2). \tag{22}$$

We now plug (20) into (21). By recalling the projection (14), the last two terms on the right–hand side of (21) are calculated as

$$\begin{aligned} &P(\alpha_1U_1+\alpha_2U_2)^2 + 2P(\alpha_1U_1+\alpha_2U_2)V + PV^2 \\ &= 2P(\alpha_1U_1+\alpha_2U_2)V|_{\mathcal{O}(|\alpha|^2)} + 2P(\alpha_1U_1+\alpha_2U_2)V|_{\mathcal{O}(|\alpha|^3)} + PV|^2_{\mathcal{O}(|\alpha|^2)} \\ &\quad + \mathcal{O}(|\alpha|^5) \\ &= 2P(\alpha_1U_1+\alpha_2U_2)\Big(\tilde{c}_1(\alpha_1^2+\alpha_2^2)+\tilde{c}_2(\alpha_1^2-\alpha_2^2)\cos 2\xi + 2\tilde{c}_2\alpha_1\alpha_2\sin 2\xi\Big) \\ &\quad + \mathcal{O}(|\alpha|^5) \\ &= (2\tilde{c}_1+\tilde{c}_2)(\alpha_1U_1+\alpha_2U_2) + \mathcal{O}(|\alpha|^5) \end{aligned} \tag{23}$$

and

$$\begin{aligned} &-2P(\alpha_1U_1+\alpha_2U_2)^3 - 6P(\alpha_1U_1+\alpha_2U_2)^2V \\ &-6P(\alpha_1U_1+\alpha_2U_2)V^2 - 2PV^3 \\ =\ &-2P(\alpha_1U_1+\alpha_2U_2)^3 - 6P(\alpha_1U_1+\alpha_2U_2)^2V|_{\mathcal{O}(|\alpha|^2)} + \mathcal{O}(|\alpha|^5) \\ =\ &-\frac{3}{2}(\alpha_1^2+\alpha_2^2)(\alpha_1U_1+\alpha_2U_2) + \mathcal{O}(|\alpha|^5), \end{aligned} \tag{24}$$

respectively. By applying (22)–(24) to (21), all small solutions of (12) satisfy the bifurcation equation

$$0 = \Big[\varepsilon^2-(1-k^2)^2+(2\tilde{c}_1+\tilde{c}_2-\frac{3}{2})(\alpha_1^2+\alpha_2^2)\Big](\alpha_1U_1+\alpha_2U_2) + O(|\alpha|^5).$$

However, as discussed in Section 2.2, the bifurcation equation in the α-vector form has the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \tilde{f}(\varepsilon, k, \alpha) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \tag{25}$$

where

$$\tilde{f}(\varepsilon, k, \alpha) = \varepsilon^2 - (1-k^2)^2 + (2\tilde{c}_1 + \tilde{c}_2 - \frac{3}{2})(\alpha_1^2 + \alpha_2^2) + O(|\alpha|^4). \tag{26}$$

We notice that the bifurcation equation (25) can also be obtained simply by the asymptotic expansion of V with respect to the parameter α_1 by setting $\alpha_2 = 0$ as done in [23] for the Brusselator model.

Proof of Theorem 1.1. We now take without loss of generality $\alpha_2 = 0$ and $\alpha_1 = \alpha$, and we solve (26) for α in terms of ε and k. Upon setting

$$\mathcal{A} = \frac{\varepsilon^2 - (1-k^2)^2}{\frac{3}{2} - 2\tilde{c}_1 - \tilde{c}_2} = \frac{2}{3 - \frac{2}{1-\varepsilon^2} - \frac{1}{(1-4k^2)^2 - \varepsilon^2}}(\varepsilon^2 - (1-k^2)^2), \tag{27}$$

the bifurcation equation $\tilde{f}(\varepsilon, k, \alpha) = 0$ becomes

$$\mathcal{A} - \alpha^2 + O(|\alpha|^4) = 0, \tag{28}$$

which is solvable for small α if and only if $\mathcal{A} \geq 0$. Indeed, by plugging $\alpha = \sqrt{\mathcal{A}}\mathcal{B}$ into (28), we see that a positive solution of (28) has the form

$$\alpha = \sqrt{\mathcal{A}} + \mathcal{O}(|\mathcal{A}|^{3/2}).$$

Let us introduce a new parameter ω defined by

$$1 - k^2 = -2\omega\varepsilon.$$

This scaling is very natural from the Ginzburg–Landau derivation of (1) (see Section 2.4). Then we arrive at

$$\mathcal{A} = \frac{9}{4}\varepsilon^2(1-4\omega^2)\Big(1 - \frac{2}{3}\omega\varepsilon + \mathcal{O}(\varepsilon^2)\Big)$$

for $\varepsilon \to 0$, so that $\mathcal{A} \geq 0$ if and only if $4\omega^2 \leq 1$, that is, $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. In conclusion, there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ a unique positive small solution of (28) exists as

$$\alpha = \sqrt{\mathcal{A}} + \mathcal{O}(|\mathcal{A}|^{3/2}) = \frac{3}{2}\varepsilon\sqrt{1-4\omega^2} - \frac{1}{2}\omega\varepsilon^2\sqrt{1-4\omega^2} + \mathcal{O}(\varepsilon^3).$$

Plugging this expansion into (20) yields

$$\begin{aligned} V(\varepsilon, \omega, \alpha, 0) &= \frac{1}{2}\Big(\frac{1}{1-\varepsilon^2} + \frac{1}{(3+8\omega\varepsilon)^2 - \varepsilon^2}\cos 2\xi\Big)\alpha^2 + \mathcal{O}(|\alpha|^3) \\ &= \frac{9}{8}\varepsilon^2(1-4\omega^2) + \frac{1}{8}\varepsilon^2(1-4\omega^2)\cos 2\xi + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Consequently, it follows from (16) that all small solutions (up to translation) of (12) bifurcating from $u \equiv 0$ have the expansion

$$\begin{aligned}\tilde{u}_{\varepsilon,\omega}(\xi) &= \alpha \cos\xi + V(\varepsilon,\omega,\alpha,0) \\ &= \frac{3}{2}\varepsilon\sqrt{1-4\omega^2}\cos\xi + \varepsilon^2\Big[\frac{9}{8}(1-4\omega^2) - \frac{1}{2}\omega\sqrt{1-4\omega^2}\cos\xi \\ &\quad + \frac{1}{8}(1-4\omega^2)\cos 2\xi\Big] + \mathcal{O}(\varepsilon^3)\end{aligned} \tag{29}$$

for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. It completes the proof of Theorem 1.1. □

Remark 2.1. If we apply the Lyapunov–Schmidt reduction to the generalized Swift–Hohenberg equation

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 u + bu^2 - su^3, \quad t \geq 0,\ u \in \mathbb{R}^1,$$

then $\mathcal{A}$ in (27) is replaced by

$$\mathcal{A} = \frac{\varepsilon^2 - (1-k^2)^2}{\frac{3}{4}s - \frac{b^2}{1-\varepsilon^2} - \frac{b^2}{2((1-4k^2)^2-\varepsilon^2)}},$$

which is equal to (27) if $b = 1$ and $s = 2$. In order to follow the proof of Theorem 1.1, the denominator of $\mathcal{A}$ must be positive in a small neighborhood of $(\varepsilon, k^2) = (0, 1)$. Thus, we obtain

$$\frac{3}{4}s - \frac{b^2}{1-\varepsilon^2} - \frac{b^2}{2((1-4k^2)^2-\varepsilon^2)} \approx \frac{3}{4}s - b^2 - \frac{b^2}{18} > 0,$$

that is, $s > \frac{38}{27}b^2$.

2.4. Ginzburg–Landau derivation

We now derive the associated amplitude equation given by Ginzburg–Landau equation. Because of the quadratic nonlinearity of (1), the ansatz $u(x,t) = \varepsilon A(T,X)e^{ix} + c.c.$ used in (3) cannot give an appropriate compatibility condition. In our case, similarly as in [23], the derivation is based on the ansatz

$$\begin{aligned}u(t,x) &\approx U_A(T,X) \\ &= \frac{1}{2}\varepsilon A(T,X)e^{ix} + c.c. \\ &\quad + \varepsilon^2(\Psi_0(T,X) + \Psi_1(T,X)e^{ix} + \Psi_2(T,X)e^{i2x} + c.c.) \\ &\quad + \varepsilon^3(\Psi_3(T,X)e^{ix} + \Psi_4(T,X)e^{i3x} + c.c.) + h.o.t.,\end{aligned}$$

where $(T,X) = (\varepsilon^2 t, \varepsilon x)$ and $\Psi_j(T,X) \in \mathbb{C}$. We substitute this ansatz into (1) and collect equal powers in ε and e^{ix}, so that we obtain

$$\Psi_0 = \frac{1}{2}|A|^2 \quad \text{and} \quad \Psi_2 = \frac{1}{36}A^2 \tag{30}$$

by collecting $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^2 e^{i2x})$ terms, respectively. Moreover, collecting the $\mathcal{O}(\varepsilon^3 e^{ix})$ terms yields the compatibility condition

$$\partial_T A = 4\partial_X^2 A + A + 2A\Psi_0 + 2\bar{A}\Psi_2 - \frac{3}{2}|A|^2 A.$$

Thus, we deduce from (30) the Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + A - \frac{4}{9}A|A|^2$$

and it has the explicit solution

$$A_\omega(X) = \frac{3}{2}\sqrt{1-4\omega^2}e^{i\omega X}$$

for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$, which agrees with $\tilde{u}_{\varepsilon,\omega}$ described in (29) at leading order of ε. We refer the reader to the existence and stability analysis of the Ginzburg-Landau equation to understand its connection with Lyapunov-Schmidt reduction in [23, Section 4].

3. Stability and instability of periodic solutions

In this section we study the stability and instability of the bifurcating periodic solutions $\tilde{u}_{\varepsilon,\omega}$ established in Section 2. To begin, linearizing (12) about $\tilde{u}_{\varepsilon,\omega}$ described in (29) yields

$$\mathcal{L}_{\varepsilon,\omega} v := -(1 + (1+2\varepsilon\omega)\partial_\xi^2)^2 v + \mathcal{F}(\tilde{u}_{\varepsilon,\omega})v \tag{31}$$

acting on $L^2(\mathbb{R})$ with densely defined domain $H^4(\mathbb{R})$, where

$$\begin{aligned}\mathcal{F}(\tilde{u}_{\varepsilon,\omega}) = 3\varepsilon\sqrt{1-4\omega^2}\cos\xi + \varepsilon^2\Big[1 - \frac{9}{2}(1-4\omega^2) - \omega\sqrt{1-4\omega^2}\cos\xi \\ - \frac{13}{2}(1-4\omega^2)\cos 2\xi\Big] + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{32}$$

We first discuss the Bloch operator which is necessary to study an eigenvalue problem of linear differential equations with spatially periodic coefficients.

3.1. Bloch operators

Since $\tilde{u}_{\varepsilon,\omega}$ is 2π–periodic on $\mathbb{R}$, every coefficient of the linear operator $\mathcal{L}_{\varepsilon,\omega}$ is also 2π–periodic. Introducing $Y = (v, \partial_\xi v, \partial_\xi^2 v, \partial_\xi^3 v)^T$, the eigenvalue problem of $\mathcal{L}_{\varepsilon,\omega}$ for $v \in H^4(\mathbb{R})$ becomes a first–order ODE system

$$\partial_\xi Y = A(\xi,\lambda)Y, \quad A(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\lambda-1+\mathcal{F}(\tilde{u}_{\varepsilon,\omega})}{(1+2\varepsilon\omega)^2} & 0 & -2/(1+2\varepsilon\omega) & 0 \end{pmatrix}, \tag{33}$$

where $A(\xi+2\pi,\lambda) = A(\xi,\lambda)$ for all $\lambda \in \mathbb{C}$. If we apply the Floquet theory [13, Section 2.1.3] to the system (33), v must take the form

$$v(\xi) = e^{i\sigma\xi}W(\xi,\lambda), \quad W(\xi+2\pi,\lambda) = W(\xi,\lambda) \tag{34}$$

for the Floquet exponent $\sigma \in \mathbb{R}$. It follows from the fact v cannot lie in $H^4(\mathbb{R})$ that the linear operator $\mathcal{L}_{\varepsilon,\omega} : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has no point spectrum, so that the spectrum must be entirely essential spectrum. To determine the essential spectrum of $\mathcal{L}_{\varepsilon,\omega}$, we substitute (34) into (31) and define the σ-dependent operator, called the Bloch operator: for $\sigma \in [-\frac{1}{2}, \frac{1}{2})$,

$$B(\varepsilon,\omega,\sigma)W := -(1+(1+2\varepsilon\omega)(\partial_\xi + i\sigma)^2)^2 W + \mathcal{F}(\tilde{u}_{\varepsilon,\omega})W \tag{35}$$

acting on $L^2_{per}([0,2\pi])$ with densely defined domain $H^4_{per}([0,2\pi])$. Here, the Bloch operators can be defined for $\sigma \in [-\frac{1}{2}, \frac{1}{2})$ because for any $\sigma \in \mathbb{R}$, $\sigma = \sigma^* + m$ for some $\sigma^* \in [-\frac{1}{2}, \frac{1}{2})$ and $m \in \mathbb{Z}$; so $W(\xi,\lambda)$ in (34) can be replaced by $e^{im\xi}W(\xi,\lambda)$. We also notice that $L^2(\mathbb{R})$–essential spectrum of $\mathcal{L}_{\varepsilon,\omega}$ is comprised of the $L^2_{per}([0,2\pi])$–eigenvalues of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in [-\frac{1}{2}, \frac{1}{2})$ (see [13, Section 3.3] for further details). Thus, the main purpose of this section is to characterize the eigenvalues of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in [-\frac{1}{2}, \frac{1}{2})$.

Let us focus on the following eigenvalue problems of $B(\varepsilon,\omega,\sigma)$ on $L^2_{per}([0,2\pi])$

$$0 = \Big[B(\varepsilon,\omega,\sigma) - \lambda\Big]W \tag{36}$$

for $\sigma \in [-\frac{1}{2}, \frac{1}{2})$. However, it is not necessary to consider (36) for all $\sigma \in [-\frac{1}{2}, \frac{1}{2})$. Recalling ε is a positive bifurcation parameter from $\varepsilon = 0$, we might consider the Bloch operators $B(\varepsilon,\omega,\sigma)$ as small perturbations of $B_0(\sigma)$ defined by

$$B_0(\sigma)\phi_m := B(0,\omega,\sigma)\phi_m = -(1+(\partial_\xi + i\sigma)^2)^2\phi_m. \tag{37}$$

Since $B_0(\sigma)$ has constant coefficients, we easily solve the eigenvalue problem of $B_0(\sigma)$ by plugging $\phi_m(\xi) = e^{im\xi}$ with $m \in \mathbb{Z}$ into (37). Then the eigenvalues $\mu_m(\sigma)$ of $B_0(\sigma)$ are given by

$$\mu_m(\sigma) = -(1-(m+\sigma)^2)^2 \le 0,$$

so that $\mu_m(\sigma) = 0$ if and only if $(m+\sigma)^2 = 1$. Since $m \in \mathbb{Z}$ and $\sigma \in [-\frac{1}{2}, \frac{1}{2})$, the eigenvalues $\mu_m(\sigma)$ are away from 0 if $m \neq \pm 1$ or σ is bounded away from 0, which affords that it is enough to consider the eigenvalues of $B(\varepsilon,\omega,\sigma)$ only for $\sigma \in \Gamma$, where

$$\Gamma = \{\sigma \in [-\frac{1}{2}, \frac{1}{2}) \mid -\eta < \sigma < \eta\}$$

for some small $\eta > 0$. Indeed, as long as ε is sufficiently small and σ is bounded away from 0, the eigenvalues of the constant–coefficient operator $-(1+(1+2\varepsilon\omega)(\partial_\xi+i\sigma)^2)^2$ have negative upper bound. Moreover, since $\mathcal{F}(\tilde{u}_{\varepsilon,\omega})$ represents a bounded small perturbation, the eigenvalues of $B(\varepsilon,\omega,\sigma)$ also have negative upper bound for $\sigma \in [-\frac{1}{2}, \frac{1}{2}) \setminus \Gamma$.

We notice that the Bloch operator $B_0(\sigma)$ on Γ has only two critical eigenfunctions $\phi_{\pm 1} = e^{\pm i\xi}$ (i.e., $m = \pm 1$) associated with small eigenvalues $\mu_{\pm 1}$, respectively. That is, for $\sigma \in \Gamma$, the eigenvectors of $B(\varepsilon,\omega,\sigma)$ can be considered as small perturbations of $e^{\pm i\xi}$ as well. Thus, it is very natural to solve the eigenvalue problem (36) in two dimension by the Lyapunov–Schmidt reduction.

3.2. Eigenvalue problems of the Bloch operators

In order to apply the Lyapunov–Schmidt reduction to (36) for $\sigma \in \Gamma$, we first set

$$U_1 := \cos\xi = \frac{e^{i\xi} + e^{-i\xi}}{2} \quad \text{and} \quad U_2 := \sin\xi = \frac{e^{i\xi} - e^{-i\xi}}{2i},$$

and define the orthogonal projection in $L^2_{per}([0,2\pi])$ onto $\text{span}\{U_1, U_2\}$:

$$PW = \frac{1}{\pi}\int_0^{2\pi} W\overline{U}_1 \; d\xi U_1 + \frac{1}{\pi}\int_0^{2\pi} W\overline{U}_2 \; d\xi U_2,$$

or equivalently as a vector form

$$\tilde{P}W = \frac{1}{\pi}\Big(\int_0^{2\pi} W\overline{U}_1 \; d\xi, \int_0^{2\pi} W\overline{U}_2 \; d\xi\Big)^T \in \mathbb{C}^2. \tag{38}$$

In this section, to be compatible with calculations in Section 2, we use the same basis functions as (13). Later, we will compare the 2×2 reduced eigenvalue equation of (36) at $(\sigma, \lambda) = (0,0)$ with the bifurcation equation (25) because $B(\varepsilon, \omega, 0)$ has a zero eigenvalue with an associated eigenfunction $\partial_\xi \tilde{u}_{\varepsilon,\omega}(\xi)$.

Decomposing $W \in L^2_{per}([0,2\pi])$ into $W = \beta_1 U_1 + \beta_2 U_2 + \mathcal{V}$ with $\beta := (\beta_1, \beta_2) \in \mathbb{C}^2$ and $P\mathcal{V} = 0$, we rewrite (36) as

$$0 = \tilde{P}\Big[B(\varepsilon,\omega,\sigma) - \lambda\Big](\beta_1 U_1 + \beta_2 U_2 + \mathcal{V}), \tag{39}$$

$$0 = (I - P)\Big[B(\varepsilon,\omega,\sigma) - \lambda\Big](\beta_1 U_1 + \beta_2 U_2 + \mathcal{V}). \tag{40}$$

As mentioned in Section 2, the second equation (40) can be uniquely solved locally for $\mathcal{V} = \mathcal{V}(\varepsilon, \omega, \sigma, \lambda, \beta)$ by the Implicit Function Theorem due to $B(0, \omega, \sigma)$ on $ran(I - P)$ is bijective. Since it is clear that $\mathcal{V}$ has the form

$$\mathcal{V} = \mathcal{V}_1(\varepsilon,\omega,\sigma,\lambda)\beta_1 + \mathcal{V}_2(\varepsilon,\omega,\sigma,\lambda)\beta_2,$$

let us find the asymptotic expansions of $\mathcal{V}_1$ and $\mathcal{V}_2$ with respect to the bifurcation parameter ε

$$\mathcal{V}_i = \mathcal{V}_i(0,\omega,\sigma,\lambda) + \partial_\varepsilon \mathcal{V}_i(0,\omega,\sigma,\lambda)\varepsilon + \mathcal{O}(\varepsilon^2), \quad i = 1,2.$$

In order to compute $\mathcal{V}_j(0,\omega,\sigma,\lambda)$ for $j = 1,2$, we differentiate (40) with respect to β_j and insert $\varepsilon = 0$:

$$\begin{aligned} 0 &= (I - P)\Big[B(0,\omega,\sigma) - \lambda\Big](U_j + \mathcal{V}_j(0,\omega,\sigma,\lambda)) \\ &= (I - P)\Big[B(0,\omega,\sigma) - \lambda\Big]\mathcal{V}_j(0,\omega,\sigma,\lambda). \end{aligned}$$

Since the operator $B(0,\omega,\sigma)$ on $ran(I - P)$ is bijective, we arrive at

$$\mathcal{V}_j(0,\omega,\sigma,\lambda) = 0, \quad i = 1,2. \tag{41}$$

Next, we compute $\partial_\varepsilon \mathcal{V}_j|_{\varepsilon=0}$ for $j=1,2$. After differentiating (40) with respect to β_j and ε, plugging $\varepsilon=0$ yields that for each $i=1,2$,

$$\begin{aligned}
&(I-P)\Big[B(0,\omega,\sigma)-\lambda\Big]\partial_\varepsilon \mathcal{V}_j|_{\varepsilon=0}\\
=&-(I-P)\Big[B_\varepsilon(0,\omega,\sigma)\Big]U_j\\
=&-(I-P)\Big[-4\omega(1+(\partial_\xi+i\sigma)^2)(\partial_\xi+i\sigma)^2+3\sqrt{1-4\omega^2}\cos\xi\Big]U_j\\
=&-(I-P)\Big[3\sqrt{1-4\omega^2}\cos\xi\Big]U_j,
\end{aligned}$$

where in the second equality we used (41). Since $P[\cos^2\xi]=0=P[\cos\xi\sin\xi]$,

$$(I-P)\Big[B(0,\omega,\sigma)-\lambda\Big](\partial_\varepsilon \mathcal{V}_1|_{\varepsilon=0})=\frac{-3\sqrt{1-4\omega^2}}{2}(1+\cos 2\xi) \tag{42}$$

and

$$(I-P)\Big[B(0,\omega,\sigma)-\lambda\Big](\partial_\varepsilon \mathcal{V}_2|_{\varepsilon=0})=\frac{-3\sqrt{1-4\omega^2}}{2}\sin 2\xi. \tag{43}$$

It follows that each leading order term takes the form

$$\partial_\varepsilon \mathcal{V}_1|_{\varepsilon=0}=h_1+h_2\cos 2\xi+h_3\sin 2\xi \quad \text{and} \quad \partial_\varepsilon \mathcal{V}_2|_{\varepsilon=0}=r_1\cos 2\xi+r_2\sin 2\xi$$

for some functions h_1, h_2, h_3, r_1 and r_2 about $(\varepsilon,\omega,\lambda)$. Plugging these forms into (42) and (43), respectively, gives

$$\begin{aligned}
&\frac{-3\sqrt{1-4\omega^2}}{2}(1+\cos 2\xi)\\
&=(I-P)\Big[B(0,\omega,\sigma)-\lambda\Big](h_1+h_2\cos 2\xi+h_3\sin 2\xi)\\
&=(I-P)\Big[-(1+(\partial_\xi+i\sigma)^2)^2-\lambda\Big](h_1+h_2\cos 2\xi+h_3\sin 2\xi)\\
&=\Big[-(1-\sigma^2)^2-\lambda\Big]h_1\\
&\quad+\Big[\Big(-(3+\sigma^2)^2-16\sigma^2-\lambda\Big)h_2+8i\sigma(3+\sigma^2)h_3\Big]\cos 2\xi\\
&\quad+\Big[\Big(-(3+\sigma^2)^2-16\sigma^2-\lambda\Big)h_3-8i\sigma(3+\sigma^2)h_2\Big]\sin 2\xi
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\frac{-3\sqrt{1-4\omega^2}}{2}\sin 2\xi\\
&=(I-P)\Big[B(0,\omega,\sigma)-\lambda\Big](r_1\cos 2\xi+r_2\sin 2\xi)\\
&=(I-P)\Big[-(1+(\partial_\xi+i\sigma)^2)^2-\lambda\Big](r_1\cos 2\xi+r_2\sin 2\xi)\\
&=\Big[\Big(-(3+\sigma^2)^2-16\sigma^2-\lambda\Big)r_2-8i\sigma(3+\sigma^2)r_1\Big]\sin 2\xi\\
&\quad+\Big[\Big(-(3+\sigma^2)^2-16\sigma^2-\lambda\Big)r_1+8i\sigma(3+\sigma^2)r_2\Big]\cos 2\xi,
\end{aligned} \tag{45}$$

which determine the expressions of h_1, h_2, h_3, r_1 and r_2. Thus, the equation (40) can be solved locally for $\mathcal{V}$:

$$
\begin{aligned}
\mathcal{V} = & \Big[(h_1 + h_2 \cos 2\xi + h_3 \sin 2\xi)\varepsilon + \mathcal{O}(\varepsilon^2)\Big]\beta_1 \\
& + \Big[(r_1 \cos 2\xi + r_2 \sin 2\xi)\varepsilon + \mathcal{O}(\varepsilon^2)\Big]\beta_2,
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
h_1 &= \frac{-3\sqrt{1-4\omega^2}}{2[-(1-\sigma^2)^2 - \lambda]} = \frac{3\sqrt{1-4\omega^2}}{2(1-\sigma^2)^2 + 2\lambda} \\
&= \frac{3\sqrt{1-4\omega^2}}{2} + \mathcal{O}(\sigma^2 + |\lambda|), \\
h_2 &= \frac{-3\sqrt{1-4\omega^2}}{2} \frac{-(3+\sigma^2)^2 - 16\sigma^2 - \lambda}{\Big[(3+\sigma^2)^2 + 16\sigma^2 + \lambda\Big]^2 - 64\sigma^2(3+\sigma^2)^2} \\
&= \frac{\sqrt{1-4\omega^2}}{6} + \mathcal{O}(\sigma^2 + |\lambda|), \\
h_3 &= \frac{-3\sqrt{1-4\omega^2}}{2} \frac{8i\sigma(3+\sigma^2)}{\Big[(3+\sigma^2)^2 + 16\sigma^2 + \lambda\Big]^2 - 64\sigma^2(3+\sigma^2)^2} \\
&= \mathcal{O}(|\sigma|(1+|\lambda|)), \\
r_1 &= \frac{-3\sqrt{1-4\omega^2}}{2} \frac{8i\sigma(3+\sigma^2)}{-\Big[(3+\sigma^2)^2 + 16\sigma^2 + \lambda\Big]^2 + 64\sigma^2(3+\sigma^2)^2} \\
&= -h_3, \quad \text{and} \\
r_2 &= \frac{-3\sqrt{1-4\omega^2}}{2} \frac{(3+\sigma^2)^2 + 16\sigma^2 + \lambda}{-\Big[(3+\sigma^2)^2 + 16\sigma^2 + \lambda\Big]^2 + 64\sigma^2(3+\sigma^2)^2} = h_2.
\end{aligned}
\tag{47}
$$

Let us find the 2×2 reduced eigenvalue problem of (36) by solving (39). Recalling (32) and (35), we first compute $P\Big[B(\varepsilon, \omega, \sigma) - \lambda\Big](\beta_1 U_1 + \beta_2 U_2)$ as follows:

$$
\begin{aligned}
& P\Big[-(1 + (1+2\omega\varepsilon)(\partial_\xi + i\sigma)^2)^2 - \lambda\Big](\beta_1 U_1 + \beta_2 U_2) \\
&= U_1\Big[-\Big((2\omega\varepsilon + (1+2\omega\varepsilon)\sigma^2)^2 + 4\sigma^2(1+2\omega\varepsilon)^2 + \lambda\Big)\beta_1 \\
&\quad + 4i\sigma(1+2\omega\varepsilon)(2\omega\varepsilon + (1+2\omega\varepsilon)\sigma^2)\beta_2\Big] \\
&\quad + U_2\Big[-4i\sigma(1+2\omega\varepsilon)(2\omega\varepsilon + (1+2\omega\varepsilon)\sigma^2)\beta_1 \\
&\quad - \Big((2\omega\varepsilon + (1+2\omega\varepsilon)\sigma^2)^2 + 4\sigma^2(1+2\omega\varepsilon)^2 + \lambda\Big)\beta_2\Big]
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
&P\Big[\mathcal{F}(\tilde{u}_{\varepsilon,\omega})\Big](\beta_1 U_1+\beta_2 U_2)\\
&= P\Big[\varepsilon^2\Big(1-\frac{9}{2}(1-4\omega^2)-\frac{13}{2}(1-4\omega^2)\cos 2\xi\Big)+\mathcal{O}(\varepsilon^3)\Big](\beta_1 U_1+\beta_2 U_2)\\
&= U_1\Big[1-\frac{9}{2}(1-4\omega^2)-\frac{13}{4}(1-4\omega^2)\Big]\varepsilon^2\beta_1\\
&\quad + U_2\Big[1-\frac{9}{2}(1-4\omega^2)+\frac{13}{4}(1-4\omega^2)\Big]\varepsilon^2\beta_2+\mathcal{O}(\varepsilon^3)(\beta_1 U_1+\beta_2 U_2)\\
&= U_1\Big[1-\frac{31}{4}(1-4\omega^2)\Big]\varepsilon^2\beta_1+U_2\Big[1-\frac{5}{4}(1-4\omega^2)\Big]\varepsilon^2\beta_2\\
&\quad +\mathcal{O}(\varepsilon^3)(\beta_1+\beta_2)(U_1+U_2),
\end{aligned}
\tag{49}
$$

where in the second equality we used $P[\cos 2\xi\cos\xi]=\frac{1}{2}U_1$ and $P[\cos 2\xi\sin\xi]=-\frac{1}{2}U_2$. Inserting (46) into $P\Big[B(\varepsilon,\omega,\sigma)-\lambda\Big]\mathcal{V}$ and using $P[\cos\xi\sin 2\xi]=\frac{1}{2}U_2$, we deduce that

$$
\begin{aligned}
&P\Big[B(\varepsilon,\omega,\sigma)-\lambda\Big]\mathcal{V}\\
&= P\Big[\mathcal{F}(\tilde{u}_{\varepsilon,\omega})\Big]\mathcal{V}\\
&= P\Big[3\varepsilon\sqrt{1-4\omega^2}\cos\xi\Big]\mathcal{V}+\mathcal{O}(\varepsilon^3)(\beta_1+\beta_2)(U_1+U_2)\\
&= U_1\Big[3\varepsilon^2\sqrt{1-4\omega^2}\Big(h_1+\frac{1}{2}h_2\Big)\beta_1+\Big(\frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}r_1\Big)\beta_2\Big]\\
&\quad + U_2\Big[\Big(\frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}h_3\Big)\beta_1+\Big(\frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}r_2\Big)\beta_2\Big]\\
&\quad +\mathcal{O}(\varepsilon^3)(\beta_1+\beta_2)(U_1+U_2).
\end{aligned}
\tag{50}
$$

Recalling (38) and applying (48)–(50) to (39), we obtain a 2×2 matrix $\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)$ satisfying

$$
\begin{pmatrix}0\\0\end{pmatrix}=\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}:=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},
\tag{51}
$$

where each entry of $\mathcal{M}$ is estimated as

$$
\begin{aligned}
m_{11} &= -(2\omega\varepsilon+(1+2\omega\varepsilon)\sigma^2)^2-4\sigma^2(1+2\omega\varepsilon)^2-\lambda+\varepsilon^2\\
&\qquad -\frac{31}{4}\varepsilon^2(1-4\omega^2)+3\varepsilon^2\sqrt{1-4\omega^2}(h_1+\frac{1}{2}h_2)+\mathcal{O}(\varepsilon^3)\\
&= -2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3)-4\sigma^2-\lambda+\mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
m_{12} &= 4i\sigma(1+2\omega\varepsilon)(2\omega\varepsilon+(1+2\omega\varepsilon)\sigma^2)+\frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}r_1+\mathcal{O}(\varepsilon^3)\\
&= 8i\sigma\omega\varepsilon+\mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|)+\sigma^3)+\mathcal{O}(\varepsilon^3),
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
m_{21} &= -4i\sigma(1+2\omega\varepsilon)(2\omega\varepsilon+(1+2\omega\varepsilon)\sigma^2) - \frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}r_1 + \mathcal{O}(\varepsilon^3) \\
&= -8i\sigma\omega\varepsilon + \mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|)+\sigma^3) + \mathcal{O}(\varepsilon^3),
\end{aligned} \tag{54}
$$

and

$$
\begin{aligned}
m_{22} &= -(2\omega\varepsilon+(1+2\omega\varepsilon)\sigma^2)^2 - 4\sigma^2(1+2\omega\varepsilon)^2 - \lambda + \varepsilon^2 \\
&\qquad - \frac{5}{4}\varepsilon^2(1-4\omega^2) + \frac{3}{2}\varepsilon^2\sqrt{1-4\omega^2}r_2 + \mathcal{O}(\varepsilon^3) \\
&= -4\sigma^2 - \lambda + \mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{55}
$$

Since all the Bloch operators $B(\varepsilon,\omega,\sigma)$ are self-adjoint, the matrix $\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)$ is Hermitian, so that the main diagonal entries are real and the off diagonal entries satisfy $m_{12}=\overline{m}_{21}$. Moreover, one can easily see that the Bloch operator satisfies

$$
B(\varepsilon,\omega,\sigma) = R_j B(\varepsilon,\omega,-\sigma) R_j, \quad j=1,2, \tag{56}
$$

where R_1 and R_2 are reflection symmetries defined by $(R_1W)(\xi)=W(-\xi)$ and $(R_2W)(\xi)=\overline{W}(\xi)$, respectively [15, 16, 23]. These two symmetries imply that

$$
\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)\begin{pmatrix}\beta_1\\-\beta_2\end{pmatrix} = \begin{pmatrix} m_{11}(\varepsilon,\omega,-\sigma,\lambda) & m_{12}(\varepsilon,\omega,-\sigma,\lambda) \\ -m_{21}(\varepsilon,\omega,-\sigma,\lambda) & -m_{22}(\varepsilon,\omega,-\sigma,\lambda)\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}
$$

and $\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)=\overline{\mathcal{M}}(\varepsilon,\omega,-\sigma,\lambda)$. From these observations we deduce that the diagonal entries are even in σ and real, while the off diagonal entries are odd in σ and purely imaginary. One more thing to investigate here is the $\mathcal{O}(\varepsilon^3)$ terms in (53)–(55). The refined estimates of the $\mathcal{O}(\varepsilon^3)$ terms are demonstrated by the following lemma.

Lemma 3.1. *At $\sigma=0$ and $\lambda=0$, the matrix $\mathcal{M}$ becomes*

$$
\mathcal{M}(\varepsilon,\omega,0,0) = \begin{pmatrix} -2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3) & 0 \\ 0 & 0\end{pmatrix}.
$$

Proof. By recalling (12) and (35), we notice that the Bloch operator $B(\varepsilon,\omega,0)$ is obtained by differentiating $N(\varepsilon,\omega,\tilde{u})$ with respect to $\tilde{u}$ and plugging $\alpha_2=0$. It follows that for each $i=1,2$,

$$
\begin{aligned}
\partial_{\alpha_i}N(\varepsilon,\omega,\alpha_1U_1+\alpha_2U_2+V)|_{\alpha_2=0} &= \partial_{\tilde{u}}N(\varepsilon,\omega,\tilde{u})(U_i+\partial_{\alpha_i}V)|_{\alpha_2=0} \\
&= B(\varepsilon,\omega,0)(U_i+\partial_{\alpha_i}V|_{\alpha_2=0}).
\end{aligned}
$$

Thus, for each $i=1,2$ we differentiate the second equation of (14) and (40) with respect to α_1 and β_1, respectively, to obtain

$$
0=(I-P)\partial_{\alpha_i}N(\varepsilon,k,\alpha_1U_1+\alpha_2U_2+V)|_{\alpha_2=0} = (I-P)B(\varepsilon,\omega,0)(U_i+\partial_{\alpha_i}V|_{\alpha_2=0})
$$

and

$$
0=(I-P)\partial_{\beta_i}B(\varepsilon,\omega,0)(\beta_1U_1+\beta_2U_2+\mathcal{V}) = (I-P)B(\varepsilon,\omega,0)(U_i+\mathcal{V}_i).
$$

Since the linear operator $B(\varepsilon,\omega,0)$ on $ran(I-P)$ is bijective,

$$
\partial_{\alpha_i}V|_{\alpha_2=0} = \mathcal{V}_i(\varepsilon,\omega,0), \quad i=1,2.
$$

We now compare the matrix $\mathcal{M}(\varepsilon,\omega,0,0)$ with (25). Noting that $(m_{1i},m_{2i})^T$ is the derivative of (39) with respect to β_i for each $i=1,2$, we deduce that

$$\begin{aligned}\begin{pmatrix} m_{11} \\ m_{21}\end{pmatrix} &= \partial_{\alpha_1}\begin{pmatrix}\tilde{f}\alpha_1 \\ \tilde{f}\alpha_2\end{pmatrix}\Big|_{\alpha_2=0} \\ &= \begin{pmatrix}\alpha_1\tilde{f}_{\alpha_1}+\tilde{f} \\ \alpha_2\tilde{f}_{\alpha_1}\end{pmatrix}\Big|_{\alpha_2=0} = \begin{pmatrix}\varepsilon^2(1-4\omega^2)+3(2\tilde{c}_1+\tilde{c}_2-\frac{3}{2})\alpha_1^2+\mathcal{O}(\alpha_1^4) \\ 0\end{pmatrix}\end{aligned}$$

and

$$\begin{pmatrix} m_{12} \\ m_{22}\end{pmatrix} = \partial_{\alpha_2}\begin{pmatrix}\tilde{f}\alpha_1 \\ \tilde{f}\alpha_2\end{pmatrix}\Big|_{\alpha_2=0} = \begin{pmatrix}\alpha_1\tilde{f}_{\alpha_2} \\ \alpha_2\tilde{f}_{\alpha_2}+\tilde{f}\end{pmatrix}\Big|_{\alpha_2=0} = \begin{pmatrix}\mathcal{O}(\alpha_2) \\ \tilde{f}\end{pmatrix}\Big|_{\alpha_2=0} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

Recalling (26) gives $m_{11}=-2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3)$, which completes the proof. □

By the reflection symmetries (56) and Lemma 3.1, we see that the $\mathcal{O}(\varepsilon^3)$ terms in (53)–(55) are absorbed into the terms $\mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|))$ and $\mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|)$, respectively; consequently, the matrix $\mathcal{M}$ in (51) can be written as

$$\begin{aligned}\mathcal{M}(\varepsilon,\omega,\sigma,\lambda) &= \begin{pmatrix} c(\varepsilon,\omega)-4\sigma^2-\lambda & 8i\sigma\omega\varepsilon \\ -8i\sigma\omega\varepsilon & -4\sigma^2-\lambda\end{pmatrix} \\ &\quad + \begin{pmatrix}\mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4) & \mathcal{O}(\varepsilon^2\sigma(1+|\lambda|)+\sigma^3) \\ \mathcal{O}(\varepsilon^2\sigma(1+|\lambda|)+\sigma^3) & \mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4)\end{pmatrix},\end{aligned} \tag{57}$$

where $c(\varepsilon,\omega):=-2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3)$.

We now solve the reduced eigenvalue problem (51) to prove Theorem 1.2.

Proof of Theorem 1.2. In order to study the stability and instability of $\tilde{u}_{\varepsilon,\omega}$ we need to solve $det\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)=0$ for λ. Recalling (57), a direct computation gives

$$\begin{aligned}\det\mathcal{M}(\varepsilon,\omega,\sigma,\lambda) &= \lambda^2-\lambda(c(\varepsilon,\omega)-8\sigma^2) \\ &\quad -(c(\varepsilon,\omega)-4\sigma^2)4\sigma^2-64\sigma^2\omega^2\varepsilon^2+F(\varepsilon,\omega,\sigma,\lambda),\end{aligned}$$

where

$$\begin{aligned}&F(\varepsilon,\omega,\sigma,\lambda) \\ &= \mathcal{O}\Big(|\lambda|(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4)+(\varepsilon^2+\sigma^2)(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4) \\ &\quad +(\varepsilon\sigma^2+\varepsilon^2|\lambda|+\sigma^4)^2+\varepsilon\sigma(\varepsilon^2\sigma(1+|\lambda|)+\sigma^3)+(\varepsilon^2\sigma(1+|\lambda|)+\sigma^3)^2\Big).\end{aligned}$$

By the Weierstrass Preparation Theorem, there is an analytic function $q(\varepsilon,\sigma,\lambda)$ in a neighborhood of $(\varepsilon,\sigma,\lambda)=(0,0,0)$ such that $q(0,0,0)=1$ and

$$q(\varepsilon,\sigma,\lambda)\ \det\mathcal{M}(\varepsilon,\omega,\sigma,\lambda)=\lambda^2+a_1\lambda+a_0=0,$$

where $a_0(\varepsilon,\sigma)$ and $a_1(\varepsilon,\sigma)$ are also analytic functions determined by

$$\begin{aligned} a_0(\varepsilon,\sigma) &= q(\varepsilon,\sigma,0)\ \det\mathcal{M}(\varepsilon,\omega,\sigma,0) \\ &= (1+\mathcal{O}(\varepsilon+\sigma))\Big((-c(\varepsilon,\omega)+4\sigma^2)4\sigma^2 - 64\sigma^2\omega^2\varepsilon^2 + F(\varepsilon,\omega,\sigma,0)\Big) \\ &= 8\sigma^2(\varepsilon^2(1-12\omega^2)+2\sigma^2) + \mathcal{O}(\sigma^2(\varepsilon+\sigma)^3) \end{aligned}$$

and

$$\begin{aligned} a_1(\varepsilon,\sigma) &= q_\lambda(\varepsilon,\sigma,0)\ \det\mathcal{M}(\varepsilon,\omega,\sigma,0) + q(\varepsilon,\sigma,0)\ \det\mathcal{M}_\lambda(\varepsilon,\omega,\sigma,0) \\ &= \mathcal{O}(1+\varepsilon+\sigma)\Big(\mathcal{O}((\varepsilon^2+\sigma^2)\sigma^2) + F(\varepsilon,\omega,\sigma,0)\Big) \\ &\quad + (1+\mathcal{O}(\varepsilon+\sigma))\Big(-c(\varepsilon,\omega) + 8\sigma^2 + F_\lambda(\varepsilon,\omega,\sigma,0)\Big) \\ &= 2\varepsilon^2(1-4\omega^2) + 8\sigma^2 + \mathcal{O}((\varepsilon+\sigma)^3). \end{aligned}$$

Since the matrix $\mathcal{M}$ is Hermitian, the both eigenvalues $\lambda(\varepsilon,\omega,\sigma)=\frac{-a_1\pm\sqrt{a_1^2-4a_0}}{2}$ are real implying $a_1^2-4a_0\geq 0$, and these are all nonpositive if and only if $a_0\geq 0$ and $a_1\geq 0$. However, for (ε,σ) sufficiently small a_1 is much larger than a_0, so that $\tilde{u}_{\varepsilon,\omega}$ is spectrally stable if and only if $a_0\geq 0$. We also notice from the existence result of $\tilde{u}_{\varepsilon,\omega}$ that $0\leq 1-4\omega^2$. Consequently, if ω satisfies

$$0 < 1-12\omega^2 \leq 1-4\omega^2, \quad \text{i.e., } \omega\in(-\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}),$$

then $a_0>0$ for (ε,σ) sufficiently small, which implies the stability of $\tilde{u}_{\varepsilon,\omega}$. On the other hand, $\tilde{u}_{\varepsilon,\omega}$ is unstable if $\omega\in[-\frac{1}{2},-\frac{1}{2\sqrt{3}})\cup(\frac{1}{2\sqrt{3}},\frac{1}{2}]$ because $a_0<0$ for (ε,σ) sufficiently small.

We now fix any $\varepsilon>0$ sufficiently small and any $\omega\in(-\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}})$ to investigate the stable spectral curves $\lambda(\varepsilon,\omega,\sigma)$ as σ varies in a small neighborhood of $\sigma=0$. Indeed, since $a_0=0$ at $\sigma=0$, one of the eigenvalues is 0. Let us say $\lambda_2(\varepsilon,\omega,0)=0$. Unfortunately, the error terms of a_0 and a_1 are not small enough to estimate $\lambda(\varepsilon,\omega,\sigma)$ directly as in [15, 16, 19]. Thus, similarly as in [23], we consider two cases: (i) $\frac{\sigma}{\varepsilon}\to 0$ as $\sigma\to 0$ and (ii) $|\frac{\sigma}{\varepsilon}|\geq C$ as $\sigma\to 0$ for some positive constant C. For the first case, $\frac{a_0}{a_1^2}$ is sufficiently small as well, so that the eigenvalues are given by

$$\lambda_1(\varepsilon,\omega,\sigma) = -a_1 + \mathcal{O}(\frac{a_0}{a_1}) = -2\varepsilon^2(1-4\omega^2) + \mathcal{O}\Big(\varepsilon^2(\varepsilon+\frac{\sigma^2}{\varepsilon^2})\Big) \tag{58}$$

and

$$\lambda_2(\varepsilon,\omega,\sigma) = -\frac{a_0}{a_1} + \mathcal{O}(\frac{a_0^2}{a_1^3}) = \frac{-4(1-12\omega^2)}{1-4\omega^2}\sigma^2 + \mathcal{O}\Big(\sigma^2(\varepsilon+\frac{\sigma^2}{\varepsilon^2})\Big). \tag{59}$$

For the second case, $\frac{a_0}{a_1^2}$ is also lower bounded away from 0, and we then see that for $j=1,2$,

$$\lambda_j(\varepsilon,\omega,\sigma) = \frac{-a_1\pm a_1\sqrt{1-\frac{4a_0}{a_1^2}}}{2} \leq -C_1a_1 \leq -C_2(\varepsilon^2+\sigma^2) \tag{60}$$

for some positive constants C_1 and C_2. These estimates (58)–(60) complete the proof. □

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