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A NOTE ON φ-PROXIMATE ORDER OF MEROMORPHIC FUNCTIONS

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Abstract. The main aim of this paper is to introduce the definition of φ-proximate order of a meromorphic function on the complex plane. By considering the concept of φ-proximate order, we will extend some previous results of Lahiri [11]. Furthermore, as an application of φ-proximate order, a result concerning the growth of composite entire and meromorphic function will be given.

1. Introduction, Definitions and Notations

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions which are available in [7, 12, 13, 20, 21, 22] and therefore we do not explain those in details. The term meromorphic function throughout this paper means meromorphic in the whole complex plane $\mathbb{C}$. This will not be recalled in next.

To study the generalized growth properties of entire and meromorphic functions, the concepts of the iterated p-order (see [10, 15]) and the (p, q)-th order (see [8]) are very useful and during the past decades, several authors made close investigations on the generalized growth properties of entire and meromorphic functions related to iterated p-order and the (p, q)-th order in some different directions.

Recently, Chyzhykov et al. [5] showed that both definitions of iterated p-order and the (p, q)-th order have the disadvantage that they do not cover arbitrary growth. (see [5, Example 1.4]). They used more general scale, called the φ-order (see [5]). In recent times, the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [2, 3, 5, 9]). In the following, we recall the definition of φ-order of entire and meromorphic functions.

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Definition 1.1. [5] *Let φ be an increasing unbounded function on $[1, +\infty)$. The φ-order of a meromorphic function $f(z)$ is defined as*

$$\rho_{[\varphi,f]} = \limsup_{r \to +\infty} \frac{\varphi(\exp(T(r,f)))}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of $f(z)$. When $f(z)$ is constant, $\rho_{[\varphi,f]}$ has to be taken to be zero.

If $f(z)$ is an entire function, then the φ-order is defined as

$$\widetilde{\rho}_{[\varphi,f]} = \limsup_{r \to +\infty} \frac{\varphi(M(r,f))}{\log r},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus of $f(z)$.

By symbol Φ, we define the class of positive unbounded increasing functions on $[1, +\infty)$, such that $\varphi(e^t)$ grows slowly, i.e., $\varphi(e^{ct}) = (1+o(1))\varphi(e^t)$ as $t_0 \leq t \to +\infty$ for each $c \in (0, +\infty)$ and Φ^1, we define the class of positive unbounded increasing functions on $[1, +\infty)$, such that $\varphi(e^{((1+o(1))t}) = ((1 + o(1))\varphi(e^t)$ as $t \to +\infty$. Clearly, $\Phi \subset \Phi^1$.

$f(z)$ is an entire function and $\varphi \in \Phi$, then

$$\limsup_{r \to +\infty} \frac{\varphi(\exp(T(r,f)))}{\log r} = \limsup_{r \to +\infty} \frac{\varphi(M(r,f))}{\log r} \quad \text{i.e., } \rho_{[\varphi,f]} = \widetilde{\rho}_{[\varphi,f]} \text{ (see [5]).}$$

Henceforth, we assume that always $\varphi \in \Phi$ unless otherwise specifically stated.

Remark 1.2. *Let $f(z)$ be an entire or a meromorphic function. If $\varphi(r) = \log^{[p]} r$ where $\log^{[p]} r = \log\left(\log^{[p-1]} r\right)$ and p is any positive integer ≥ 2, then the above definition reduces to the definition of iterated p-order (see* [10, 15]*). Also for $\varphi(r) = \log^{[2]} r$, Definition 1.1 reduces to the classical growth indicator known as order. Further, one can see that $\varphi(r) = \log^{[p]} r$ $(p \geq 2)$ belongs to the class Φ and $\varphi(r) = \log r \notin \Phi$.*

Historically, Valiron [20] introduced the concept of a positive continuous function called the proximate order for an entire function having finite order. Existence of such a proximate order was also established by Valiron [20]. The proof of Valiron [20] was simplified by Shah [16]. Later Lahiri [11] generalized the idea of the proximate order for a meromorphic function with finite iterated p-order and proved the existence of such a generalized proximate order. In fact some works related to the proximate order have also been explored in [6, 11, 14, 16].

Since the proximate order is not linked with φ-order it therefore seems reasonable to define suitably the φ-proximate order of a meromorphic function and prove its existence which we attempt in this paper. With this in view we use the following definition of the φ-proximate order.

Definition 1.3. *Let $f(z)$ be a meromorphic function of finite φ-order $\rho_{[\varphi,f]}$. A function $\rho_{[\varphi,f]}(r)$ is said to be a φ-proximate order of $f(z)$ if the following hold:*

(i) $\rho_{[\varphi,f]}(r)$ is nonnegative and continuous for $r > r_0$, say,

(ii) $\rho_{[\varphi,f]}(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_{[\varphi,f]}(r-0)$ and $\rho'_{[\varphi,f]}(r+0)$ exist,

(iii) $\lim\limits_{r\to+\infty} \rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}$,

(iv) $\lim\limits_{r\to+\infty} \frac{\rho'_{[\varphi,f]}(r)}{g'(r)} = 0$ where $g(r) = \log \varphi(\exp(r))$, and

(v) $\limsup\limits_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1.$

It is easy to see that if $\varphi(r) = \log^{[p]} r$ where $\log^{[p]} r = \log\left(\log^{[p-1]} r\right)$ and p is any positive integer ≥ 2 then φ-proximate order coincides with the generalized proximate order as introduced by Lahiri [11]. In order to prove the existence of φ-proximate order, we have followed some of the techniques as used by Lahiri [11].

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [1] *If $f(z)$ is a meromorphic function and $g(z)$ is an entire function then for all sufficiently large values of r,*

$$T(r, f(g)) \leqslant \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *For a meromorphic function $f(z)$ with finite φ-order $\rho_{[\varphi,f]}$, the φ-proximate order $\rho_{[\varphi,f]}(r)$ of $f(z)$ exists.*

Proof. First of all let us suppose that $f(z)$ is nonconstant. Our technique of proof is purely based on the construction of φ-proximate order.

Let $\sigma_\varphi(r) = \frac{\varphi(\exp(T(r,f)))}{\log r}$. Then

$$\rho_{[\varphi,f]} = \limsup_{r\to+\infty} \sigma_\varphi(r).$$

Now we consider the following two cases:

(I) $\sigma_\varphi(r) > \rho_{[\varphi,f]}$ for at least a sequence of values of r tending to infinity, and **(II)** $\sigma_\varphi(r) \le \rho_{[\varphi,f]}$ for all sufficiently large values of r.

Case I. Let $\sigma_\varphi(r) > \rho_{[\varphi,f]}$ for at least a sequence of values of r tending to infinity.

In this case let us define that

$$\Theta_\varphi(r) = \max_{x\ge r}\{\sigma_\varphi(x)\}.$$

Clearly $\Theta_\varphi(x)$ exists and is non-increasing.

Let $R > \log(\varphi^{-1}(\exp(\exp 1)))$ and $\sigma_\varphi(R) > \rho_{[\varphi,f]}$. Then for $r \ge R_1 > R$, say, we get $\sigma_\varphi(r) \le \sigma_\varphi(R)$. Since $\sigma_\varphi(r)$ is continuous, there exists $r_1 \in [R, R_1]$ such that

$$\sigma_\varphi(r_1) = \max_{R\le x\le R_1}\{\sigma_\varphi(x)\}.$$

Clearly $r_1 > \log(\varphi^{-1}(\exp(\exp 1)))$ and $\Theta_\varphi(r_1) = \max_{x\ge r_1}\{\sigma_\varphi(x)\} = \sigma_\varphi(r_1)$. Such values of $r = r_1$ exist for a sequence of values of r tending to infinity.

Let $\rho_{[\varphi,f]}(r_1) = \Theta_\varphi(r_1)$ and t_1 be the smallest integer not less than $1 + r_1$ such that $\Theta_\varphi(r_1) > \Theta_\varphi(t_1)$. Let us define $\rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}(r_1)$ for $r_1 < e \le t_1$.

Observing that
(i) $\Theta_\varphi(r)$ and $\rho_{[\varphi,f]}(r_1) - \log^{[2]}(\varphi(\exp(r)))+ \log^{[2]}(\varphi(\exp(t_1)))$ are continuous functions of r,
(ii) $\rho_{[\varphi,f]}(r_1) - \log^{[2]}(\varphi(\exp(r))) + \log^{[2]}(\varphi(\exp(t_1))) \to -\infty$ as $r \to +\infty$ lies in the first quadrant,
(iii) $\rho_{[\varphi,f]}(r_1) - \log^{[2]}(\varphi(\exp(r)))+ \log^{[2]}(\varphi(\exp(t_1))) > \Theta_\varphi(t_1)$ for $r(> t_1)$ sufficiently close to t_1 and
(iv) $\Theta_\varphi(r)$ is non-increasing.
So, we can define u_1 as follows

$$\begin{aligned} u_1 &> t_1 \\ \rho_{[\varphi,f]}(r) &= \rho_{[\varphi,f]}(r_1) - \log^{[2]}(\varphi(\exp(r))) + \log^{[2]}(\varphi(\exp(t_1))) \\ \text{for } t_1 &\le r \le u_1, \\ \rho_{[\varphi,f]}(r) &= \Theta_\varphi(r) \text{ for } r = u_1, \\ \text{and } \rho_{[\varphi,f]}(r) &> \Theta_\varphi(r) \text{ for } t_1 \le r < u_1. \end{aligned}$$

Let r_2 be the smallest value of r for which $r_2 \ge u_1$ and $\Theta_\varphi(r_2) = \sigma_\varphi(r_2)$. If $r_2 > u_1$ then let $\rho_{[\varphi,f]}(r) = \Theta_\varphi(r)$ for $u_1 \le r \le r_2$. As it can be easily shown that $\Theta_\varphi(r)$ is constant in $u_1 \le r \le r_2$, $\rho_{[\varphi,f]}(r)$ is constant in $u_1 \le r \le r_2$. Therefore, we repeat this procedure indefinitely and obtain that $\rho_{[\varphi,f]}(r)$ is differentiable in adjacent intervals. Further $\rho'_{[\varphi,f]}(r) = 0$ or $h'(r)$ where $h(r) = \log(g(r)) = \log\log\varphi(\exp(r))$ and $\rho_{[\varphi,f]}(r) \ge \Theta_\varphi(r) \ge \sigma_\varphi(r)$ for all $r \geq r_1$. Also $\rho_{[\varphi,f]}(r) = \sigma_\varphi(r)$ for a sequence of values of r tending to infinity and $\rho_{[\varphi,f]}(r)$ is non-increasing for $r \geq r_1$ and

$$\rho_{[\varphi,f]}(r) = \limsup_{r\to+\infty} \sigma_\varphi(r) = \limsup_{r\to+\infty} \Theta_\varphi(r).$$

So

$$\limsup_{r\to+\infty} \rho_{[\varphi,f]}(r) = \liminf_{r\to+\infty} \rho_{[\varphi,f]}(r) = \lim_{r\to+\infty} \rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}$$

and

$$\lim_{r\to+\infty} \frac{\rho'_{[\varphi,f]}(r)}{g'(r)} = 0 \text{ where } g(r) = \log \varphi(\exp(r)).$$

Further we have $\exp(\varphi(\exp(T(r,f)))) = r^{\sigma_\varphi(r)} = r^{\rho_{[\varphi,f]}(r)}$ for a sequence of values of r tending to infinity and $\exp(\varphi(\exp(T(r,f)))) \leq r^{\rho_{[\varphi,f]}(r)}$ for the remaining r's. So we get that

$$\limsup_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1.$$

Continuity of $\rho_{[\varphi,f]}(r)$ for $r \geq r_1$ follows easily from its construction which is complete in the Case I.

II. Let $\sigma_\varphi(r) \leq \rho_{[\varphi,f]}$ for all sufficiently large values of r.

In this case, we separate our proof for the following two sub cases:
Sub case (A). Let $\sigma_\varphi(r) = \rho_{[\varphi,f]}$ for at least a sequence of values of r tending to infinity.
Sub case (B). Let $\sigma_\varphi(r) < \rho_{[\varphi,f]}$ for all sufficiently large values of r.

In Sub case (A), we take $\rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}$ for all sufficiently large values of r.

In Sub case (B) let

$$\xi_\varphi(r) = \max_{X\leq x\leq r} \{\sigma_\varphi(x)\},$$

where $X > \log(\varphi^{-1}(\exp(\exp 1)))$ is such that $\sigma_\varphi(r) < \rho_{[\varphi,f]}$ whenever $x \geq X$. We note that $\xi_\varphi(r)$ is nondecreasing and for all $r \geq X$ sufficiently large, the roots of $\xi_\varphi(x) = \rho_{[\varphi,f]} + \log(\varphi^{-1}(\exp(\exp x))) - \log(\varphi^{-1}(\exp(\exp r)))$ is less than r. For a suitable large value $v_1 > X$, we define

$$\begin{aligned} \rho_{[\varphi,f]}(v_1) &= \rho_{[\varphi,f]}, \\ \rho_{[\varphi,f]}(r) &= \rho_{[\varphi,f]} + \log(\varphi^{-1}(\exp(\exp r))) - \log(\varphi^{-1}(\exp(\exp v_1))) \\ \text{for } s_1 &\leq r \leq v_1, \end{aligned}$$

where $s_1 < v_1$ is such that $\xi_\varphi(s_1) = \rho_{[\varphi,f]}(s_1)$. In fact s_1 is given by the largest positive root of

$$\xi_\varphi(x) = \rho_{[\varphi,f]} + \log(\varphi^{-1}(\exp(\exp x))) - \log(\varphi^{-1}(\exp(\exp v_1)).$$

If $\xi_\varphi(s_1) \neq \sigma_\varphi(s_1)$ let $\omega_1\,(< s_1)$ be the upper bound of point ω at which $\xi_\varphi(\omega) = \sigma_\varphi(\omega)$ and $\omega < s_1$.Clearly at ω_1, $\xi_\varphi(\omega_1) = \sigma_\varphi(\omega_1)$. We define $\rho_{[\varphi,f]}(r) = \xi_\varphi(r)$ for $\omega_1 \le r \le s_1$. It is easy to show that $\xi_\varphi(r)$ is constant in $\omega_1 \le r \le s_1$ and so $\rho_{[\varphi,f]}(r)$ is constant in $\omega_1 \le r \le s_1$. If $\xi_\varphi(s_1) = \sigma_\varphi(s_1)$ we take $\omega_1 = s_1$.

We choose $v_2 > v_1$ suitably large and let $\rho_{[\varphi,f]}(v_2) = \rho_{[\varphi,f]}$, $\rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]} + \log(\varphi^{-1}(\exp(\exp r))) - \log(\varphi^{-1}(\exp(\exp v_2)))$ for $s_2 \le r \le v_2$ where $s_2 < v_2$ is such that $\xi_\varphi(s_2) = \rho_{[\varphi,f]}(s_2)$ .If $\xi_\varphi(s_2) \neq \sigma_\varphi(s_2)$ let $\rho_{[\varphi,f]}(r) = \xi_\varphi(r)$ for $\omega_2 \le r \le s_2$, where ω_2 has the similar property as that of ω_1. As above $\rho_{[\varphi,f]}(r)$ is constant in $[\omega_2, s_2]$. If $\xi_\varphi(s_2) = \sigma_\varphi(s_2)$ we take $\omega_2 = s_2$.

Let $\rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}(\omega_2) - \log(\varphi^{-1}(\exp(\exp r))) + \log(\varphi^{-1}(\exp(\exp \omega_2)))$ for $q_1 \le r \le \omega_2$ where $q_1\,(< \omega_2)$ is the point of intersection of $y = \rho_{[\varphi,f]}$ with $y = \rho_{[\varphi,f]}(\omega_2) - \log(\varphi^{-1}(\exp(\exp x))) + \log(\varphi^{-1}(\exp(\exp \omega_2)))$. It is also possible to choose v_2 so large that $v_1 < q_1$. Let $\rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}$ for $v_1 \le r \le q_1$. We repeat this process. Now we can show that for all $r \ge v_1$, $\rho_{[\varphi,f]} \ge \rho_{[\varphi,f]}(r) \ge \xi_\varphi(r) \ge \sigma_\varphi(r)$ and $\rho_{[\varphi,f]}(r) = \sigma_\varphi(r)$ for $r = \omega_1, \omega_2, \ldots$ .So we obtain that

$$\limsup_{r\to+\infty} \rho_{[\varphi,f]}(r) = \liminf_{r\to+\infty} \rho_{[\varphi,f]}(r) = \lim_{r\to+\infty} \rho_{[\varphi,f]}(r) = \rho_{[\varphi,f]}.$$

Since $\exp(\varphi(\exp(T(r,f)))) = r^{\sigma_\varphi(r)} = r^{\rho_{[\varphi,f]}(r)}$ for a sequence of values of r tending to infinity and $\exp(\varphi(\exp(T(r,f)))) \le r^{\rho_{[\varphi,f]}(r)}$ for the remaining r's, therefore it follows that

$$\limsup_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1.$$

Also $\rho_{[\varphi,f]}(r)$ is differentiable in adjacent intervals and $\rho'_{[\varphi,f]}(r) = 0$ or $\pm\left(\frac{1}{h'(r)}\right)$ where $h(r) = \log(g(r)) = \log\log\varphi(e^r)$. So

$$\lim_{r\to+\infty} \frac{\rho'_{[\varphi,f]}(r)}{g'(r)} = 0 \text{ where } g(r) = \log\varphi(e^r).$$

Continuity of $\rho_{[\varphi,f]}(r)$ follows from its construction.

If $f(z) = C$, is a constant, then $\rho_{[\varphi,f]} = 0$ and we take $\sigma_\varphi(r) = \frac{\log(\varphi(e^C))}{\log r}$ and proceed as above.

This completes the proof. □

Corollary 3.2. *Let $f(z)$ be an entire function. Then for any $\delta(>0)$ the function $r^{\rho_{[\varphi,f]}+\delta-\rho_{[\varphi,f]}(r)}$ is ultimately an increasing function of r.*

Proof. As

$$\begin{aligned}&\frac{d}{dr} r^{\rho_{[\varphi,f]}+\delta-\rho_{[\varphi,f]}(r)}\\ =\;& \{\rho_{[\varphi,f]} + \delta - \rho_{[\varphi,f]}(r) - r\log r\,\rho'_{[\varphi,f]}(r)\}.r^{\rho_{[\varphi,f]}+\delta-\rho_{[\varphi,f]}(r)-1} > 0,\end{aligned}$$

for all sufficiently large positive numbers of r, the corollary is proved. □

As applications of φ-proximate order, we prove the following theorems.

Theorem 3.3. *Let $f(z)$ be a nonconstant meromorphic function of finite φ-order $\rho_{[\varphi,f]}$ with $f(0) \neq 0, +\infty$, and a be any complex number, finite or infinite. Then for a φ-proximate order $\rho_{[\varphi,f]}(r)$ of $f(z)$ and for all large r*

$$\exp(\varphi(\exp(n(r,a)))) \leq Ar^{\rho_{[\varphi,f]}(r)}$$

where A is a suitable constant independent of a.

Proof. From Nevalinna's first fundamental theorem we obtain that

$$m(r,a) + N(r,a) = T(r,f) + O(1),$$

which gives that

$$N(r,a) \leq T(r,f) + O(1).$$

Now replacing r by θr $(\theta > 1)$, we get from above that

$$N(\theta r,a) \leq T(\theta r,f) + O(1),$$

and so

$$n(r,a)\log\theta \leq \int_0^{\theta r} \frac{n(t,a)}{t} dt \leq T(\theta r,f) + O(1).$$

Therefore for any arbitrary $\varepsilon_1 > 0$, we get that

$$\begin{aligned}
\varphi\left(\exp(n(r,a)\log\theta)\right) &\leq \varphi\left(\exp(T(\theta r,f)(1+o(1)))\right)\\
\textit{i.e.},\ \varphi\left(\exp(n(r,a))\right) &\leq (1+o(1))\varphi\left(\exp(T(\theta r,f))\right)\\
\textit{i.e.},\ \varphi\left(\exp(n(r,a))\right) &\leq \varphi\left(\exp(T(\theta r,f))\right) + \log\varepsilon_1\\
\textit{i.e.},\ \exp\left(\varphi\left(\exp(n(r,a))\right)\right) &\leq \exp\left(\varphi\left(\exp(T(\theta r,f))\right) + \log\varepsilon_1\right)\\
\textit{i.e.},\ \exp\left(\varphi\left(\exp(n(r,a))\right)\right) &\leq \varepsilon_1 \exp\left(\varphi\left(\exp(T(\theta r,f))\right)\right). \qquad (1)
\end{aligned}$$

Since $\limsup\limits_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1$, for a given $\varepsilon > 0$, we get from (1) for all sufficiently large values of r that

$$\begin{aligned}
\exp\left(\varphi\left(\exp(n(r,a))\right)\right) &\leq \varepsilon_1 (1+\varepsilon) (\theta r)^{\rho_{[\varphi,f]}(\theta r)}\\
&= \frac{\varepsilon_1 (1+\varepsilon) (\theta r)^{\rho_{[\varphi,f]}+1}}{(\theta r)^{\rho_{[\varphi,f]}+1-\rho_{[\varphi,f]}(\theta r)}}.
\end{aligned}$$

Since by Corollary 3.2, $r^{\rho_{[\varphi,f]}+1-\rho_{[\varphi,f]}(r)}$ is increasing for all large r, it follows from above that for large r,

$$\exp\left(\varphi\left(\exp(n(r,a))\right)\right) \leq Ar^{\rho_{[\varphi,f]}(r)},$$

where A is a suitable constant independent of a. Hence the theorem follows. □

Theorem 3.4. *Let $f(z)$ be a nonconstant entire function of finite φ-order $\rho_{[\varphi,f]}$ and φ-proximate order $\rho_{[\varphi,f]}(r)$. Then*

$$\liminf_{r\to+\infty} \frac{\exp\left(\varphi(M(r,f))\right)}{\exp(\varphi(\exp(T(r,f))))} = 1,$$

where $\exp(\varphi(\exp)) \in \Phi$.

Proof. Using the inequality

$$T(r,f) \leq \log^{+} M(r,f) \leq \frac{R+r}{R-r} T(Rr,f), \ \ (0 \leq r < R) \{cf. \ [7, \text{ p. } 18]\},$$

for a nonconstant entire function $f(z)$, we get that

$$T(r,f) \leq \log M(r,f) \leq \frac{\eta+1}{\eta-1} T(\eta r, f), \tag{2}$$

where $\eta > 1$.

From the first part of (2) we obtain that

$$\liminf_{r\to+\infty} \frac{\exp(\varphi(M(r,f)))}{\exp(\varphi(\exp(T(r,f))))} \geq 1. \tag{3}$$

Now let $0 < \varepsilon < 1$. Since $\limsup_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1$, so it follows for all sufficiently large values of r that

$$\exp(\varphi(\exp(T(\eta r, f)))) < (1+\varepsilon)(\eta r)^{\rho_{[\varphi,f]}(\eta r)} \tag{4}$$

Since $\exp(\varphi(\exp)) \in \Phi$, therefore from the second part of (2) and (4) we get for all sufficiently large values of r that

$$\begin{aligned} \exp(\varphi(M(r,f))) &< (1+o(1)) \exp(\varphi(\exp(T(\eta r, f)))) \\ \text{i.e., } \exp(\varphi(M(r,f))) &< (1+o(1))(1+\varepsilon)(\eta r)^{\rho_{[\varphi,f]}(\eta r)} \\ \text{i.e., } \exp(\varphi(M(r,f))) &< \frac{(1+o(1))(1+\varepsilon)(\eta r)^{\rho_{[\varphi,f]}+\delta}}{(\eta r)^{\rho_{[\varphi,f]}+\delta-\rho_{[\varphi,f]}(\eta r)}}. \end{aligned} \tag{5}$$

In view of Corollary 3.2, $r^{\rho_{[\varphi,f]}+\delta-\rho_{[\varphi,f]}(r)}$ is ultimately an increasing function of r, and so it follows from (5) for all sufficiently large values of r that

$$\exp(\varphi(M(r,f))) < (1+o(1))(1+\varepsilon)\, \eta^{\rho_{[\varphi,f]}+\delta} r^{\rho_{[\varphi,f]}(r)}. \tag{6}$$

Since $\limsup_{r\to+\infty} \frac{\exp(\varphi(\exp(T(r,f))))}{r^{\rho_{[\varphi,f]}(r)}} = 1$, so it follows for a sequence values of r tending to infinity that

$$\exp(\varphi(\exp(T(r,f)))) > (1-\varepsilon)(r)^{\rho_{[\varphi,f]}(r)}. \tag{7}$$

Therefore from (6) and (7), we get for a sequence values of r tending to infinity that

$$\exp(\varphi(M(r,f))) < \frac{(1+o(1))(1+\varepsilon)}{(1-\varepsilon)} \eta^{\rho_{[\varphi,f]}+\delta} \exp(\varphi(\exp(T(r,f)))).$$

Since ε, η are arbitrary, so it follows from above that

$$\text{i.e., } \liminf_{r\to+\infty} \frac{\exp(\varphi(M(r,f)))}{\exp(\varphi(\exp(T(r,f))))} \leq 1. \tag{8}$$

Hence the theorem follows from (3) and (8). □

Theorem 3.5. *Let $f(z)$ be a nonconstant entire function of finite φ-order $\rho_{[\varphi,f]}$ and φ-proximate order $\rho_{[\varphi,f]}(r)$. Then for any $A>0$*

$$\liminf_{r\to+\infty}\frac{\exp\left(\varphi(M(r,f))\right)}{\exp(\varphi(\exp(T(r,f))))\left(\varphi(\exp(T(r,f)))\right)^{A}}=0,$$

where $\exp(\varphi(\exp))\in\Phi$.

Proof. Since $\left(\varphi(\exp(T(r,f)))\right)^{A}\to+\infty$ as $r\to+\infty$, and in view of Theorem 3.4 we get that

$$\liminf_{r\to+\infty}\frac{\exp\left(\varphi(M(r,f))\right)}{\exp(\varphi(\exp(T(r,f))))}<+\infty,$$

so the conclusion of the theorem follows. □

Now let us recall that Sheremeta [18] introduced the concept of generalized order of entire functions considering two continuous non-negative functions defined on $(-\infty,+\infty)$. For details about generalized order one may see [18]. Several researchers made close investigations on the properties of entire functions related to generalized order as introduced by Sheremeta [18] in some different direction. For the purpose of further applications, recently Biswas et al. [4] rewrite the definition of the generalized order of entire and meromorphic functions after giving a minor modification to the original definition introduced by Sheremeta [18] which is as follows:

Definition 3.6. [4] *Let $\varphi_1\in\Phi$ and $\varphi_2(cr)=(1+o(1))\varphi_2(r)$ as $r_0\leq r\to+\infty$ for each $c\in(0,+\infty)$. Then the generalized order (φ_1,φ_2) of a meromorphic function $f(z)$ is defined as:*

$$\rho_{[\varphi_1,\varphi_2;f]}=\limsup_{r\to+\infty}\frac{\varphi_1(\exp(T(r,f)))}{\varphi_2(r)}.$$

If $f(z)$ is an entire function, then

$$\rho_{[\varphi_1,\varphi_2;f]}=\limsup_{r\to+\infty}\frac{\varphi_1(M(r,f))}{\varphi_2(r)}.$$

Theorem 3.7. *Let $f(z)$ be meromorphic and $g(z)$ be entire such that $\rho_{[\varphi_1,\varphi_2;f]}$ and $\rho_{[\varphi_2,g]}$ are finite. Then*

$$\liminf_{r\to+\infty}\frac{\varphi_1\left(\exp\left(T\left(r,f\circ g\right)\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}\leqslant\rho_{[\varphi_1,\varphi_2;f]}\cdot 2^{\rho_{[\varphi_2,g]}},$$

where φ_1, $\exp\varphi_2\in\Phi$.

Proof. Let $\varepsilon\,(>0)$ is arbitrary. Since $T\left(r,g\right)\leqslant\log^{+}M\left(r,g\right)$, therefore we have from Lemma 2.1 for all sufficiently large values of r that

$$\varphi_1\left(\exp\left(T\left(r,f\circ g\right)\right)\right)\leqslant(1+o(1))\left(\rho_{[\varphi_1,\varphi_2;f]}+\varepsilon\right)\varphi_2\left(M\left(r,g\right)\right)$$

$$i.e.,\ \frac{\varphi_1\left(\exp\left(T\left(r,f\circ g\right)\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}\leqslant\frac{(1+o(1))\left(\rho_{[\varphi_1,\varphi_2;f]}+\varepsilon\right)\varphi_2\left(M\left(r,g\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}.$$

Since $\varepsilon\,(>0)$ we get from above that

$$\liminf_{r\to+\infty}\frac{\varphi_1\left(\exp\left(T\left(r,f\circ g\right)\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}\leqslant\rho_{[\varphi_1,\varphi_2;f]}\liminf_{r\to+\infty}\frac{\varphi_2\left(M\left(r,g\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}. \tag{9}$$

As $\limsup\limits_{r\to+\infty}\frac{\exp(\varphi_2(\exp(T(r,g))))}{r^{\rho_{[\varphi_2,g]}(r)}}=1$, for given $\varepsilon\,(0<\varepsilon<1)$ we obtain for all sufficiently large values of r that

$$\exp(\varphi_2(\exp(T(r,g))))<(1+\varepsilon)r^{\rho_{[\varphi_2,g]}(r)}$$

and for a sequence values of r tending to infinity that

$$\exp(\varphi_2(\exp(T(r,g))))>(1-\varepsilon)r^{\rho_{[\varphi_2,g]}(r)}.$$

Since $\log M(r,g)\leq 3T(2r,g)$ {cf. [7, p. 18]}, $\varphi_2(r)<\exp\varphi_2(r)$ and $\exp\varphi_2\in\Phi$, for a sequence values of r tending to infinity we get for any $\delta\,(>0)$ that

$$\begin{aligned}\frac{\varphi_2\left(M\left(r,g\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))} &< \frac{(1+o(1))\exp(\varphi_2(\exp(T(2r,g))))}{\exp(\varphi_2(\exp(T(r,g))))}\\ &< \frac{(1+o(1))(1+\varepsilon)}{(1-\varepsilon)}\cdot\frac{(2r)^{\rho_{[\varphi_2,g]}+\delta}}{(2r)^{\rho_{[\varphi_2,g]}+\delta-\rho_{[\varphi_2,g]}(2r)}}\cdot\frac{1}{r^{\rho_{[\varphi_2,g]}(r)}}\\ &< \frac{(1+o(1))(1+\varepsilon)}{(1-\varepsilon)}\cdot 2^{\rho_{[\varphi_2,g]}+\delta}\end{aligned}$$

because $r^{\rho_{[\varphi_2,g]}+\delta-\rho_{[\varphi_2,g]}(r)}$ is ultimately an increasing function of r by Corollary 3.2. Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are both arbitrary, we get from above that

$$\liminf_{r\to+\infty}\frac{\varphi_2\left(M\left(r,g\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}\leq 2^{\rho_{[\varphi_2,g]}}. \tag{10}$$

Therefore from (9) and (10) it follows that

$$\liminf_{r\to\infty}\frac{\varphi_1\left(\exp\left(T\left(r,f\circ g\right)\right)\right)}{\exp(\varphi_2(\exp(T(r,g))))}\leqslant\rho_{[\varphi_1,\varphi_2;f]}\cdot 2^{\rho_{[\varphi_2,g]}}.$$

This proves the theorem. □

4. Concluding Remarks

The main aim of this paper is actually to extend and to modify the notion of proximate order to φ-proximate order of higher dimensions in case of meromorphic functions and establish its existence. However, the notion of lower proximate order and proximate type of entire functions are not unknown and were used in [17] and [19] respectively. Accordingly, those outcomes may also be extended by using the concepts of φ-proximate lower order and φ-proximate type of entire and meromorphic functions. Moreover, it is interesting to study about the similar properties of φ-proximate order and φ-proximate type of entire and meromorphic functions of several complex variables which are left to the interested readers or the involved authors for future study in this research subject.

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