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CROSS-INTERCALATES AND GEOMETRY OF SHORT EXTREME POINTS IN THE LATIN POLYTOPE OF DEGREE 3

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Abstract. The polytope of tristochastic tensors of degree three, the Latin polytope, has two kinds of extreme points. Those that are at a maximum distance from the barycenter of the polytope correspond to Latin squares. The remaining extreme points are said to be short. The aim of the paper is to determine the geometry of these short extreme points, as they relate to the Latin squares.

The paper adapts the Latin square notion of an intercalate to yield the new concept of a cross-intercalate between two Latin squares. Cross-intercalates of pairs of orthogonal Latin squares of degree three are used to produce the short extreme points of the degree three Latin polytope. The pairs of orthogonal Latin squares fall into two classes, described as parallel and reversed, each forming an orbit under the isotopy group. In the inverse direction, we show that each short extreme point of the Latin polytope determines four pairs of orthogonal Latin squares, two parallel and two reversed.

1. Introduction

For a positive integer n (called the *degree* here to distinguish from the *order* of a tensor, and to extend the terminology of permutations), an $n \times n$ Latin square with symbols from an alphabet $\{a_1, \dots, a_n\}$ may be identified with an ordered n-tuple

$$T = (T_1, \dots, T_n) \tag{1.1}$$

of permutation matrices, where for $1 \leq i \leq n$, the permutation matrix T_i specifies the positions of the symbol a_i in the Latin square. More precisely, for

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$1 \leq i, j, k \leq n$, the entry $[T_i]_{jk}$ of T_i is 1 if and only if the symbol a_i appears in the j-th row and the k-th column of the Latin square.

Permutation matrices are the vertices of the *Birkhoff polytope* Ω_n consisting of all *bistochastic* matrices, matrices having non-negative real entries, where all the rows and all the columns sum to 1. Bistochastic matrices may be considered as relaxations of permutation matrices. In this paper, we are concerned with *tristochastic tensors* or *approximate Latin squares* [11, Defn. 3.4(b)], which are comparable relaxations of Latin squares. Thus a tristochastic tensor (of *degree* n) is an ordered list of n bistochastic $n \times n$-matrices whose sum is J_n, the all ones $n \times n$-matrix.

The set of all tristochastic tensors of degree n forms a polytope, the *Latin polytope* Λ_n [11, §3.3]. Latin squares of degree n are extreme points of Λ_n, but they are not the only ones. Fischer and Swart found 54 non-Latin extreme points of Λ_3 by a computer search [4, p. 184]. Now consider the set $\mathbf{2}_n^n$ of matrices of degree n with entries from the set $\mathbf{2} = \{0, 1\}$. For an element $A = [a_{ij}]_{1 \leq i,j \leq n}$ of $\mathbf{2}_n^n$, set $z(A) = \big|\{(i,j) \mid a_{ij} = 0\}\big|$. Then

$$L_n = n! \sum_{A \in \mathbf{2}_n^n} (-1)^{z(A)} \binom{\operatorname{per} A}{n}$$

is the number of Latin squares of degree n [15], while asymptotically,

$$L_n^{\frac{3}{2}+o(1)} \tag{1.2}$$

is a lower bound for the number of extreme points of Λ_n [12, Th. 1.5].

Jurkat and Ryser presented the prototype [9, Th. 3.3] for what became a series of extremality criteria for elements of Λ_n and its higher-order analogues (compare [3, 10, 12], etc.), and it is their criterion which will be used in this paper. They made the following comments about the extreme points of Λ_n in comparison with those of Ω_n:

> "But the corresponding situation for 3-dimensional extremal stochastic matrices is vastly more complicated. In fact these matrices are not known to us explicitly for general n"

[9, p. 195], and then:

> "But we have been unable to obtain an explicit description of the extremal stochastic matrices"

[9, p. 217]. Now, fifty years later, there are various possible approaches to understanding the non-Latin extreme points of the Latin polytopes, such as the method of [10, §4], or the graph-theoretical approach of [12] that underlies the lower bound (1.2).

Our goal is to further a more geometric approach to a Latin polytope and its extreme points. Latin squares represent the extreme points that are global maximizers of the distance from the barycenter, so we characterize the non-Latin extreme points as being *short*. The current paper initiates the geometric study by focusing on the Latin polytope Λ_3, correlating its short extreme points with certain pairs of mutually orthogonal Latin squares (MOLS) as summarized in Figure 1. A standard concept from the theory of Latin squares, the notion of an intercalate within a single Latin square, is adapted to create what are known as *cross-intercalates* of pairs of Latin squares. In our geometry, pairs of MOLS of degree 3 split into two classes that are described as *parallel* and *reversed*. The cross-intercalates split into two classes, *row* and *column* cross-intercalates. Each short extreme point of Λ_3 is then obtained by a row cross-intercalate change from two parallel pairs of MOLS, and by a column cross-intercalate change from two reversed pairs of MOLS.

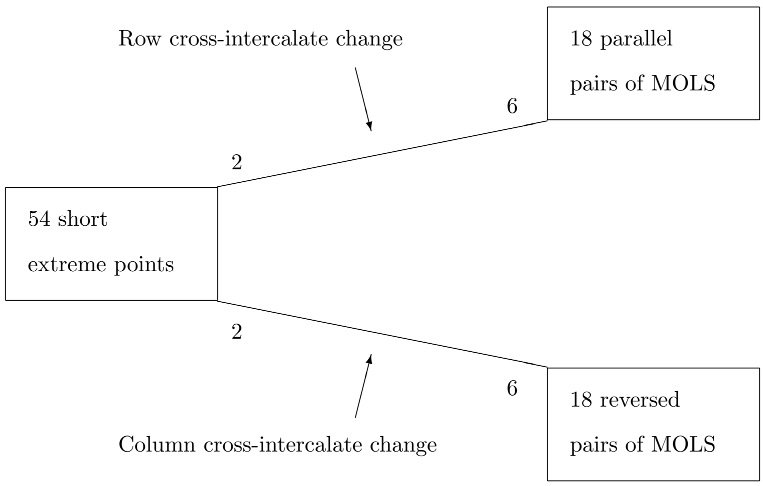

FIGURE 1. Geometry of short extreme points and mutually orthogonal Latin squares in degree 3.

The plan of the paper is as follows. Section 2 reviews background material, including the isotopy group that acts on a Latin polytope (§2.3), along with the Jurkat/Ryser criterion for extremality of a tristochastic tensor (Theorem 2.2). It deals with the *metric* geometry of spaces of tensors, contrasted with the *projective* geometry introduced in [7] and further studied in [8]. The parallel and reversed classes of degree 3 MOLS are presented in Section 3. In particular, it is shown that each of these classes forms an orbit under the action of the isotopy group on Λ_3 (Theorem 3.2). The cross-intercalates are introduced in Section 4. Working with the Jurkat/Ryser extremality condition, Section 5 shows how to construct one short extreme point of Λ_3 by making a cross-intercalate change in a pair of MOLS. Section 6 then shows how to obtain the full set of short extreme points by exploiting the action of the isotopy group.

The final Section 7 investigates the reverse of the process exhibited in Section 5, showing how each short extreme point determines two parallel pairs and two reversed pairs of MOLS via appropriate cross-intercalate changes (Theorem 7.3, Figure 1).

Readers are referred to [16] and [17, Ch. 11] for aspects of quasigroup and Latin square theory that are not otherwise explained explicitly in the paper.

2. Background

2.1. Metric spaces of tensors

2.1.1. *Stochastic vectors.* The set

$$\Pi_n = \left\{ (p_1, \dots, p_n) \,\middle|\, \forall\ 1 \leq i \leq n\,,\ p_i \in [0,1] \text{ and } \sum_{i=1}^{n} p_i = 1 \right\}$$

is the set of probability distributions that are available on an n-element set. Topologically, it forms an $(n-1)$-dimensional simplex Δ_{n-1}. Here, the vertices or extreme points are the "crisp" probability distributions where each weight p_i lies in the set $\{0, 1\}$, not just the closed interval $[0, 1]$. Considering the elements of Π_n supported on a specific n-element set $A = \{ a_1, \dots, a_n \}$, it is often convenient to identify the symbol a_i with the crisp distribution having $p_i = 1$, for $1 \leq i \leq n$.

As a subset of Euclidean space $\mathbb{R}^n$, the set Π_n inherits the Euclidean metric given by the squared norm $||\mathbf{x}||^2 = \mathbf{x}\mathbf{x}^*$, where $\mathbf{x}^*$ denotes the (conjugate) transpose of the $(1 \times n)$-matrix $\mathbf{x}$. Elements $\mathbf{p} = (p_1, \dots, p_n)$ of Π_n are described as *stochastic vectors.*

The barycenter of the simplex Π_n is the uniform distribution

$$\left(\frac{1}{n}, \dots, \frac{1}{n} \right).$$

The crisp distributions are global maximizers, over Π_n, of the distance from the barycenter, yielding

$$\sqrt{\left(1 - \frac{1}{n}\right)^2 + (n-1)\frac{1}{n^2}} = \sqrt{\frac{n-1}{n}}$$

as the maximum distance.

2.1.2. *Bistochastic matrices.* An $n \times n$ matrix $T = [t_{ij}]_{1 \leq i,j \leq n}$ is said to be *bistochastic* if each row and each (transposed) column is a stochastic vector. The set Ω_n of all bistochastic matrices of degree n is described as the *Birkhoff polytope.* As a subset of Euclidean space $\mathbb{R}^{n^2}$, the set Ω_n inherits the Euclidean metric given by the squared norm $||T||^2 = \operatorname{tr}(TT^*)$, where T^* denotes the (conjugate) transpose of the $(n \times n)$-matrix T.

The bistochastic matrices $T = [t_{ij}]_{1\le i,j\le n}$ are specified uniquely by arbitrary $(n-1)\times(n-1)$ arrays $[t_{ij}]_{1\le i,j\le n-1}$ of non-negative real numbers such that

$$\sum_{k=1}^{n-1} t_{ik} \le 1 \quad \text{and} \quad \sum_{k=1}^{n-1} t_{kj} \le 1$$

for $1 \le i,j \le n-1$. Thus Ω_n forms an $(n-1)^2$-dimensional polytope. Birkhoff gave a succinct proof [1] that the vertices or extreme points of Ω_n are precisely the permutation matrices of degree n, namely the bistochastic matrices whose entries lie in the set $\{0,1\}$, not just the closed interval $[0,1]$.

The barycenter of the polytope Ω_n is the matrix $\frac{1}{n}J_n$, where J_n is the $n\times n$ all-ones matrix. The permutation matrices are global maximizers, over Ω_n, of the distance from the barycenter, yielding

$$\sqrt{n\Big(1-\frac{1}{n}\Big)^2 + (n^2-n)\frac{1}{n^2}} = \sqrt{n-1}$$

as the maximum value.

2.1.3. *Tristochastic tensors.* A (real) 3-tensor of degree n is a three-dimensional array $T = [t_{ijk}]_{1\le i,j,k\le n}$ of real numbers. We write such a 3-tensor as a *stack*

$$T = (T_1, \dots, T_n) \tag{2.1}$$

or ordered list of $(n\times n)$-matrices

$$T_1 = [T_{1jk}]_{1\le j,k\le n}, \dots, T_n = [T_{njk}]_{1\le j,k\le n}$$

known as the *layers* of the stack. The 3-tensors of degree n lie in Euclidean space $\mathbb{R}^{n^3}$. Thus the squared norm of the stack (2.1) is given by

$$||T||^2 = \sum_{i=1}^{n} ||T_i||^2 = \sum_{i=1}^{n} \mathrm{tr}(T_i T_i^*)$$

in terms of the matrix norms of its layers. A *tristochastic tensor* is a real 3-tensor (2.1) whose layers are all bistochastic, with $\sum_{i=1}^n T_i = J_n$.

2.2. The Latin polytope

2.2.1. *Approximate Latin squares.* In analogy with Π_n and Ω_n, the set of tristochastic tensors of degree n forms a polytope Λ_n, described as the *Latin polytope.* Elements of Λ_n are interpreted as (*weak*) *approximate Latin squares* [11, Defn. 3.4]. A tristochastic tensor $T = [t_{ijk}]_{1\le i,j,k\le n}$ with $t_{ijk}\in\{0,1\}$ for all $1\le i,j,k\le n$ corresponds to a Latin square written as a formal linear combination $\sum_{i=1}^n a_i T_i$ of the layers T_i, with symbols from the alphabet A. In general, each tristochastic tensor $T = [t_{ijk}]_{1\le i,j,k\le n}$ is identified with an approximate Latin square $\sum_{i=1}^n a_i T_i$ comprising symbols from the alphabet A, as illustrated by (5.2) below, for example, on the alphabet $\{a,b,c\}$.

2.2.2. *Latin squares as global extreme points.* The barycenter of the Latin polytope Λ_n is the *uniform approximate Latin square* $\mathsf{UL}(n) = \frac{1}{n}(J_n, \dots, J_n)$ [7, Defn. 2.8]. The distance of a Latin square from the barycenter is

$$\sqrt{n^2\Big(1 - \frac{1}{n}\Big)^2 + (n^3 - n^2)\frac{1}{n^2}} = \sqrt{n(n-1)}\,.$$

It was shown inductively in [9, pp. 200–201], and more directly in [3, Lemma 3.2], [11, Theorem 3.13], that Latin squares of degree n are extreme points of the Latin polytope Λ_n. They are global maximizers, over Λ_n, of the distance from the barycenter.

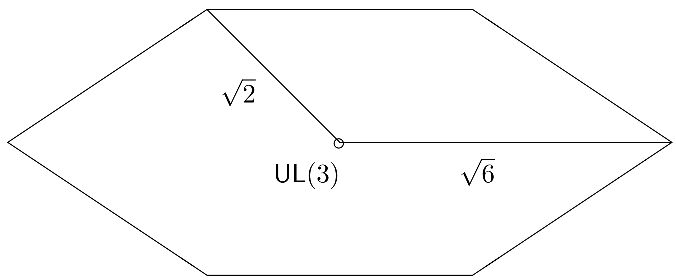

FIGURE 2. Short and long extreme points.

2.2.3. *Short extreme points.* In the polytopes Π_n and Ω_n, every extreme point is at maximal distance from the barycenter. This is no longer the case for the Latin polytope Λ_n if $n > 2$. The computer search reported in [4, p. 184] showed that in addition to the Latin squares, the Latin polytope Λ_3 of degree 3 has extreme points like

$$\left(\begin{bmatrix} \frac12 & \frac12 & 0 \\ 0 & 0 & 1 \\ \frac12 & \frac12 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac12 & \frac12 \\ \frac12 & \frac12 & 0 \\ \frac12 & 0 & \frac12 \end{bmatrix}, \begin{bmatrix} \frac12 & 0 & \frac12 \\ \frac12 & \frac12 & 0 \\ 0 & \frac12 & \frac12 \end{bmatrix}\right),$$

lying at a distance of

$$\sqrt{1 \times \frac{4}{9} + 16 \times \frac{1}{36} + 10 \times \frac{1}{9}} = \sqrt{2}$$

from the barycenter $\mathsf{UL}(3)$. Extreme points of this type, which are not global maximizers of the distance from the barycenter, are described as being *short.*[1] Figure 2 may help a reader appreciate the difference between the long and short extreme points of Λ_3, where the Latin squares lie at a distance of $\sqrt{6}$ from the barycenter $\mathsf{UL}(3)$.

[1] The term "exotic" was used in [11, Defn. 3.14(a)].

2.3. Isotopy group action

2.3.1. *Symmetries of the Birkhoff polytope.* Consider a quasigroup Q. Its (*combinatorial*) *multiplication group* is the subgroup

$$\mathrm{Mlt}\, Q = \langle L(q), R(q) \mid q \in Q \rangle_{Q!}$$

of the group $Q!$ of bijections from Q to Q generated by all *left multiplications* $L(q)\colon Q \to Q; x \mapsto qx$ and *right multiplications* $R(q)\colon Q \to Q; x \mapsto xq$ with elements q of Q. Its subgroups

$$\mathrm{LMlt}\, Q = \langle L(q) \mid q \in Q \rangle_{Q!} \quad \text{and} \quad \mathrm{RMlt}\, Q = \langle R(q) \mid q \in Q \rangle_{Q!}$$

are respectively known as the *left* and *right multiplication groups* of Q.

If Q is a group, considered as a quasigroup, its multiplication group $\mathrm{Mlt}\, Q$ is given by the exact sequence

$$\{1\} \longrightarrow Z(Q) \xrightarrow{\Delta} Q \times Q \xrightarrow{T} \mathrm{Mlt}\, Q \longrightarrow \{1\}$$

of groups with $\Delta : z \mapsto (z, z)$ and $T : (x, y) \mapsto L(x)^{-1}R(y)$ [16, Ex. 2.1]. The symmetric group S_n is abelian for $n \le 2$, so $\mathrm{Mlt}\, S_n \cong S_n$. For $n > 2$, one has $Z(S_n) = \{1\}$ and $\mathrm{Mlt}\, S_n \cong S_n \times S_n$.

The natural action of the combinatorial multiplication group $\mathrm{Mlt}\, S_n$ on (the permutation matrices that faithfully represent) S_n yields an isometric action of $\mathrm{Mlt}\, S_n$ on the Birkhoff polytope Ω_n. Here, the left multiplications permute matrix rows, while the right multiplications permute matrix columns. These two actions commute mutually, in accord with the associativity of S_n.

2.3.2. *Isotopy of the Latin polytope.* Let $T = (T_1, \dots, T_n)$ be a tristochastic tensor of degree n, and let α be an element of $\mathrm{Mlt}\, S_n$. Then the diagonal extension of the multiplication group action on Ω_n from §2.3.1 yields an action

$$\alpha\colon (T_1, \dots, T_n) \mapsto (T_1^\alpha, \dots, T_n^\alpha)$$

of α on T. In other words, the isometric action of $\mathrm{Mlt}\, S_n$ on Ω_n extends diagonally to an isometric action of $\mathrm{Mlt}\, S_n$ on Λ_n.

An additional action of S_n on Λ_n is furnished by the permutations of the layers of each stack. The group generated by the diagonal action of $\mathrm{Mlt}\, S_n$ on Λ_n, together with the layer permutations, is called the *isotopy group* of Λ_n. It acts isometrically. We identify the following subgroups of the isotopy group:

- The *row subgroup* corresponding to the left multiplication group of S_n;
- The *column subgroup* corresponding to the right multiplication group of S_n;
- The *multiplication subgroup*, the join of the row and column subgroups;
- The *symbol subgroup* corresponding to the layer permutations.

These subgroups act in the expected way to yield isotopies of weak approximate Latin squares. The row subgroup permutes rows, while the column subgroup permutes columns, and then the symbol subgroup permutes symbols.

2.4. The Jurkat/Ryser extremality condition

In this section, we recall the general necessary and sufficient condition [9, Th. 3.3] given by Jurkat and Ryser for extremality of a tristochastic tensor. (Other conditions that are more limited in scope were presented in [3, §3.2], on the basis of a concept of permanent for 3-tensors.)

2.4.1. *The lines of a stack.* Consider a tristochastic tensor or stack

$$T = (T_1, \dots, T_n) = [t_{ijk}]_{1 \le i,j,k \le n}$$

of degree n. The *rows* of the stack are the rows

$$ij* := \{t_{ij1}, \dots, t_{ijn}\} \tag{2.2}$$

of the matrices forming the layers of the stack, for $1 \le i, j \le n$. The *columns* of the stack are the columns

$$i{*}k := \{t_{i1k}, \dots, t_{ink}\} \tag{2.3}$$

of the matrices forming the layers of the stack, for $1 \le i, k \le n$. The *piles* of the stack are the sets

$$*jk := \{t_{1jk}, \dots, t_{njk}\} \tag{2.4}$$

of corresponding matrix entries, for $1 \le j, k \le n$. Together, the sets (2.2)–(2.4) are described as the *lines* of the stack.

2.4.2. *The incidence matrix of a stack.* Let

$$T = (T_1, \dots, T_n) = [t_{ijk}]_{1 \le i,j,k \le n}$$

be a tristochastic tensor or stack. Its *incidence matrix* has rows indexed by the lines of the stack, and columns indexed by the non-zero entries t_{ijk} of the stack. The incidence matrix column indexed by a non-zero entry t_{ijk} has an entry of 1 for each of the three lines (namely $ij*$, $i{*}k$, and $*jk$) containing t_{ijk}.

Example 2.1. The body of Table 1 displays the incidence matrix of the stack (5.5).

2.4.3. *Jurkat/Ryser relations.*

Theorem 2.2 ([9, Th. 3.3])**.** *A tristochastic tensor of degree n is an extreme point of the Latin polytope Λ_n if and only if the columns of its incidence matrix are linearly independent.*

For a tristochastic tensor that is not extremal, non-trivial relations holding between the columns of its incidence matrix will be described as *Jurkat/Ryser relations.*

3. Pairs of mutually orthogonal Latin squares

3.1. Parallel and reversed classes

This section concerns itself with the case of degree 3, which will be the main focus of the paper. It will occasionally be convenient to have the abbreviated notation

$$(1) = r_1\,,\ (1\ 2\ 3) = r_2\,,\ (1\ 3\ 2) = r_3\,,\ (1\ 2) = s_1\,,\ (2\ 3) = s_2\,,\ (3\ 1) = s_3$$

for the elements of S_3 acting naturally on $\{1, 2, 3\}$, and to impose the two lexicographic orders

$$(1) < (1\ 2\ 3) < (1\ 3\ 2)\,,\ (1\ 2) < (2\ 3) < (3\ 1)$$

or

$$r_1 < r_2 < r_3 \quad \text{and } s_1 < s_2 < s_3\,,$$

extending to the cyclic orderings

$$r_1 < r_2 < r_3 < r_1 < r_2 \quad \text{and } s_1 < s_2 < s_3 < s_1 < s_2\,. \tag{3.1}$$

Identifying permutations as usual here with their permutation matrices, the pairs of MOLS of degree 3 comprise an element of each of the symbol subgroup orbits of the two stacks (r_1, r_2, r_3) and (s_1, s_2, s_3). Thus altogether, there are $6 \times 6 = 36$ unordered pairs of MOLS. They fall into two classes, defined as follows.

Definition 3.1. Suppose that $\{(p_1, p_2, p_3), (q_1, q_2, q_3)\}$ is a pair of MOLS of degree 3, with $\{p_1, p_2, p_3\} = A_3$ and $\{q_1, q_2, q_3\} = S_3 \smallsetminus A_3$.

(a) The pair is *parallel* if the respective stacks are $(p_1 < p_2 < p_3)$ and $(q_1 < q_2 < q_3)$ or $(p_1 > p_2 > p_3)$ and $(q_1 > q_2 > q_3)$ under the cyclic orderings of (3.1).

(b) The pair is *reversed* if the respective stacks are $(p_1 < p_2 < p_3)$ and $(q_1 > q_2 > q_3)$ or $(p_1 > p_2 > p_3)$ and $(q_1 < q_2 < q_3)$ under the cyclic orderings of (3.1).

3.2. Isotopy action on pairs of MOLS

We now show that the classes introduced in Definition 3.1 form orbits under the isotopy group. In preparation, we note the two Cayley diagrams

$$\begin{tikzcd} s_2 \arrow[r, equal] \arrow[d, dash] & r_1 \arrow[r, dash] & s_1 \arrow[d, equal] \\ r_2 \arrow[r, equal] & s_3 \arrow[r, dash] & r_3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} s_2 \arrow[r, equal, dashed] \arrow[d, dash, dashed] & r_1 \arrow[r, dash, dashed] & s_1 \arrow[d, equal, dashed] \\ r_3 \arrow[r, equal, dashed] & s_3 \arrow[r, dash, dashed] & r_2 \end{tikzcd} \tag{3.2}$$

of S_3 with respect to its generating set $\{s_1, s_2\}$. Solid edges are used to denote right multiplications, while dashed edges are used to denote left multiplications, singly for s_1 and doubly for s_2.

Theorem 3.2. *The unordered pairs of MOLS form two orbits under the action of the isotopy group: the class of parallel pairs, and the class of reversed pairs.*

Proof. The symbol subgroup preserves the two classes. The Cayley diagram

$$\begin{tikzcd}
r_1r_2r_3, s_1s_2s_3 \arrow[r, equal] \arrow[d, equal] & s_2s_3s_1, r_3r_1r_2 \arrow[r, dash] \arrow[d, equal] & r_2r_3r_1, s_3s_1s_2 \arrow[d, equal] \\
s_2s_1s_3, r_2r_1r_3 \arrow[r, equal] \arrow[d, dash] & r_1r_3r_2, s_3s_2s_1 \arrow[r, dash] \arrow[d, dash] & s_1s_3s_2, r_3r_2r_1 \arrow[d, dash] \\
r_3r_1r_2, s_3s_1s_2 \arrow[r, equal] \arrow[d, equal] & s_1s_2s_3, r_2r_3r_1 \arrow[r, dash] \arrow[d, equal] & r_1r_2r_3, s_2s_3s_1 \arrow[d, equal] \\
s_3s_2s_1, r_3r_2r_1 \arrow[r, equal] \arrow[d, dash] & r_2r_1r_3, s_1s_3s_2 \arrow[r, dash] \arrow[d, dash] & s_2s_1s_3, r_1r_3r_2 \arrow[d, dash] \\
r_2r_3r_1, s_2s_3s_1 \arrow[r, equal] \arrow[d, equal] & s_3s_1s_2, r_1r_2r_3 \arrow[r, dash] \arrow[d, equal] & r_3r_1r_2, s_1s_2s_3 \arrow[d, equal] \\
s_1s_3s_2, r_1r_3r_2 \arrow[r, equal] & r_3r_2r_1, s_2s_1s_3 \arrow[r, dash] & s_3s_2s_1, r_2r_1r_3
\end{tikzcd}$$

of the multiplication subgroup action with respect to its generating set

$$\{L(s_1), L(s_2), R(s_1), R(s_2)\}$$

then shows that the parallel class forms an orbit. Here, the conventions of (3.2) are used for the generators, and the missing actions of $R(s_1)$ and $R(s_2)$, for example $R(s_1)$ at the vertex $r_1r_2r_3, s_1s_2s_3$, are trivial. Furthermore, the actions of $L(s_1)$ and $L(s_2)$ on the vertices in the top and bottom rows are not shown, since they are implicit from the braid relation $L(s_1)L(s_2)L(s_1) = L(s_2)L(s_1)L(s_2)$. In the diagram, a vertex label $p_1p_2p_3, q_1q_2q_3$ denotes the unordered pair $\{(p_1, p_2, p_3), (q_1, q_2, q_3)\}$ of stacks. A similar procedure shows that the reversed class forms an orbit. $\square$

4. Intercalates and cross-intercalates

4.1. Intercalates and intercalate changes

An *intercalate*[2] in a Latin square $T = (T_1, \dots, T_n) = \sum_{i=1}^{n} a_i T_i$ is defined by indices

$$1 \leq i_1 \neq i_2\,,\ j_1 \neq j_2\,,\ k_1 \neq k_2 \leq n$$

[2]Compare [5], [6], [13]; the term was introduced in [14].

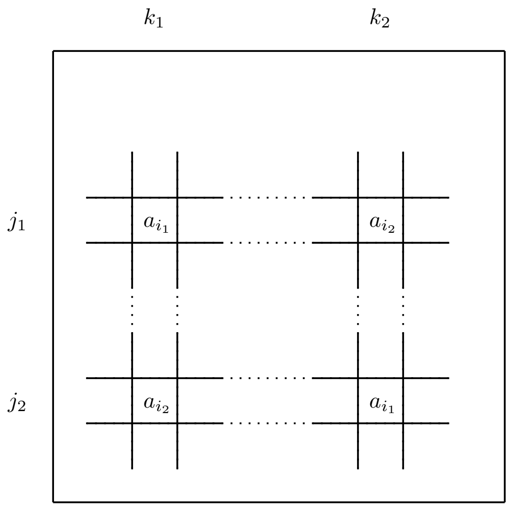

FIGURE 3. An intercalate in a Latin square.

such that $t_{i_1,j_1,k_1} = t_{i_1,j_2,k_2} = t_{i_2,j_1,k_2} = t_{i_2,j_2,k_1} = 1$ (Figure 3). An *intercalate change* is a transformation of Latin squares of the form

$$\begin{bmatrix} a_{i_1} & a_{i_2} \\ a_{i_2} & a_{i_1} \end{bmatrix} \mapsto \begin{bmatrix} a_{i_2} & a_{i_1} \\ a_{i_1} & a_{i_2} \end{bmatrix}$$

where unmarked terms are not changed.

4.2. Cross-intercalates and cross-intercalate changes

Definition 4.1. Suppose that $T^{(1)} = \left(T_1^{(1)}, \dots, T_n^{(1)}\right) = \sum_{i=1}^n a_i T_i^{(1)}$ and $T^{(2)} = \left(T_1^{(2)}, \dots, T_n^{(2)}\right) = \sum_{i=1}^n b_i T_i^{(2)}$ are Latin squares, with

$$T_i^{(1)} = [t_{ijk}^{(1)}]_{1\le j,k\le n} \quad \text{and} \quad T_i^{(2)} = [t_{ijk}^{(2)}]_{1\le j,k\le n}$$

for $1 \le i \le n$. Then a *cross-intercalate between* $T^{(1)}$ *and* $T^{(2)}$, or a *cross-intercalate in* the weak approximate Latin square $\frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)}$, is defined by indices

$$1 \le i_1 \ne i_2\,,\ j_1 \ne j_2\,,\ k_1 \ne k_2 \le n$$

such that $t^{(1)}_{i_1,j_1,k_1} = t^{(1)}_{i_1,j_2,k_2} = t^{(2)}_{i_2,j_1,k_2} = t^{(2)}_{i_2,j_2,k_1} = 1$ (Figure 4).

Lemma 4.2. *Under the action of the isotopy group, cross-intercalates are mapped to cross-intercalates.*

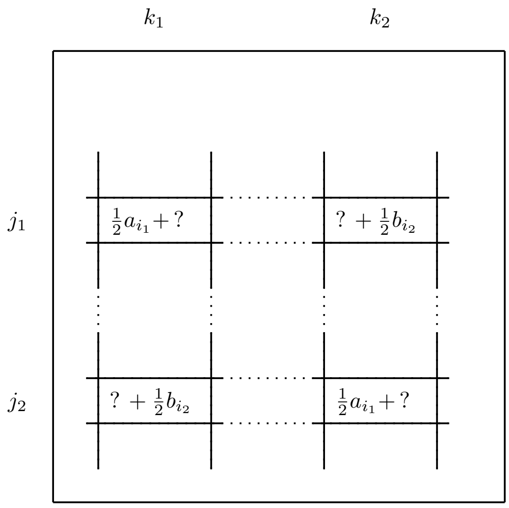

FIGURE 4. A cross-intercalate in an approximate Latin square.

A *cross-intercalate change* will now be defined as a transformation of weak approximate Latin squares of the form

$$\begin{bmatrix} \frac{1}{2}a_{i_1}+? & ? + \frac{1}{2}b_{i_2} \\ ? + \frac{1}{2}b_{i_2} & \frac{1}{2}a_{i_1}+? \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{2}b_{i_2}+? & ? + \frac{1}{2}a_{i_1} \\ ? + \frac{1}{2}a_{i_1} & \frac{1}{2}b_{i_2}+? \end{bmatrix}$$

where unmarked terms, or terms marked by ?, are not changed. Note that, starting from an approximate Latin square, the result is again an approximate Latin square: all the row, column, and pile sums are 1. Summarizing, to contrast with the situation considered in the following section, cross-intercalate changes are always possible when a cross-intercalate is given.

4.3. Column and row cross-intercalate changes

Suppose that we have a cross-intercalate in the weak approximate Latin square $\frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)}$, as in Definition 4.1. As observed in the previous section, a cross-intercalate change may be performed to obtain a new approximate Latin square. In this section, rather than obtaining a single weak approximate Latin square as the result of the cross-intercalate change, we consider the possibility, not always feasible, of obtaining a pair of genuine Latin squares from the cross-intercalate change. More precisely, new squares $S^{(1)}$ and $S^{(2)}$ may potentially be created from the squares $T^{(1)}$ and $T^{(2)}$ by interchanging the entries a_{i_1} and b_{i_2} at the four cross-intercalate points indicated in Figure 4.

For this question, it is necessary to distinguish two different kinds of cross-intercalate changes, namely *row cross-intercalate changes* as indicated by

$$\begin{bmatrix} \frac{1}{2}a_{i_1} + ? & ? + \frac{1}{2}b_{i_2} \\ ? + \frac{1}{2}b_{i_2} & \frac{1}{2}a_{i_1} + ? \end{bmatrix}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\begin{bmatrix} b_{i_2} & a_{i_1} \\ & \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & \\ a_{i_1} & b_{i_2} \end{bmatrix}$$

or *column cross-intercalate changes* as indicated by

$$\begin{bmatrix} \frac{1}{2}a_{i_1} + ? & ? + \frac{1}{2}b_{i_2} \\ ? + \frac{1}{2}b_{i_2} & \frac{1}{2}a_{i_1} + ? \end{bmatrix}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\begin{bmatrix} b_{i_2} & \\ a_{i_1} & \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & a_{i_1} \\ & b_{i_2} \end{bmatrix}.$$

In each case, the potential squares $S^{(1)}$ and $S^{(2)}$ are represented on the left and right hand sides of the display respectively. Various examples of this type appear in the proof of Theorem 7.3. In some cases, the squares $S^{(1)}$ and $S^{(2)}$ exist, while in other cases, they do not.

5. Construction of a short extreme point

5.1. Means of orthogonal Latin squares

Consider the respective tristochastic tensors

$$T^{(1)} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right)$$

and

$$T^{(2)} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

together representing a reversed pair

$$\begin{bmatrix} b & a & c \\ c & b & a \\ a & c & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix} \tag{5.1}$$

of mutually orthogonal Latin squares. The tristochastic tensor

$$\frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)}$$

represents the approximate Latin square

$$\begin{bmatrix} \frac{1}{2}b + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}c & \frac{1}{2}c + \frac{1}{2}b \\ \frac{1}{2}c + \frac{1}{2}c & \frac{1}{2}b + \frac{1}{2}b & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}b & \frac{1}{2}c + \frac{1}{2}a & \frac{1}{2}b + \frac{1}{2}c \end{bmatrix} \tag{5.2}$$

which contains the cross-intercalate

$$\begin{bmatrix} \frac{1}{2}\boxed{b} + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}\boxed{c} & \frac{1}{2}c + \frac{1}{2}b \\ \frac{1}{2}\boxed{c} + \frac{1}{2}c & \frac{1}{2}b + \frac{1}{2}\boxed{b} & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}b & \frac{1}{2}c + \frac{1}{2}a & \frac{1}{2}b + \frac{1}{2}c \end{bmatrix} \tag{5.3}$$

identified by the boxed entries.

5.2. The Jurkat/Ryser relation of the mean

The mean of the tristochastic tensors $T^{(1)}$ and $T^{(2)}$ is

$$\frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)} = \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right). \tag{5.4}$$

Its non-zero entries may be identified as follows:

$$\left(\begin{bmatrix} v_1 & v_2 & 0 \\ 0 & 0 & v_3 \\ v_4 & v_5 & 0 \end{bmatrix}, \begin{bmatrix} v_6 & 0 & v_7 \\ 0 & v_8 & 0 \\ v_9 & 0 & v_{10} \end{bmatrix}, \begin{bmatrix} 0 & v_{11} & v_{12} \\ v_{13} & 0 & 0 \\ 0 & v_{14} & v_{15} \end{bmatrix} \right). \tag{5.5}$$

The locations of the non-zero entries $v_1, \ldots, v_{15}$ on the respective lines $11*$, $12*$, …, $*33$ of the 3-tensor are presented in Table 1. Identifying the entries with their 27-dimensional column vectors from the table, it is seen that the Jurkat/Ryser relation

$$(v_1 - v_4 + v_5 - v_2) - (v_6 - v_9 + v_{10} - v_7) + (v_{11} - v_{14} + v_{15} - v_{12}) = 0 \tag{5.6}$$

holds, witnessing that the mean $\frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)}$ is not an extreme point of Λ_3. In fact, the span of the vectors $v_1, \ldots, v_{15}$ has dimension 14, so up to scalar multiples, (5.6) is the only relation holding.

Table 1. Line incidence with non-zero entries in the mean.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
$11*$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$12*$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$13*$	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$21*$	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
$22*$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$23*$	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
$31*$	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
$32*$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$33*$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$1*1$	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$1*1$	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
$1*1$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$2*1$	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
$2*2$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$2*3$	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
$3*1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$3*2$	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
$3*3$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
$*11$	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$*12$	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
$*13$	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
$*21$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$*22$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$*23$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$*31$	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
$*32$	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
$*33$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

It is convenient to display the relation (5.6) by the directed graph

$$
\begin{tikzcd}
v_1 & v_2 \arrow[l] \arrow[d] & & v_6 \arrow[r] \arrow[d] & v_7 & & v_{11} & v_{12} \arrow[l] \arrow[d] \\
v_4 \arrow[u] \arrow[r] & v_5 & & v_9 & v_{10} \arrow[l] \arrow[u] & & v_{14} \arrow[u] \arrow[r] & v_{15}
\end{tikzcd}
\tag{5.7}
$$

where vertices with in-degree two appear in (5.6) with a positive sign, while vertices with out-degree two appear in (5.6) with a negative sign.

5.3. The cross-intercalate change

In order to obtain an extreme point of the Latin polytope Λ_3 from the mean (5.4) of the orthogonal Latin squares, the relation (5.6) must be broken, in such a way that no new Jurkat/Ryser relations are introduced in the process. The symbol interchange $b \leftrightarrow c$ in the boxes of the cross-intercalate (5.3) will destroy the two rightmost cycles appearing in the undirected reduct of the directed graph (5.7), essentially by removal of the vertices v_6 and v_{11} from the picture.

Interchange of the symbols b and c in the boxes of the cross-intercalate of (5.3) yields the approximate Latin square

$$\begin{bmatrix} \frac{1}{2}\boxed{c}+\frac{1}{2}a & \frac{1}{2}a+\frac{1}{2}\boxed{b} & \frac{1}{2}c+\frac{1}{2}b \\ \frac{1}{2}\boxed{b}+\frac{1}{2}c & \frac{1}{2}b+\frac{1}{2}\boxed{c} & \frac{1}{2}a+\frac{1}{2}a \\ \frac{1}{2}a+\frac{1}{2}b & \frac{1}{2}c+\frac{1}{2}a & \frac{1}{2}b+\frac{1}{2}c \end{bmatrix}$$

with tristochastic tensor

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right). \tag{5.8}$$

Its non-zero entries may be identified as

$$\left(\begin{bmatrix} u_1 & u_2 & 0 \\ 0 & 0 & u_3 \\ u_4 & u_5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & u_6 & u_7 \\ u_8 & u_8' & 0 \\ u_9 & 0 & u_{10} \end{bmatrix}, \begin{bmatrix} u_{11} & 0 & u_{12} \\ u_{13} & u_{13}' & 0 \\ 0 & u_{14} & u_{15} \end{bmatrix} \right)$$

to contrast with the previous arrangement

$$\left(\begin{bmatrix} v_1 & v_2 & 0 \\ 0 & 0 & v_3 \\ v_4 & v_5 & 0 \end{bmatrix}, \begin{bmatrix} v_6 & 0 & v_7 \\ 0 & v_8 & 0 \\ v_9 & 0 & v_{10} \end{bmatrix}, \begin{bmatrix} 0 & v_{11} & v_{12} \\ v_{13} & 0 & 0 \\ 0 & v_{14} & v_{15} \end{bmatrix} \right)$$

from (5.5). The cross-intercalate change acts as

$$v_i \mapsto \begin{cases} u_i, u_i' & \text{for } i \in \{8, 13\}; \\ u_i & \text{otherwise} \end{cases}$$

on the entries of (5.5). The cycle elements v_6, v_{11} from (5.7), respectively incident with the columns $2*1$ and $3*2$, are transformed to u_6, u_{11}, which are respectively incident with the columns $2*2$ and $3*1$. As such, they no longer form cycles of the type displayed in (5.7), and thus the Jurkat/Ryser relation (5.6) is broken. An analysis of the analogue of Table 1 for the vectors $u_1, \dots, u_8, u_8', \dots, u_{13}, u_{13}', \dots, u_{15}$ shows no Jurkat/Ryser relations appearing. Thus by Theorem 2.2, (5.8) is a short extreme point of Λ_3, located at a distance of

$$\sqrt{2} = \sqrt{1 \times \frac{16}{36} + 16 \times \frac{1}{36} + 10 \times \frac{4}{36}}$$

from the barycenter $(\frac{1}{3}J_3, \frac{1}{3}J_3, \frac{1}{3}J_3)$ of the polytope.

6. Short extreme points

Following the construction of the single short extreme point (5.8) that was presented in the previous section, the isometric action of the isotopy group on the Latin polytope generates further short extreme points.

Lemma 6.1. *The orbit of* (5.8) *under the action of the isotopy group has* 54 *elements.*

Proof. Consider the tristochastic tensor (5.8):

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right).$$

Consider the ordered pair obtained by deleting the first layer. The invertible transformation

$$\theta \colon 0 \mapsto 1\,,\ \frac{1}{2} \mapsto 0$$

of entries, applied to the second and third layers, produces the ordered pair

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

of permutation matrices. The second matrix corresponds to an element of A_3, while the first corresponds to an element of the coset $S_3 \smallsetminus A_3$. Given the second and third layers, the first layer of (5.8) is uniquely determined by the condition that the sum of the three layers is J_3.

Now consider an arbitrary element T' of the orbit of (5.8) under the isotopy group. Within the orbit of T' under the symbol subgroup, there is a unique tensor T such that the inverse image under θ of the second layer lies in A_3, and the inverse image under θ of the third layer lies in $S_3 \smallsetminus A_3$. Overall, there are $|A_3| \times |S_3 \smallsetminus A_3| = 9$ such tensors T, each of which generates an orbit of size 6 under the symbol subgroup. Thus the full orbit of (5.8) under the isotopy group contains $9 \times 6 = 54$ elements. □

The following result was presented in [4, p. 184] and later papers as the outcome of a computer search. Bóna's work [2] implies an analytical verification that there are 54 short extreme points. Now, making use of the Latin square concepts of orthogonality and cross-intercalate, we have obtained an *analytical* determination of these extreme points:

Theorem 6.2. *The* 54 *tensors in the orbit of* (5.8) *under the action of the isotopy group constitute the full set of non-Latin extreme points of the Latin polytope* Λ_3.

7. Inverse cross-intercalate changes

Section 5 employed a certain column cross-intercalate change in a reversed pair of mutually orthogonal Latin squares in order to construct a particular short extreme point of Λ_3. We now bring the inverse process into play, and associate cross-intercalates between a pair of mutually orthogonal Latin squares to a given short extreme point. We refer to a pair of mutually orthogonal Latin squares (MOLS) with an identified cross-intercalate as a MOLS/C-I.

Lemma 7.1. *Consider a pair of mutually orthogonal Latin squares.*

(a) *There are* 6 *cross-intercalates in the pair of MOLS: column cross-intercalates in a reversed pair, and row cross-intercalates in a parallel pair.*
(b) *Of these* 6 *cross-intercalates in each pair of MOLS, two involve any given* 2*-element subset of the symbol set.*

Proof. By Theorem 3.2, the reversed pairs of MOLS form one orbit under the isotopy group, while the parallel pairs form another. It will suffice to examine a particular example, say the reversed pair (5.1). For compactness, this pair will be represented as

$$\begin{bmatrix} ba & ac & cb \\ cc & bb & aa \\ ab & ca & bc \end{bmatrix}.$$

Six column intercalate pairs may then be presented as

$$\begin{bmatrix} b)a & a(c & cb \\ c)c & b(b & aa \\ ab & ca & bc \end{bmatrix} \qquad \begin{bmatrix} ba & ac & cb \\ c(c & b)b & aa \\ a(b & c)a & bc \end{bmatrix} \qquad \begin{bmatrix} b(a & ac & c)b \\ c(c & bb & a)a \\ ab & ca & bc \end{bmatrix} \tag{7.1}$$

$$\begin{bmatrix} ba & ac & cb \\ c)c & bb & a(a \\ a)b & ca & b(c \end{bmatrix} \qquad \begin{bmatrix} ba & a)c & c(b \\ cc & b)b & a(a \\ ab & ca & bc \end{bmatrix} \qquad \begin{bmatrix} ba & ac & cb \\ cc & b(b & a)a \\ ab & c(a & b)c \end{bmatrix}$$

using a convention whereby the top left matrix in the array represents the column cross-intercalate of (5.3). Similarly, six row intercalate pairs may be found in any parallel pair of MOLS. □

Corollary 7.2. *There are* $2 \times 6 \times 3 \times 6 = 216$ *MOLS/C-I structures.*

A comparison of Corollary 7.2 with Theorem 6.2 shows that 216 MOLS/C-I structures are to be associated with 54 short extreme points. The following result shows that the association is regular.

Theorem 7.3. *Each short extreme point of* Λ_3 *is obtained by a cross-intercalate change from precisely four MOLS/C-I structures, namely two column cross-intercalate changes in reversed pairs of MOLS, and two row cross-intercalate changes in parallel pairs of MOLS.*

Proof. Because the isotopy group acts transitively on the set of short extreme points, it suffices to consider the single short extreme point (5.8), represented by the approximate Latin square

$$\begin{bmatrix} \frac{1}{2}c+\frac{1}{2}a & \frac{1}{2}a+\frac{1}{2}b & \frac{1}{2}c+\frac{1}{2}b \\ \frac{1}{2}b+\frac{1}{2}c & \frac{1}{2}b+\frac{1}{2}c & \frac{1}{2}a+\frac{1}{2}a \\ \frac{1}{2}a+\frac{1}{2}b & \frac{1}{2}c+\frac{1}{2}a & \frac{1}{2}b+\frac{1}{2}c \end{bmatrix}. \tag{7.2}$$

Since only the symbol a appears (in the second row) with a coefficient of 1, it cannot feature in a cross-intercalate change. Thus only cross-intercalate changes involving the symbols b and c need be considered. There are five such potential cross-intercalates in (7.2), each to be examined separately. We present the first case here, and defer the remaining four cases to the Appendix for the benefit of readers who would like to see the specific details of each case.

Case I:

$$\begin{bmatrix} \frac{1}{2}\boxed{c}+\frac{1}{2}a & \frac{1}{2}a+\frac{1}{2}\boxed{b} & \frac{1}{2}c+\frac{1}{2}b \\ \frac{1}{2}\boxed{b}+\frac{1}{2}c & \frac{1}{2}b+\frac{1}{2}\boxed{c} & \frac{1}{2}a+\frac{1}{2}a \\ \frac{1}{2}a+\frac{1}{2}b & \frac{1}{2}c+\frac{1}{2}a & \frac{1}{2}b+\frac{1}{2}c \end{bmatrix}.$$

This was the original cross-intercalate used to create the short extreme point (5.8). It came from a column cross-intercalate change using the MOLS/C-I indicated compactly as

$$\begin{bmatrix} b)a & a(c & cb \\ c)c & b(b & aa \\ ab & ca & bc \end{bmatrix}$$

employing the notation taken from the proof of Lemma 7.1. On the other hand, an attempt to implement a row cross-intercalate change would lead to the configuration

$$\begin{bmatrix} b & c & \\ & & a \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ c & b & a \\ & & \end{bmatrix}$$

of partial Latin squares to be completed, a task which fails at the top right-hand corner of the first square. $\square$

Lemma 7.1 and Theorem 7.3 are illustrated by Figure 1.

8. Conclusion and future work

We have determined the geometry of the short extreme points of the Latin polytope Λ_3, showing how they are obtained by cross-intercalate changes to the means of pairs of mutually orthogonal Latin squares. An immediate next step in the current program is to conduct a comparable geometric examination of the extreme points of Λ_4, representatives for which are listed without provenance in the appendix of [10].

In particular, the key question is the extent to which the Latin squares alone continue to govern the shorter extreme points, in ways that are comparable to the geometry observed in Λ_3. Of course, the absence of mutually orthogonal pairs of Latin squares in degree 6, and possibly also the asymptotics of (1.2), point to a greater diversity of construction methods than that needed for degree 3. Nevertheless, each effective method should imply higher-level relationships between Latin squares, like the parallel and reversed classes that are introduced in Section 3 of the current paper. Relational structure of this type could provide additional tools to tackle difficult questions, such as the possible number of MOLS of a given degree.

Appendix

This Appendix presents the remaining cases for the proof of Theorem 7.3.

Case II:

$$\begin{bmatrix} \frac{1}{2}c + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}b & \frac{1}{2}c + \frac{1}{2}b \\ \frac{1}{2}b + \frac{1}{2}\boxed{c} & \frac{1}{2}\boxed{b} + \frac{1}{2}c & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}\boxed{b} & \frac{1}{2}\boxed{c} + \frac{1}{2}a & \frac{1}{2}b + \frac{1}{2}c \end{bmatrix}.$$

This column cross-intercalate derives from the reversed MOLS/C-I written as

$$\begin{bmatrix} ac & ba & cb \\ b)b & c(c & aa \\ c)a & a(b & bc \end{bmatrix}$$

in the notation of the proof of Lemma 7.1. An attempt to implement a row cross-intercalate change would lead to the configuration

$$\begin{bmatrix} & & \\ b & c & a \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & a \\ c & b & \end{bmatrix}$$

of partial Latin squares to be completed, doomed to fail in the bottom right-hand corner of the second square.

Case III:

$$\begin{bmatrix} \frac{1}{2}c + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}\boxed{b} & \frac{1}{2}\boxed{c} + \frac{1}{2}b \\ \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}b & \frac{1}{2}\boxed{c} + \frac{1}{2}a & \frac{1}{2}\boxed{b} + \frac{1}{2}c \end{bmatrix}.$$

This row cross-intercalate derives from the MOLS/C-I written as

$$\begin{bmatrix} ac & c)a & b)b \\ cb & bc & aa \\ ba & a(b & c(c \end{bmatrix}$$

in the notation of the proof of Lemma 7.1. An attempt to implement a column cross-intercalate change would lead to the configuration

$$\begin{bmatrix} & c & \\ & & a \\ & b & \end{bmatrix}, \begin{bmatrix} & & b \\ & & a \\ & & c \end{bmatrix}$$

of partial Latin squares to be completed, failing at the middle entry of the first square.

Case IV:

$$\begin{bmatrix} \frac{1}{2}\boxed{c} + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}b & \frac{1}{2}c + \frac{1}{2}\boxed{b} \\ \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}\boxed{b} & \frac{1}{2}c + \frac{1}{2}a & \frac{1}{2}b + \frac{1}{2}\boxed{c} \end{bmatrix}.$$

This (row) cross-intercalate derives from the MOLS/C-I written as

$$\begin{bmatrix} b)a & ab & c)c \\ cb & bc & aa \\ a(c & ca & b(b \end{bmatrix}$$

in the notation of the proof of Lemma 7.1. An attempt to implement a column cross-intercalate change would lead to the configuration

$$\begin{bmatrix} b & & \\ & & a \\ c & & \end{bmatrix}, \begin{bmatrix} & & c \\ & & a \\ & & b \end{bmatrix}$$

of partial Latin squares to be completed, impossible for the first column of the first square.

Case V:

$$\begin{bmatrix} \frac{1}{2}\boxed{c} + \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2}\boxed{b} & \frac{1}{2}c + \frac{1}{2}b \\ \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}b + \frac{1}{2}c & \frac{1}{2}a + \frac{1}{2}a \\ \frac{1}{2}a + \frac{1}{2}\boxed{b} & \frac{1}{2}\boxed{c} + \frac{1}{2}a & \frac{1}{2}b + \frac{1}{2}c \end{bmatrix}.$$

A column cross-intercalate change would lead to the configuration

$$\begin{bmatrix} b & & \\ & & a \\ c & & \end{bmatrix}, \begin{bmatrix} & c & \\ & & a \\ & b & \end{bmatrix}$$

of partial Latin squares to be completed, but completion is impossible in the second row of either square. A row cross-intercalate change would lead to the configuration

$$\begin{bmatrix} b & c & \\ & & a \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & a \\ c & b & \end{bmatrix}$$

of partial Latin squares to be completed, but in this case completion is impossible in the third column of either square.

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