UNIQUENESS RELATED TO HIGHER ORDER DIFFERENCE OPERATORS OF ENTIRE FUNCTIONS

Xinmei Liu [a] and Junfan Chen [b,*]

Abstract. In this paper, by using the difference analogue of Nevanlinna's theory, the authors study the shared-value problem concerning two higher order difference operators of a transcendental entire function with finite order. The following conclusion is proved: Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \rho(f)$, where $a(z)(\in S(f))$ is an entire function and satisfies $\rho(a(z)) < 1$, and let $\eta(\in \mathbb{C})$ be a constant such that $\Delta_\eta^{n+1} f(z) \not\equiv 0$. If $\Delta_\eta^{n+1} f(z)$ and $\Delta_\eta^n f(z)$ share $\Delta_\eta^n a(z)$ CM, where $\Delta_\eta^n a(z) \in S\left(\Delta_\eta^{n+1} f(z)\right)$, then $f(z)$ has a specific expression $f(z) = a(z) + Be^{Az}$, where A and B are two non-zero constants and $a(z)$ reduces to a constant.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and suppose that $f(z)$ is a meromorphic function in $\mathbb{C}$. Here and in the sequel it is assumed that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory of meromorphic functions (see [18]) such as $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and $T(r, f)$. In addition, we denote by $S(r, f)$ any function satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. If a meromorphic function $a(z)(\not\equiv \infty)$ satisfies $T(r, a(z)) = S(r, f)$, then $a(z)$ is called a small function of $f(z)$, and we denote by $S(f)$ the set of functions which are small compared to $f(z)$. Throughout this paper, we define the order $\rho(f)$ of growth of $f(z)$ as

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.$$

If $\rho(f) < \infty$, then the function f is called *meromorphic function* of finite order.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that $f(z)$ and $g(z)$ share the value a CM provided that $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities, that $f(z)$ and $g(z)$ share the value ∞ CM provided that $f(z)$ and $g(z)$ have the same poles counting multiplicities. Using the same method, we can define that $f(z)$ and $g(z)$ share the function $a(z)$ CM, where $a(z) \in S(f) \cap S(g)$. Moreover, we need the following two definitions.

Definition 1.1 ([18])**.** Let $f(z)$ be a transcendental meromorphic function whose non-zero zeros are $z_1, z_2, \cdots, z_n, \cdots$, appearing often according to their multiplicities. Let $|z_n| = r_n$, and $r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots$. We call the *infimum* of the positive numbers τ that converge the series $\sum_{n=1}^{\infty} \frac{1}{|r_n|^\tau}$ as the exponent of the convergence of the zeros of $f(z)$, denoted by

$$\lambda = \inf_{\tau > 0} \left(\sum_{n=1}^{+\infty} \frac{1}{|z_n|^\tau} \right) < +\infty.$$

If the transcendental meromorphic function $f(z)$ has no zeros or finitely many zeros, then the exponent of the convergence of the zeros of $f(z)$ is required to be 0. If $f(z)$ has infinitely many zeros but the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^\tau}$ does not converge for any $\tau > 0$, then the exponent of the convergence of the zeros of $f(z)$ is ∞.

Definition 1.2. Let $f(z)$ be a meromorphic function. Then its difference operator is defined as

$$\Delta_\eta f(z) = f(z + \eta) - f(z), \ \Delta_\eta^k f(z) = \Delta_\eta(\Delta_\eta^{k-1} f(z)), \ k \in \mathbb{N}, \ k \geq 2,$$

where η is a non-zero constant, $\Delta_\eta f(z)$ is usually regarded as a difference analogue of f'.

According to the definition of difference operator, the expression of n-order difference operator can be deduced by induction as

$$\Delta_\eta^n f(z) = \sum_{j=0}^{n} (-1)^{n-j} C_n^j f(z + j\eta).$$

The uniqueness theory of meromorphic functions is an important part of complex analysis. Recently many scholars have begun to study the uniqueness of meromorphic functions and their difference operators sharing values or small functions, and have obtained many meaningful results. For example, one can refer to the literatures

(see [2, 3, 4, 5, 9, 13, 16]). In this paper, we will study the shared-value problem concerning two higher order difference operators of a transcendental entire function with finite order. Now we recall the following results.

In 1977, Rubel and Yang [17] studied the uniqueness of an entire function and its derivative sharing two values, and proved the following theorem.

Theorem 1.1 ([17]). *Let $f(z)$ be a nonconstant entire function. If $f(z)$ and $f'(z)$ share a,b CM, where a, b are two distinct finite complex values, then $f(z) \equiv f'(z)$.*

In 1986, Jank, Mues and Volkman [14] generalized Theorem 1.1 and obtained the following theorem.

Theorem 1.2 ([14]). *Let $f(z)$ be a nonconstant meromorphic function, let a be a nonzero complex number. If $f(z)$, $f'(z)$ and $f''(z)$ share a CM, then $f(z) \equiv f'(z)$.*

In 2013, Chen and Yi [6] considered the problem that $f(z)$ and $\Delta_\eta f(z)$ share one value a CM and proved the following theorem.

Theorem 1.3 ([6]). *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $\eta(\in \mathbb{C})$ be a constant such that $f(z + \eta) \not\equiv f(z)$. If $\Delta_\eta f(z) = f(z + \eta) - f(z)$ and $f(z)$ share the value a CM, then*

$$a = 0, \ \frac{f(z + \eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant.

The following theorem, studied and proved by Farissi, Latreuch and Asiri [10] in 2016, can be regarded as the difference analogue of Theorem 1.2.

Theorem 1.4 ([10]). *Let $f(z)$ be a nonconstant entire function of finite order, let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function of period η. If $f(z)$, $\Delta_\eta f(z)$ and $\Delta_\eta^2 f(z)$ share $a(z)$ CM, then $\Delta_\eta f(z) \equiv f(z)$.*

In 2021, Chen and Zhang [7] studied the CM sharing value problem of $\Delta_\eta^2 f(z)$ and $\Delta_\eta f(z)$. They generalized sharing one value in Theorem C to the case of sharing a small function $a(z)$ of order less than 1, and proved the following theorem.

Theorem 1.5 ([7]). *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \rho(f)$, where $a(z)(\in S(f))$ is an entire function and satisfies $\rho(a(z)) < 1$, and let $\eta(\in \mathbb{C})$ be a constant such that $\Delta_\eta^2 f(z) \not\equiv 0$. If $\Delta_\eta^2 f(z)$ and $\Delta_\eta f(z)$ share $\Delta_\eta a(z) CM$, where $\Delta_\eta a(z) \in S\left(\Delta_\eta^2 f(z)\right)$, then $f(z) = a(z) + Be^{Az}$, where A and B are two non-zero constants and $a(z)$ reduces to a constant.*

In this paper, we consider extending the condition that $\Delta_\eta^2 f(z)$ and $\Delta_\eta f(z)$ share $\Delta_\eta a(z)$ CM in Theorem 1.5 to the condition that $\Delta_\eta^{n+1} f(z)$ and $\Delta_\eta^n f(z)$ share $\Delta_\eta^n a(z)$ CM, where the following theorem is established.

Theorem 1.6. *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \rho(f)$, where $a(z)(\in S(f))$ is an entire function and satisfies $\rho(a(z)) < 1$, and let $\eta(\in \mathbb{C})$ be a constant such that $\Delta_\eta^{n+1} f(z) \not\equiv 0$. If $\Delta_\eta^{n+1} f(z)$ and $\Delta_\eta^n f(z)$ share $\Delta_\eta^n a(z)$ CM, where $\Delta_\eta^n a(z) \in S\left(\Delta_\eta^{n+1} f(z)\right)$, then*

$$f(z) = a(z) + Be^{Az},$$

where A and B are two non-zero constants and $a(z)$ reduces to a constant.

Example 1.1. Let $f(z) = e^{4z}$, and let $\eta = 1$, where $a(z) = 0$, $A = 4$, $B = 1$. Then we can get

$$\Delta_\eta f(z) = f(z + \eta) - f(z) = e^{4(z+\eta)} - e^{4z} = e^{4z}(e^{4\eta} - 1) = e^{4z}(e^4 - 1),$$

$$\Delta_\eta^2 f(z) = f(z + 2\eta) - 2f(z + \eta) + f(z) = e^{4(z+2\eta)} - 2e^{4(z+\eta)} + e^{4z}$$
$$= e^{4z}(e^{8\eta} - 2e^{4\eta} + 1) = e^{4z}(e^{4\eta} - 1)^2 = e^{4z}(e^4 - 1)^2.$$

By mathematical induction, we deduce

$$\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z)) = e^{4z}(e^{4\eta} - 1)^n = e^{4z}(e^4 - 1)^n,$$

$$\Delta_\eta^{n+1} f(z) = \Delta_\eta(\Delta_\eta^n f(z)) = e^{4z}(e^{4\eta} - 1)^{n+1} = e^{4z}(e^4 - 1)^{n+1}.$$

Example 1.2. Let $f(z) = e^{4z} + 1$, and let $\eta = 1$, where $a(z) = 1$, $A = 4$, $B = 1$. Then we can get

$$\Delta_\eta f(z) = f(z + \eta) - f(z) = e^{4(z+\eta)} + 1 - e^{4z} - 1 = e^{4z}(e^{4\eta} - 1) = e^{4z}(e^4 - 1),$$

$$\Delta_\eta^2 f(z) = f(z + 2\eta) - 2f(z + \eta) + f(z) = e^{4(z+2\eta)} + 1 - 2e^{4(z+\eta)} - 2 + e^{4z} + 1$$
$$= e^{4z}(e^{8\eta} - 2e^{4\eta} + 1) = e^{4z}(e^{4\eta} - 1)^2 = e^{4z}(e^4 - 1)^2.$$

By mathematical induction, we deduce

$$\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z)) = e^{4z}(e^{4\eta} - 1)^n = e^{4z}(e^4 - 1)^n,$$

$$\Delta_\eta^{n+1} f(z) = \Delta_\eta(\Delta_\eta^n f(z)) = e^{4z}(e^{4\eta} - 1)^{n+1} = e^{4z}(e^4 - 1)^{n+1}.$$

Remark 1.1. Example 1.1 and Example 1.2 show the existence of such transcendental entire function $f(z)$ of finite order satisfying the condition in Theorem 1.6.

Example 1.3. Let $f(z) = e^{4z} + z^{n+1}$, and let $\eta = 1$, where $a(z) = z^{n+1}$, $A = 4$, $B = 1$, $n \in \mathbb{N}$. Then we can get

$$\Delta_\eta f(z) = f(z + \eta) - f(z) = e^{4(z+\eta)} + (z + \eta)^{n+1} - e^{4z} - z^{n+1}$$

$$= e^{4z}(e^{4\eta} - 1) + C_{n+1}^1 z^n + \cdots + C_{n+1}^n z + 1,$$

$$\Delta_\eta^2 f(z) = f(z + 2\eta) - 2f(z + \eta) + f(z)$$

$$= e^{4(z+2\eta)} + (z + 2\eta)^{n+1} - 2e^{4(z+\eta)} - 2(z + \eta)^{n+1} + e^{4z} + z^{n+1}$$

$$= e^{4z}(e^8 - 2e^4 + 1) + (z + 2)^{n+1} - 2(z + 1)^{n+1} + z^{n+1}$$

$$= e^{4z}(e^4 - 1)^2 + 2C_{n+1}^2 z^{n-1} + \cdots + (2^n - 2)C_{n+1}^n z + 2^{n+1} - 2,$$

$$\Delta_\eta^3 f(z) = \Delta_\eta\left(\Delta_\eta^2 f(z)\right) = \Delta_\eta^2 f(z + \eta) - \Delta_\eta^2 f(z)$$

$$= e^{4(z+\eta)}(e^4 - 1)^2 + 2C_{n+1}^2 (z + \eta)^{n-1} + \cdots + (2^n - 2)C_{n+1}^n (z + \eta) + 2^{n+1}$$

$$- 2 - e^{4z}(e^4 - 1)^2 - 2C_{n+1}^2 z^{n-1} - \cdots - (2^n - 2)C_{n+1}^n z - 2^{n+1} + 2$$

$$= e^{4z}(e^4 - 1)^3 + 2C_{n+1}^2 \left[(z + 1)^{n-1} - z^{n-1}\right]$$

$$+ (2^3 - 2)C_{n+1}^3 \left[(z + 1)^{n-2} - z^{n-2}\right] + \cdots + (2^n - 2)C_{n+1}^n$$

$$= e^{4z}(e^4 - 1)^3 + 2C_{n+1}^2 C_{n-1}^1 z^{n-2}$$

$$+ \cdots + \left[2C_{n+1}^2 + (2^3 - 2)C_{n+1}^3 + \cdots + (2^n - 2)C_{n+1}^n\right].$$

By mathematical induction, we deduce

$$\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z)) = e^{4z}(e^4 - 1)^n + c(z),$$

$$\Delta_\eta^{n+1} f(z) = \Delta_\eta(\Delta_\eta^n f(z)) = e^{4z}(e^4 - 1)^{n+1} + C,$$

where $c(z)$ is a polynomial of deg $c(z) = 1$, and C is a nonzero constant.

Remark 1.2. Example 1.3 shows that if $a(z)$ is a polynomial, then $\Delta_\eta^n f(z)$ and $\Delta_\eta^{n+1} f(z)$ do not share 0 CM, that is, the conclusion that $a(z)$ reduces to a constant in Theorem 1.6 is accurate.

Remark 1.3. The idea used in this work comes from [7].

2. Some Lemmas

The following properties of the order and the exponent of the convergence of the zeros of canonical products play a key role in the proof of this paper.

Lemma 2.1 ([18]). *Let $f(z)$ be a nonconstant periodic meromorphic function. Then $f(z)$ has the order $\rho(f) \geq 1$.*

Lemma 2.2 ([18]). *Let $f(z)$ be a meromorphic function of finite order ρ, and*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots \ (c_k \neq 0)$$

in the neighborhood of $z = 0$. Suppose that $a_1, a_2, \cdots$ are nonzero zeros of $f(z)$ and $b_1, b_2, \cdots$ are nonzero poles of $f(z)$. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ is a canonical product of nonzero zeros of $f(z)$, $P_2(z)$ is a canonical product of nonzero poles of $f(z)$, and $Q(z)$ is a polynomial of degree at most ρ.

Lemma 2.3 ([18]). *The order ρ of the canonical product $P(z)$ is equal to the exponent of the convergence of the zeros of $P(z)$.*

Based on the definition of the ε-set [12], Bergweiler and Langley [1] came to the following conclusions.

Lemma 2.4 ([1]). *Let g be a transcendental meromorphic function of order less than 1, and let $h > 0$. Then there exists an ε-set E such that*

$$g(z + c) - g(z) = cg'(z)(1 + o(1)) \quad as \quad z \to \infty \quad in \quad \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leq h$.

Lemma 2.5 ([1]). *Let g be a transcendental meromorphic function of order< 1, and let h be a normal number. Then there exists a ε-set E such that for all c satisfying $|c| \leq h$, when $z \to \infty$ and $z \in \mathbb{C}\backslash E$, we have*

$$\frac{g'(z + c)}{g(z + c)} \to 0, \ \frac{g(z + c)}{g(z)} \to 1.$$

Further, the set E can be chosen such that for sufficiently large $|z| \notin E$, the function g has no zeros or poles in $|\zeta - z| \leq h$.

Lemma 2.6 ([7]). *Let $g(z)$ be a transcendental meromorphic function which satisfies $\rho(g) < 1$, and let $\eta \in \mathbb{C}\backslash\{0\}$. Then $\Delta_\eta^n g(z)$ and $G(z) = \frac{\Delta_\eta g(z)}{g(z)} = \frac{g(z+\eta)-g(z)}{g(z)}$ are both transcendental such that*

$$\liminf_{r \to \infty} \frac{T(r, \Delta_\eta^n g(z))}{r} = 0,$$

where $n \geq 1$ is an integer.

Lemma 2.7 ([7]). *Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_n P_0 \not\equiv 0$ and satisfy*

$$P_n(z) + \cdots + P_0(z) \not\equiv 0.$$

Then every finite order transcendental meromorphic solution $g(z)(\not\equiv 0)$ of the equation

$$P_n(z)g(z + n\eta) + P_{n-1}(z)g(z + (n-1)\eta) + \cdots + P_0(z)g(z) = 0$$

satisfies $\rho(g) \geq 1$, and $g(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and satisfies $\lambda(g - a) = \rho(g)$, where $\eta \in \mathbb{C}\backslash\{0\}$.

Lemma 2.8 ([7]). *Let $G(z)$, $P_n(z), \cdots, P_0(z)$ be polynomials such that $GP_n P_0 \not\equiv 0$. Then every finite order transcendental meromorphic solution $g(z)(\not\equiv 0)$ of the equation*

$$P_n(z)g(z + n\eta) + P_{n-1}(z)g(z + (n-1)\eta) + \cdots + P_0(z)g(z) = G$$

satisfies $\lambda(g) = \rho(g) \geq 1$, where $\eta \in \mathbb{C}\backslash\{0\}$.

The difference analogue of Clunie's lemma described below also plays an important role in the proof of this paper.

Lemma 2.9 ([15]). *Let g be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z, g)P(z, g) = Q(z, g),$$

where $U(z, g)$, $P(z, g)$, $Q(z, g)$ are difference polynomials in $g(z)$ and its shifts such that the total degree $\deg U(z, g) = n$, and $\deg Q(z, g) \leq n$. Moreover, we assume that $U(z, g)$ contains just one term of maximal total degree in $g(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m(r, P(z, g)) = O(r^{\rho - 1 + \varepsilon}) + S(r, g),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.10 ([8]). *Let g be a meromorphic function with a finite order ρ, η be a nonzero constant. Let $\varepsilon > 0$ be given. Then there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,*

$$\exp\{-r^{\rho - 1 + \varepsilon}\} \leq \left|\frac{g(z + \eta)}{g(z)}\right| \leq \exp\{r^{\rho - 1 + \varepsilon}\}.$$

Lemma 2.11 ([11, 18]). *Suppose that $n \geq 2$ and let $f_1(z), \cdots, f_n(z)$ be meromorphic functions and $g_1(z), \cdots, g_n(z)$ be entire functions such that*

(1) $\sum\limits_{j=1}^{n} f_j(z) \exp\{g_j(z)\} = 0$;

(2) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;

(3) when $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}, \quad r \to \infty, \ r \notin E,$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0$, $j = 1, \cdots, n$.

3. Proof of Theorem 1.6

Since $f(z)$ is a transcendental entire function of finite order, and $a(z)(\in S(f))$ is an entire function satisfying $\rho(a(z)) < 1$, by Lemma 2.2, we can write $f(z)$ in the form

$$(3.1) \qquad\qquad f(z) - a(z) = B(z)e^{h(z)},$$

where $B(z)(\not\equiv 0)$ is an entire function and $h(z)$ is a polynomial satisfying $\deg h = k$. And due to $\lambda(f - a(z)) < \rho(f)$, from Lemma 2.3, we know that $\lambda(B(z)) = \rho(B(z))$. Then we can get that

$$(3.2) \qquad \lambda(B) = \rho(B) = \lambda(f - a(z)) = \rho_1 < k = \rho(f) = \deg h,$$

implying that $k \geq 1$. If $k = 0$, then a contradiction can be obtained by comparing the order of growth of both sides of (3.1). According to the hypothesis of Theorem 1.1, $\Delta_\eta^{n+1} f(z)$ and $\Delta_\eta^n f(z)$ share $\Delta_\eta^n a(z)$ CM, we have

$$(3.3) \qquad \frac{\Delta_\eta^{n+1} f(z) - \Delta_\eta^n a(z)}{\Delta_\eta^n f(z) - \Delta_\eta^n a(z)} = e^{P(z)},$$

where $P(z)$ is a polynomial.

Substituting (3.1) into (3.3) yields

$$(3.4) \quad \frac{\Delta_\eta^{n+1} f(z) - \Delta_\eta^n a(z)}{\Delta_\eta^n f(z) - \Delta_\eta^n a(z)} = \frac{\sum\limits_{j=0}^{n+1}(-1)^{n+1-j} C_{n+1}^j B(z+j\eta) e^{h(z+j\eta)} + u(z)}{\sum\limits_{j=0}^{n}(-1)^{n-j} C_n^j B(z+j\eta) e^{h(z+j\eta)}} = e^{P(z)},$$

where $u(z) = \Delta_\eta^{n+1} a(z) - \Delta_\eta^n a(z)$ and

$$\rho(u(z)) \leq \max \left\{ \rho\left(\Delta_\eta^n a(z)\right), \ \rho\left(\Delta_\eta^{n+1} a(z)\right) \right\} \leq \rho(a(z)) < 1.$$

Now suppose that $\deg P(z) = s$. Set

(3.5)
$$h(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0,$$
$$P(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0,$$

where $k = \rho(f) \geq 1$, $a_k(\neq 0)$, $a_{k-1}, \cdots, a_0$, $b_s(\neq 0)$, $b_{s-1}, \cdots, b_0$ are constants and

$$0 \leq \deg P = s \leq \deg h = k.$$

CASE 1: If $\deg P(z) = s = 0$, then $e^{P(z)}$ is always a nonzero constant. We assume $e^{P(z)} = D$. Then (3.4) can be rewritten as

(3.6)
$$\frac{\Delta_\eta^{n+1} f(z) - \Delta_\eta^n a(z)}{\Delta_\eta^n f(z) - \Delta_\eta^n a(z)} = \frac{\sum\limits_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z+j\eta) e^{h(z+j\eta)} + u(z)}{\sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j B(z+j\eta) e^{h(z+j\eta)}} = D,$$

namely,

(3.7)
$$\sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z+j\eta) e^{h(z+j\eta)-h(z)}$$
$$- D \sum_{j=0}^{n} (-1)^{n-j} C_n^j B(z+j\eta) e^{h(z+j\eta)-h(z)} = -u(z) e^{-h(z)}.$$

Combining $\rho(B) < k$ with $\deg(h(z+j\eta) - h(z)) = k - 1 < k$ $(j = 0, \ 1, \cdots, n+1)$, it can be obtained that the order of growth of the left-hand side of (3.7) is less than k. However, since $\rho(u(z)) < 1$ and $\deg h(z) = k$, the order of growth of the right-hand side of (3.7) is equal to k. Therefore, by comparing the order of growth of the both sides of (3.7), we get $u(z) \equiv 0$, namely,

(3.8)
$$u(z) = \Delta_\eta^{n+1} a(z) - \Delta_\eta^n a(z) \equiv 0.$$

Secondly, we assume that $a(z)$ is a nonconstant entire function. Let

(3.9)
$$W(z) = a(z + \eta) - 2a(z).$$

Then $W(z)$ is also a nonconstant entire function. Combining $u(z) \equiv 0$ with (3.8), we get

$$
\begin{aligned}
u(z) &= \Delta_\eta^{n+1} a(z) - \Delta_\eta^n a(z) = \Delta_\eta \left(\Delta_\eta^n a(z)\right) - \Delta_\eta^n a(z) \\
&= \Delta_\eta^n a(z+\eta) - \Delta_\eta^n a(z) - \Delta_\eta^n a(z) = \Delta_\eta^n a(z+\eta) - 2\Delta_\eta^n a(z) \equiv 0.
\end{aligned}
$$

Thus according to the definitions of difference operator and n-order difference operator expressions, we have

$$
\begin{aligned}
\Delta_\eta^n W(z) &= \Delta_\eta^n(a(z+\eta) - 2a(z)) \\
&= \sum_{j=0}^n (-1)^{n-j} C_n^j [a(z+\eta+j\eta) - 2a(z+j\eta)] \\
&= \sum_{j=0}^n (-1)^{n-j} C_n^j a(z+\eta+j\eta) - 2\sum_{j=0}^n (-1)^{n-j} C_n^j a(z+j\eta) \\
&= \Delta_\eta^n a(z+\eta) - 2\Delta_\eta^n a(z) \equiv 0.
\end{aligned}
$$

By

$$
\Delta_\eta^n W(z) = \Delta_\eta \left(\Delta_\eta^{n-1} W(z)\right) = \Delta_\eta^{n-1} W(z+\eta) - \Delta_\eta^{n-1} W(z) \equiv 0,
$$

we get $\Delta_\eta^{n-1} W(z+\eta) \equiv \Delta_\eta^{n-1} W(z)$. Let $H(z) = \Delta_\eta^{n-1} W(z)$. Then we can get $H(z+\eta) \equiv H(z)$, which means that $H(z)$ is a periodic function with period η. From Lemma 2.1 we can obtain $\rho(H(z)) \geq 1$. And because of the $\rho\left(\Delta_\eta^{n-1} W(z)\right) \leq \rho(W(z)) \leq \rho(a(z)) < 1$, $\rho(H(z)) \leq \rho(W(z)) < 1$ is contradictory. Thus, W is a constant.

If $W \neq 0$, then (3.9) is changed into the following form

(3.10) $$\frac{a(z+\eta)}{a(z)} - 2 = \frac{W}{a(z)}.$$

Thus, according to Lemma 2.5, there exists a ε-set E_0 with finite logarithmic measure such that for all $z \to \infty$ satisfying $z \in \mathbb{C}\backslash E_0$,

$$\frac{a(z+\eta)}{a(z)} \to 1,$$

and when $|z| = r \to +\infty$, $z \in \{\xi \in \mathbb{C} / |a(\xi)| = M(r,a)\}$ and $z \notin E_0$,

$$\frac{W}{a(z)} \to 0.$$

Therefore, a contradiction can be obtained from (3.10).

If $W = 0$, then (3.9) is changed into the following form

$$\frac{a(z+\eta)}{a(z)} = 2.$$

Thus, we can also obtain a contradiction from Lemma 2.5. Therefore, $a(z) \equiv a$ is constant, and (3.6) is changed into the following form

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z+j\eta)e^{h(z+j\eta)} = D\sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)},$$

that is,

(3.11)
$$\sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z+j\eta)e^{h(z+j\eta)}$$
$$- D\sum_{j=1}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)} = (-1)^n(1+D)B(z)e^{h(z)}.$$

Next we assume that $\rho(f) = \deg h = k \geq 2$. Then we will derive the contradiction from the following two cases.

SUBCASE 1.1: If $D = -1$, then from (3.11) we get

$$\sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z+j\eta)e^{h(z+j\eta)} + \sum_{j=1}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)} = 0,$$

namely,

(3.12)
$$\sum_{j=1}^{n}(-1)^{n+1-j}B(z+j\eta)e^{h(z+j\eta)}[C_{n+1}^j - C_n^j]$$
$$+ B(z+(n+1)\eta)e^{h(z+(n+1)\eta)} = 0.$$

According to the combination number formula $C_n^j = \dfrac{n!}{j!(n-j)!}$, we can get $C_{n+1}^j - C_n^j = C_n^{j-1}$. By combining (3.12), we get

$$\sum_{j=1}^{n+1}(-1)^{n+1-j}C_n^{j-1}B(z+j\eta)e^{h(z+j\eta)} = 0,$$

that is,

(3.13) $$\sum_{j=2}^{n+1}(-1)^{n+1-j}C_n^{j-1}\frac{B(z+j\eta)}{B(z+\eta)}e^{h(z+j\eta)-h(z+\eta)} = (-1)^{n+1}.$$

SUBCASE 1.1.1: If $n = 1$, then (3.13) is changed into the following form

(3.14) $$e^{h(z+2\eta)-h(z+\eta)} = \frac{B(z+\eta)}{B(z+2\eta)}.$$

Thus, $S_1(z) = \frac{B(z+\eta)}{B(z+2\eta)}$ is a nonconstant entire function.

By Lemma 2.10, (3.2) and (3.14), we see that for ε_1 $(0 < 3\varepsilon_1 < \deg h - \rho_1)$, there exists a set $E_1 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$,

$$\exp\{-r^{\rho_1 - 1 + \varepsilon_1}\} \le \left| \frac{B(z+\eta)}{B(z+2\eta)} \right| \le \exp\{r^{\rho_1 - 1 + \varepsilon_1}\}.$$

Then

$$m\left(r, \frac{B(z+\eta)}{B(z+2\eta)}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{B(z+\eta)}{B(z+2\eta)} \right| d\theta$$
$$\le \frac{1}{2\pi} \int_0^{2\pi} \log^+ e^{r^{\rho_1 - 1 + \varepsilon_1}} d\theta = r^{\rho_1 - 1 + \varepsilon_1},$$

which yields

$$T(r, S_1(z)) = m(r, S_1(z)) = m\left(r, \frac{B(z+\eta)}{B(z+2\eta)}\right) \le r^{\rho_1 - 1 + \varepsilon_1}.$$

According to the definition of the order, we know $\rho(S_1(z)) \le \rho_1 - 1 + \varepsilon_1 < \deg h - 1$. In this way, a contradiction can be obtained by comparing the order of growth of the both sides of (3.14).

SUBCASE 1.1.2: If $n \ge 2$, then we have $P_1(z) = e^{h(z+2\eta) - h(z+\eta)}$. Because of $\deg h = k \ge 2$, we know that $\rho(P_1(z)) = \deg h - 1 \ge 1$, that is, $P_1(z)$ is a transcendental entire function. Thus, for $j = 2, 3, \cdots, n+1$, we have

$$e^{h(z+j\eta) - h(z+\eta)} = e^{h(z+j\eta) - h(z+(j-1)\eta)} e^{h(z+(j-1)\eta) - h(z+(j-2)\eta)} \cdots e^{h(z+2\eta) - h(z+\eta)}$$
$$= P_1(z + (j-2)\eta) P_1(z + (j-3)\eta) \cdots P_1(z).$$

Hence, (3.13) can be rewritten as

(3.15) $$L(z, P_1(z)) \cdot P_1(z) = (-1)^{n+1},$$

where

(3.16)
$$L(z, P_1(z)) = \frac{B(z + (n+1)\eta)}{B(z+\eta)} P_1(z + (n-1)\eta) P_1(z + (n-2)\eta) \cdots P_1(z+\eta)$$
$$- C_n^{n-1} \frac{B(z+n\eta)}{B(z+\eta)} P_1(z + (n-2)\eta) P_1(z + (n-3)\eta) \cdots P_1(z+\eta)$$
$$+ \cdots + (-1)^{n-1} C_n^1 \frac{B(z+2\eta)}{B(z+\eta)}.$$

Since $n \geq 2$, we can get $\deg(P_1(z)L(z, P_1(z))) = n - 1 \geq 1$. Therefore, by Lemma 2.9, for $j = 2, 3, \cdots, n + 1$, if the coefficient of $L(z, P_1(z))$ satisfies

$$m\left(r, \frac{B(z + j\eta)}{B(z + \eta)}\right) = S(r, P_1(z)),$$

then

$$m(r, P_1(z)) = S(r, P_1(z)).$$

Now let's prove $m\left(r, \frac{B(z+j\eta)}{B(z+\eta)}\right) = S(r, P_1(z))$.

By Lemma 2.10, (3.2) and (3.16), we see that for ε_2 $(0 < 3\varepsilon_2 < \deg h - \rho_1)$, there exists a set $E_2 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$,

$$(3.17) \quad \exp\{-r^{\rho_1 - 1 + \varepsilon_2}\} \leq \left|\frac{B(z + j\eta)}{B(z + \eta)}\right| \leq \exp\{r^{\rho_1 - 1 + \varepsilon_2}\} \quad (j = 2, 3, \cdots, n + 1).$$

Thus, by (3.17), for $j = 2, 3, \cdots, n + 1$, we have

$$m\left(r, \frac{B(z + j\eta)}{B(z + \eta)}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{B(z + j\eta)}{B(z + \eta)}\right| d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ e^{r^{\rho_1 - 1 + \varepsilon_2}} d\theta = r^{\rho_1 - 1 + \varepsilon_2}.$$

Since $(-1)^{n+1} \neq 0$ and $P_1(z)$ is a transcendental entire function, $L(z, P_1(z)) \not\equiv 0$ can be obtained from (3.15). And since $P_1(z)$ is of regular growth and $\rho(P_1(z)) = k - 1$, for any given ε_2 $(0 < 3\varepsilon_2 < \deg h - \rho_1)$ and all $r > r_0(> 0)$, we have

$$(3.18) \qquad\qquad T(r, P_1(z)) > r^{\deg h - 1 - \varepsilon_2}.$$

Thus

$$\frac{m\left(r, \frac{B(z+j\eta)}{B(z+\eta)}\right)}{T(r, P_1(z))} \leq \frac{r^{\rho_1 - 1 + \varepsilon_2}}{r^{\deg h - 1 - \varepsilon_2}} \to 0 \quad (j = 2, 3, \cdots, n + 1),$$

namely,

$$m\left(r, \frac{B(z + j\eta)}{B(z + \eta)}\right) = o(T(r, P_1)) = S(r, P_1) \quad (j = 2, 3, \cdots, n + 1).$$

Since $P_1(z)$ is a transcendental entire function, it is obvious that

$$T(r, P_1(z)) = m(r, P_1(z)) = S(r, P_1(z)),$$

which is a contradiction.

Subcase 1.2: If $D \neq -1$, then from (3.11), we can get

$$
\begin{aligned}
&\sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j \frac{B(z+j\eta)}{B(z)}e^{h(z+j\eta)-h(z)} \\
&- D\sum_{j=1}^{n}(-1)^{n-j}C_n^j \frac{B(z+j\eta)}{B(z)}e^{h(z+j\eta)-h(z)} = (-1)^n(1+D).
\end{aligned}
$$
(3.19)

Set $P_2(z) = e^{h(z+\eta)-h(z)}$. Then $\rho(P_2(z)) = \deg h - 1 \geq 1$, $P_2(z)$ is a transcendental entire function. Thus, for $j = 1,\ 2,\cdots, n+1$, we have

$$
e^{h(z+j\eta)-h(z)} = e^{h(z+j\eta)-h(z+(j-1)\eta)}e^{h(z+(j-1)\eta)-h(z+(j-2)\eta)}\cdots e^{h(z+\eta)-h(z)}
$$

$$
= P_2(z+(j-1)\eta)P_2(z+(j-2)\eta)\cdots P_2(z).
$$

Hence (3.19) is changed into the following form

$$
L(z, P_2(z)) \cdot P_2(z) = (-1)^n(1+D),
$$
(3.20)

where

$$
\begin{aligned}
L(z, P_2(z)) =& \left(\frac{B(z+(n+1)\eta)}{B(z)}P_2(z+n\eta)P_2(z+(n-1)\eta)\cdots P_2(z+\eta)\right. \\
& - C_{n+1}^n \frac{B(z+n\eta)}{B(z)}P_2(z+(n-1)\eta)P_2(z+(n-2)\eta)\cdots P_2(z+\eta) \\
& + \cdots + (-1)^n C_{n+1}^1 \frac{B(z+\eta)}{B(z)}\right) - D\left(\frac{B(z+n\eta)}{B(z)}P_2(z+(n-1)\eta)\right. \\
& P_2(z+(n-2)\eta)\cdots P_2(z+\eta) - C_n^{n-1}\frac{B(z+(n-1)\eta)}{B(z)}P_2(z+(n-2)\eta) \\
& \left. P_2(z+(n-3)\eta)\cdots P_2(z+\eta) + \cdots + (-1)^{n-1}C_n^1 \frac{B(z+\eta)}{B(z)}\right).
\end{aligned}
$$

Thus, according to (3.20), we can derive a contradiction by means similar to the Subcase 1.1.2.

Therefore, combining the discussion of Subcase 1.1 with Subcase 1.2, we can get $\rho(f) = \deg h(z) = k = 1$. Let $h(z) = Az + A_0$, where $A(\neq 0)$, A_0 are constants. Combining (3.1) with (3.2), we can get

$$
f(z) = a(z) + B(z)e^{Az+A_0} = a(z) + B_1(z)e^{Az},
$$
(3.21)

where $B_1 = B(z)e^{A_0}(\not\equiv 0)$ is an entire function such that

$$
\rho(B_1(z)) = \lambda(B_1(z)) = \lambda(f(z)-a(z)) < \rho(f) = 1.
$$

Finally, we need to prove that $B_1(z)$ is constant. Substituting (3.21) into (3.6), by $u(z) \equiv 0$ we can get

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B_1(z+j\eta)e^{A(z+j\eta)} = D\sum_{j=0}^{n}(-1)^{n-j}C_n^j B_1(z+j\eta)e^{A(z+j\eta)},$$

namely,

$$(3.22) \quad \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B_1(z+j\eta)e^{Aj\eta} - D\sum_{j=0}^{n}(-1)^{n-j}C_n^j B_1(z+j\eta)e^{Aj\eta} = 0.$$

Obviously, the sum of all coefficients of (3.22) is 0. If not, then from Lemma 2.7, $\rho(B_1(z)) \geq 1$, which is contradictory. Thus we get

$$(3.23) \quad \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j e^{Aj\eta} - D\sum_{j=0}^{n}(-1)^{n-j}C_n^j e^{Aj\eta} = 0,$$

namely,

$$(3.24) \quad (e^{A\eta}-1)^{n+1} - D(e^{A\eta}-1)^n = 0.$$

So from (3.24) we can derive either $e^{A\eta} = 1$ or $e^{A\eta} = 1+D$.

If $e^{A\eta} = 1$, then (3.22) can be written as

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B_1(z+j\eta) - D\sum_{j=0}^{n}(-1)^{n-j}C_n^j B_1(z+j\eta) = 0,$$

that is,

$$\Delta_\eta^{n+1}B_1(z) - D\Delta_\eta^n B_1(z) = 0.$$

According to the definition of difference operator, we get

$$(3.25) \quad \begin{aligned} \Delta_\eta^{n+1}B_1(z) - D\Delta_\eta^n B_1(z) &= \Delta_\eta\left(\Delta_\eta^n B_1(z)\right) - D\Delta_\eta^n B_1(z) \\ &= \Delta_\eta^n B_1(z+\eta) - \Delta_\eta^n B_1(z) - D\Delta_\eta^n B_1(z) \\ &= \Delta_\eta^n B_1(z+\eta) - (1+D)\Delta_\eta^n B_1(z) = 0. \end{aligned}$$

Combining (3.25) with Lemma 2.7, we know $\rho\left(\Delta_\eta^n B_1(z)\right) \geq 1$. This contradicts

$$\rho\left(\Delta_\eta^n B_1(z)\right) \leq \rho(B_1(z)) < 1.$$

Thus, we have $e^{A\eta} = 1+D$.

Note that (3.23) can be rewritten as

$$(3.26) \quad \sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j e^{Aj\eta} - D\sum_{j=1}^{n}(-1)^{n-j}C_n^j e^{Aj\eta} = (-1)^n(1+D).$$

Then combining (3.22) with (3.26), we can get

(3.27)
$$\sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j e^{Aj\eta}[B_1(z+j\eta)-B_1(z)]$$
$$-D\sum_{j=1}^{n}(-1)^{n-j}C_n^j e^{Aj\eta}[B_1(z+j\eta)-B_1(z)] = 0.$$

Thus, according to Lemma 2.4, there exists $n+1$ ε-set E_k^*, such that for $k = 1, 2, \cdots, n+1$, when $z \to \infty$ and $z \in \mathbb{C}\backslash E_k^*$,

$$B_1(z+k\eta)-B_1(z) = k\eta B_1'(z)(1+o(1)).$$

So (3.27) is changed into the following form

$$\sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j e^{Aj\eta}\left[j\eta B_1'(z)(1+o(1))\right]$$
$$-D\sum_{j=1}^{n}(-1)^{n-j}C_n^j e^{Aj\eta}\left[j\eta B_1'(z)(1+o(1))\right] = 0,$$

namely,

(3.28)
$$\left(\sum_{j=1}^{n+1}(-1)^{n+1-j}jC_{n+1}^j e^{Aj\eta} - D\sum_{j=1}^{n}(-1)^{n-j}jC_n^j e^{Aj\eta}\right)(\eta B_1'(z)(1+o(1))) = 0.$$

Let

(3.29)
$$K = \sum_{j=1}^{n+1}(-1)^{n+1-j}jC_{n+1}^j e^{Aj\eta} - D\sum_{j=1}^{n}(-1)^{n-j}jC_n^j e^{Aj\eta}.$$

Obviously $K \neq 0$. If $K = 0$, then for $j = 1, 2, \cdots, n+1$, we have

$$\begin{cases} jC_{n+1}^j = \frac{j(n+1)!}{j!(n+1-j)!} = \frac{(n+1)n!}{(j-1)!(n+1-j)!} = (n+1)C_n^{j-1}, \\ jC_n^j = \frac{jn!}{j!(n-j)!} = \frac{n(n-1)!}{(j-1)!(n-j)!} = nC_{n-1}^{j-1}. \end{cases}$$

Thus (3.29) can be rewritten as

(3.30)
$$K = (n+1)\sum_{j=1}^{n+1}(-1)^{n+1-j}C_n^{j-1} e^{Aj\eta} - nD\sum_{j=1}^{n}(-1)^{n-j}C_{n-1}^{j-1} e^{Aj\eta}.$$

Let $k = j-1$. Then (3.30) can be written as

$$K = (n+1)e^{A\eta}\sum_{k=0}^{n}(-1)^{n-k}C_n^k e^{Ak\eta} - nDe^{A\eta}\sum_{k=0}^{n-1}(-1)^{n-1-k}C_{n-1}^k e^{Ak\eta},$$

namely,

$$(3.31) \qquad K = (n+1)e^{A\eta}(e^{A\eta} - 1)^n - nDe^{A\eta}(e^{A\eta} - 1)^{n-1} = 0.$$

Substituting $e^{A\eta} = 1 + D$ into (3.31), we get

$$(3.32) \qquad \begin{aligned} K &= (n+1)e^{A\eta}(e^{A\eta} - 1)^n - nDe^{A\eta}(e^{A\eta} - 1)^{n-1} \\ &= (n+1)e^{A\eta}D^n - nDe^{A\eta}D^{n-1} = 0, \end{aligned}$$

where D, $e^{A\eta}$ are nonzero constants. This means that $n + 1 = n$, a contradiction.

To sum up, we can get $K \neq 0$. Since both η and $(1 + o(1))$ are non-zero constants, $B_1'(z) \equiv 0$ can be obtained from (3.28). That is, it can be proved that $B_1(z) \equiv B$ is a constant.

So Theorem 1.6 is proved in Case 1.

CASE 2: If $\deg P(z) = s \geq 1$, then we can discuss it in the following two cases.

SUBCASE 2.1: If $1 \leq s = k$, then (3.4) can be written as

$$(3.33) \qquad \begin{aligned} &\sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z + j\eta)e^{h(z+j\eta)} + u(z) \\ &= \left(\sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z + j\eta)e^{h(z+j\eta)}\right)e^{P(z)}. \end{aligned}$$

SUBCASE 2.1.1: If $a_k + b_k = 0$, then (3.33) can be written as

$$(3.34) \qquad \begin{aligned} &\left(\sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z + j\eta)e^{h(z+j\eta)-h(z)} - u(z)e^{-h(z)-P(z)}\right)e^{P(z)} \\ &= \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z + j\eta)e^{h(z+j\eta)-h(z)}. \end{aligned}$$

Let

$$(3.35) \qquad H_1(z) = \sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z + j\eta)e^{h(z+j\eta)-h(z)} - u(z)e^{-h(z)-P(z)}.$$

Since $\deg P(z) = s = k = \deg h(z)$ and $a_k + b_k = 0$, it follows that $\deg(h(z + j\eta) - h(z)) = \deg(-h(z) - P(z)) = k - 1$ $(j = 1, 2, \cdots, n+1)$. Since $\rho(B(z)) < \rho(f) = \deg h(z) = k$, the order of growth of the both sides of (3.34) can be compared to $H_1(z) \equiv 0$. If $H_1(z) \equiv 0$, then by (3.34), we can get

$$(3.36) \qquad \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z + j\eta)e^{h(z+j\eta)-h(z)} \equiv 0,$$

namely,

$$(3.37) \qquad \sum_{j=1}^{n+1}(-1)^{n+1-j}C_{n+1}^j \frac{B(z+j\eta)}{B(z)}e^{h(z+j\eta)-h(z)} = (-1)^n.$$

Similar to the method in Subcase 1.1.2, it can be deduced that $\rho(f) = \deg h = 1$. Let $h(z) = a_1 z + a_0$ $(a_1 \neq 0)$. Then by (3.36), we can get

$$(3.38) \qquad \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z+j\eta)e^{a_1 j\eta} \equiv 0.$$

Obviously, the sum of the coefficients of (3.38) is 0. If not, then we can assume that the coefficient sum of (3.38) is not 0. Thus, we know from Lemma 2.7 that $\rho(B(z)) \geq 1$, which contradicts $\rho(B(z)) < \rho(f) = 1$. Therefore

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j e^{a_1 j\eta} = (e^{a_1\eta}-1)^{n+1} = 0,$$

which implies $e^{a_1\eta} = 1$. Substituting this into (3.38), we can get

$$(3.39) \qquad \Delta_\eta^{n+1}B(z) = \sum_{j=0}^{n+1}(-1)^{n+1-j}C_{n+1}^j B(z+j\eta) \equiv 0.$$

If $B(z)$ is a transcendental entire function, then it follows from Lemma 2.6 that $\Delta_\eta^{n+1}B(z)$ is also a transcendental entire function, which contradicts $\Delta_\eta^{n+1}B(z) \equiv 0$.

If $B(z)$ is non-zero polynomial, then by (3.35), $H_1(z) \equiv 0$, which means

$$
\begin{aligned}
u(z) &= \Delta_\eta^{n+1}a(z) - \Delta_\eta^n a(z) \\
&= \Delta_\eta^n a(z+\eta) - \Delta_\eta^n a(z) - \Delta_\eta^n a(z) \\
&= \Delta_\eta^n a(z+\eta) - 2\Delta_\eta^n a(z) \\
&= \left(\sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)-h(z)}\right)e^{h(z)+P(z)}.
\end{aligned}
$$

$$(3.40)$$

Assume that

$$\sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)-h(z)} \equiv 0.$$

Then we have

$$\Delta_\eta^n B(z)e^{h(z)} = \sum_{j=0}^{n}(-1)^{n-j}C_n^j B(z+j\eta)e^{h(z+j\eta)} \equiv 0,$$

that is,

$$\Delta_\eta^{n+1} B(z)e^{h(z)} = \Delta_\eta \left(\Delta_\eta^n B(z)e^{h(z)} \right)$$

(3.41)
$$= \sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z+j\eta)e^{h(z+j\eta)} \equiv 0.$$

Combining (3.1) with (3.4), we get

(3.42) $$\Delta_\eta^{n+1} f(z) - \Delta_\eta^n a(z) = u(z) = \Delta_\eta^{n+1} a(z) - \Delta_\eta^n a(z) \equiv 0,$$

which contradicts the condition $\Delta_\eta^n a(z) \in S\left(\Delta_\eta^{n+1} f(z)\right)$ in Theorem 1.1.

Assume that

$$\sum_{j=0}^n (-1)^{n-j} C_n^j B(z+j\eta)e^{h(z+j\eta)-h(z)} \not\equiv 0.$$

Since $b_1 + a_1 = 0$, $\deg h(z) = 1$, $e^{h(z)+P(z)}$ and $e^{h(z+j\eta)-h(z)}$ $(j = 0, 1, \cdots, n)$ are non-zero constants, and

$$\left(\sum_{j=0}^n (-1)^{n-j} C_n^j B(z+j\eta)e^{h(z+j\eta)-h(z)} \right) e^{h(z)+P(z)}$$

is a nonzero polynomial. Therefore, according to Lemma 2.8, $\rho\left(\Delta_\eta^n a(z)\right) \geq 1$, which contradicts $\rho\left(\Delta_\eta^n a(z)\right) \leq \rho(a(z)) < 1$. To sum up, $H_1(z) \not\equiv 0$ is contradictory.

SUBCASE 2.1.2: If $a_k + b_k \neq 0$, then (3.33) can be written as

(3.43) $$G_{11}(z)e^{P(z)} + G_{12}(z)e^{-h(z)} + G_{13}(z)e^{h_0(z)} = 0,$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} G_{11}(z) = -\sum_{j=0}^n (-1)^{n-j} C_n^j B(z+j\eta)e^{h(z+j\eta)-h(z)}, \\ G_{12}(z) = u(z), \\ G_{13}(z) = \sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z+j\eta)e^{h(z+j\eta)-h(z)}. \end{cases}$$

Since $\rho(B(z)) < \rho(f) = k$, $\deg(h(z+j\eta) - h(z)) = k-1 < k$ $(j = 1, 2, \cdots, n+1)$,

$$\rho\left(G_{1m}(z)\right) < k, m = 1, 2, 3,$$

$$\deg(P+h) = \deg(P-h_0) = \deg(-h-h_0) = k,$$

which means that for $m = 1,\ 2,\ 3$, we have

(3.44)
$$\begin{cases} T(r, G_{1m}) = o\left(T(r, e^{P+h})\right), \\ T(r, G_{1m}) = o\left(T(r, e^{P})\right), \\ T(r, G_{1m}) = o\left(T(r, e^{-h})\right). \end{cases}$$

Applying Lemma 2.11 to (3.43), we can get $G_{1m}(z) \equiv 0$ ($m = 1,\ 2,\ 3$). If $G_{13}(z) \equiv 0$, then

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z + j\eta) e^{h(z+j\eta)-h(z)} \equiv 0.$$

So we can derive

$$\sum_{j=1}^{n+1} (-1)^{n+1-j} C_{n+1}^j \frac{B(z + j\eta)}{B(z)} e^{h(z+j\eta)-h(z)} = (-1)^n.$$

Using the method similar to Subcase 2.1.1, we can also get a contradiction.

SUBCASE 2.2: If $1 \le s < k$, then (3.4) can be written as

(3.45)
$$\sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z + j\eta) e^{h(z+j\eta)-h(z)}$$
$$- \left(\sum_{j=0}^{n} (-1)^{n-j} C_n^j B(z + j\eta) e^{h(z+j\eta)-h(z)} \right) e^{P(z)} = -u(z) e^{-h(z)}.$$

By comparing the order of growth of both sides of (3.45), it is obvious that $u(z) \equiv 0$. Therefore, (3.45) can be rewritten as

(3.46)
$$\sum_{j=0}^{n+1} (-1)^{n+1-j} C_{n+1}^j B(z + j\eta) e^{h(z+j\eta)-h(z)}$$
$$- \left(\sum_{j=0}^{n} (-1)^{n-j} C_n^j B(z + j\eta) e^{h(z+j\eta)-h(z)} \right) e^{P(z)} = 0.$$

In Subcase 2.2, we consider the following two subcases.

SUBCASE 2.2.1: Suppose that $\deg P = s < k - 1 = \deg h - 1$. Set $P_2(z) = e^{h(z+\eta)-h(z)}$. Then $\rho(P_2(z)) = \deg h - 1 \ge 1$, that is, $P_2(z)$ is a transcendental entire function. Thus, for $j = 1, 2, \cdots, n + 1$, we have

$$e^{h(z+j\eta)-h(z)} = e^{h(z+j\eta)-h(z+(j-1)\eta)} e^{h(z+(j-1)\eta)-h(z+(j-2)\eta)} \cdots e^{h(z+\eta)-h(z)}$$

$$= P_2(z + (j - 1)\eta) P_2(z + (j - 2)\eta) \cdots P_2(z).$$

Then (3.46) is changed into the following form

$$(3.47) \qquad L_1(z, P_2(z)) \cdot P_2(z) = (-1)^n \left(1 + e^{P(z)}\right),$$

where

$$L_1(z, P_2(z)) = \left(\frac{B(z + (n+1)\eta)}{B(z)} P_2(z + n\eta) P_2(z + (n-1)\eta) \cdots P_2(z + \eta)\right.$$

$$- C_{n+1}^n \frac{B(z + n\eta)}{B(z)} P_2(z + (n-1)\eta) P_2(z + (n-2)\eta) \cdots P_2(z + \eta)$$

$$+ \cdots + (-1)^n C_{n+1}^1 \frac{B(z + \eta)}{B(z)}\right) - e^{P(z)} \left(\frac{B(z + n\eta)}{B(z)} P_2(z + (n-1)\eta)\right.$$

$$P_2(z + (n-2)\eta) \cdots P_2(z + \eta) - C_n^{n-1} \frac{B(z + (n-1)\eta)}{B(z)} P_2(z + (n-2)\eta)$$

$$P_2(z + (n-3)\eta) \cdots P_2(z + \eta) + \cdots + (-1)^{n-1} C_n^1 \frac{B(z + \eta)}{B(z)}\right).$$

Since $\rho\left((-1)^n(1 + e^{P(z)})\right) = \deg P = s < k - 1$, we can get

$$m\left(r, (-1)^n(1 + e^{P(z)})\right) = S(r, P_2(z)).$$

And similar to the discussion of Subcase 1.1.2, we know

$$m\left(r, \frac{B(z + j\eta)}{B(z)}\right) = S(r, P_2(z)) \quad (j = 1, 2, \cdots, n + 1).$$

Therefore, by using methods similar to Subcase 1.1.2, we can get

$$T(r, P_2(z)) = m(r, P_2(z)) = S(r, P_2(z)),$$

which is a contradiction.

SUBCASE 2.2.2: Suppose that $1 \leq \deg P = s = k - 1 = \deg h - 1$. Set $V(z) = \sum_{j=0}^{n} (-1)^{n-j} C_n^j B(z + j\eta) e^{h(z+j\eta) - h(z)}$. Then $\rho_2 = \rho(V) \leq \max\{\rho(B), \ k - 1\} < k$, and (3.46) can be rewritten as

$$(3.48) \qquad e^{P(z)} = \frac{V(z + \eta)}{V(z)} e^{h(z+\eta) - h(z)} - 1.$$

Obviously, $\frac{V(z+\eta)}{V(z)}$ is an entire function. By Lemma 2.10, we see that for ε_3 ($0 < 3\varepsilon_3 < \deg h - \rho_2$), there exists a set $E_3 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$,

$$(3.49) \qquad \exp\{-r^{\rho_2 - 1 + \varepsilon_3}\} \leq \left|\frac{V(z + \eta)}{V(z)}\right| \leq \exp\{r^{\rho_2 - 1 + \varepsilon_3}\} \ (j = 1, \ 2, \cdots, n).$$

Since $\frac{V(z+\eta)}{V(z)}$ is an entire function, by (3.49), we can get

$$T\left(r, \frac{V(z+\eta)}{V(z)}\right) = m\left(r, \frac{V(z+\eta)}{V(z)}\right) \leq r^{\rho_2 - 1 + \varepsilon_3},$$

namely,

$$\rho\left(\frac{V(z+\eta)}{V(z)}\right) \leq \rho_2 - 1 + \varepsilon_3 < k - 1.$$

Since

$$T\left(r, \frac{V(z)}{V(z+\eta)}\right) + O(1) = T\left(r, \frac{V(z+\eta)}{V(z)}\right) \leq r^{\rho_2 - 1 + \varepsilon_3},$$

we get

$$N\left(r, \frac{V(z)}{V(z+\eta)}\right) \leq T\left(r, \frac{V(z)}{V(z+\eta)}\right) \leq r^{\rho_2 - 1 + \varepsilon_3}.$$

Combining (3.48) with the second main theorem of Nevanlinna, we can get

$$T\left(r, e^{P(z)}\right) \leq \overline{N}\left(r, e^{P(z)}\right) + \overline{N}\left(r, \frac{1}{e^{P(z)}}\right) + \overline{N}\left(r, \frac{1}{e^{P(z)} + 1}\right) + S(r, e^{P(z)})$$

$$\leq \overline{N}\left(r, \frac{1}{\frac{V(z+\eta)}{V(z)} e^{h(z+\eta) - h(z)}}\right) + S(r, e^{P(z)})$$

$$\leq N\left(r, \frac{1}{\frac{V(z+\eta)}{V(z)} e^{h(z+\eta) - h(z)}}\right) + S(r, e^{P(z)})$$

$$\leq r^{\rho_2 - 1 + \varepsilon_3} + S(r, e^{P(z)}) = S(r, e^{P(z)}),$$

which contradicts that $e^{P(z)}$ is transcendental.

Thus, combining Case 1 with Case 2, Theorem 1.6 is proved. $\qquad \square$

Acknowledgment

References

1. W. Bergweiler & J.K. Langley: Zeros of differences of meromorphic functions. Math. Proc. Camb. Phil. Soc. **142** (2007), 133-147.
2. K.S. Charak, R.J. Korhonen & G. Kumar: A note on partial sharing of values of meromorphic functions with their shifts. J. Math. Anal. Appl. **435** (2016), 1241-1248.
3. Z.X. Chen: Complex Differences and Difference Equations. Science Press, Beijing, 2014.

4. Z.X. Chen: On difference counterpart of Brück's conjecture. Acta. Math. Sci. Ser. B **3** (2014), 653-659.

5. C.X. Chen & Z.X. Chen: Entire functions and their higher order differences. Taiwan J. Math. **18** (2014), 711-729.

6. Z.X. Chen & H.X. Yi: On sharing values of meromorphic functions and their differences. Results Math. **63** (2013), 557-565.

7. C.X. Chen & R.R. Zhang: Uniqueness theorems related to difference operators of entire functions. Ann. Math. Ser. A (Chinese version) **42** (2021), 11-22.

8. Y.M. Chiang & S.J. Feng: On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane. Ramanujan J. **16** (2008), 105-129.

9. B.M. Deng, D. Liu & M.L. Fang. Unicity of meromorphic functions and their difference operators. Ann. Math. Ser. A (Chinese version) **39** (2018), 341-348.

10. A.E. Farissi, Z. Latreuch & A. Asiri: On the uniqueness theory of entire functions and their difference operators. Complex Anal. Oper. Theory **10** (2016), 1317-1327.

11. F. Gross: Factorization of Meromorphic Functions. Washington: U.S. Government Printing Office, 1972.

12. W.K. Hayman: Meromorphic Functions. Oxford: Clarendon Press, 1964.

13. Z.B. Huang & R.R. Zhang: Uniqueness of the differences of meromorphic functions. Analysis Math. **44** (2018), 461-473.

14. G. Jank, E. Mues & L. Volkman: Meromorphe funktionen die mit ihrerersten und zweiten ableitung einen wertteilen. Complex Var. Theory Appl. **6** (1986), 51-71.

15. I. Laine & C.C. Yang: Clunie theorems for difference and q-difference polynomials. J. Lond Math. Soc. **76** (2007), 556-566.

16. X.M. Li, Y. Liu & H.X. Yi: Meromorphic functions sharing four values with their difference operators. Comput. Methods Funct. Theory **21** (2021), 317-341.

17. L.A. Rubel & C.C. Yang: Values shared by an entire function and its derivative. Lect. Notes Math. Berlin, Springer-Verlag. **599** (1977), 101-103.

18. C.C. Yang & H.X. Yi: Uniqueness Theory of Meromorphic Functions. Kluwer Academic Publishers, Dordrecht, 2003.

[a]Professor: School of Mathematics and Statistics, Fujian Normal University, Fuzhou 350117, P.R. China
Email address: xinmeiliu1999@163.com

[b]Professor: School of Mathematics and Statistics, Fujian Normal University, Fuzhou 350117, P.R. China
Email address: junfanchen@163.com