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CLAIRAUT POINTWISE SLANT RIEMANNIAN SUBMERSION FROM NEARLY KÄHLER MANIFOLDS

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Abstract. In the present article, we introduce pointwise slant Riemannian submersion from nearly Kähler manifold to Riemannian manifold. We established the conditions for fibers to be totally geodesic. We also find necessary and sufficient conditions for pointwise slant submersion φ to be a harmonic and totally geodesic. Further, we study clairaut pointwise slant Riemannian submersion from nearly Kähler manifold to Riemannian manifold. We derive the clairaut conditions for φ such that φ is a clairaut map. Finally, one example is constructed which demonstrates existence of clairaut pointwise slant submersion from nearly Kähler manifold to Riemannian manifold.

1. Introduction

The geometry of Riemannian manifolds can be investigated through smooth maps by comparing the geometric structures with the Riemannian manifolds whose geometrical structures are well known. The isometric immersion and Riemannian submersion are two such basic maps which are being studied thoroughly. The idea of Riemannian submersion was first introduced by O'Neill[13] and Gray [4]. The Riemannian submersion from almost hermitian manifolds (from Riemannian manifolds) to Riemannian manifolds (to almost hermitian manifolds) have been studied [21], the accumulated work can be seen in [14]. Later, Fischer [3] defined Riemannian maps as a generalization of isometric immersion and Riemannian submersion which links geometric optics and physical optics. But, still the theory of Riemannian submersions needs to be explored. Various classes of Riemannian submersion from Kähler manifolds to Riemannian manifolds on the basis of $Ker\varphi_*$ behaviour under $(1,1)$ tensor field J, viz. anti-invariant Riemannian submersion, semi-invariant Riemannian submersions, generic Riemannian submersions, slant submersion, semi-slant

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Riemannian submersions, hemi-slant Riemannian submersion, pointwise slant submersions were studied in detail [14]. Similarly, the Riemannian submersion from almost contact manifolds (Riemannian manifolds) to Riemannian manifolds (to almost contact manifolds) are studied.

A new characterization of Riemannian submersion, namely Clairaut submersion was first discussed by Bishop [2]. Later, the Clairaut submersion have been studied on Kähler manifold, nearly Kähler manifold, Sasakian manifold and Kenmotsu manifold etc. Various classes of Clairaut submersions from Kähler manifolds to Riemannian manifolds viz. Clairaut anti invariant Riemannian submersion [10], Clairaut semi-invariant Riemannian submersion [19], Clairaut pointwise slant submersion [14] etc., were defined. For more detail, the following research articles can be studied [22]-[23], [17]-[12].

In this article, we introduce pointwise slant submersion from nearly Kähler manifold to Riemannian manifold. Further, we introduce the clairaut pointwise slant Riemannian submersion from nearly Kähler manifold to Riemannian manifold. This paper is organised in the following sections: In Section 2, we recall all basic terminologies which are being used throughout the paper. In Section 3, we introduce pointwise slant submersion from nearly Kähler manifold to Riemannian manifold. We find conditions for the fibers of φ to be totally geodesic. Further, we derive conditions for pointwise slant submersion φ to be harmonic and totally geodesic. In Section 4, we introduce clairaut pointwise slant Riemannian submersion from nearly Kähler manifold to Riemannian manifold. We also establish the necessary and sufficient condition on φ to be clairaut map. Some other important results are also established.

2. Preliminaries

Suppose $\bar{M}$ and N be two Riemannian manifolds with dimension m and n respectively, where $m \geq n$. A differentiable map $\varphi : \bar{M} \longrightarrow N$ is said to be submersion if φ is surjective and the differential map $d\varphi$ is surjective for all points of $\bar{M}$.

Definition 2.1. [14] *Let $(\bar{M}^m, g_{\bar{M}})$ and (N^n, g_N) be Riemannian manifolds and $m > n$. A Riemannian submersion $\varphi : \bar{M} \longrightarrow N$ is a surjective map of $\bar{M}$ onto N satisfying the following axioms:*

(i) *φ has maximal rank.*
(ii) *The differential φ preserves the lengths of horizontal vectors.*

For each $q \in N$, $\varphi^{-1}(q)$ is an $(m-n)$-dimensional submanifold of $\bar{M}$ which are called fibers. For any $p \in \bar{M}$, a vector field $\mathcal{V}_p$ is tangent to fibers and $\mathcal{H}_p$ vertical to fiber and known as vertical and horizontal vector field, respectively. A vector field Y on M is said to be basic vector field if X is horizontal and φ related to vector field X_*. The projection morphism on the distribution

$ker\varphi_*$ and $(ker\varphi_*)^\perp$ are denoted by $\mathcal{V}$ and $\mathcal{H}$ respectively. Therefore, we can decompose $T_p\bar{M}$ in the following sense

$$T_p\bar{M} = \mathcal{V}_p \oplus \mathcal{H}_p.$$

For vector fields $X^{'}$, $Y^{'}$ on $\bar{M}$, the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ are defined by

$$\mathcal{A}_{X'}Y^{'} = \mathcal{H}\nabla_{\mathcal{H}X'}\mathcal{V}Y^{'} + \mathcal{V}\nabla_{\mathcal{H}X'}\mathcal{H}Y^{'} \tag{1}$$

$$\mathcal{T}_{X'}Y^{'} = \mathcal{V}\nabla_{\mathcal{V}X'}\mathcal{H}Y^{'} + \mathcal{H}\nabla_{\mathcal{V}X'}\mathcal{V}Y^{'} \tag{2}$$

where ∇ denotes metric connection on $\bar{M}$.
A Riemannian submersion is said to have totally geodesics fibers if and only if, the tensor $\mathcal{T}$ vanishes identically. The tensors $\mathcal{T}_{X'}$ and $\mathcal{A}_{X'}$ are skew symmetric operators on $(\Gamma(T\bar{M}), g_{\bar{M}})$ which are reversing the horizontal and vertical vectors.
The tensor $\mathcal{T}$ is vertical and $\mathcal{A}$ is horizontal. Also, for all $V, W \in \Gamma(Ker\varphi_*)$ and $X^{'}, Y^{'} \in \Gamma(Ker\varphi_*)^\perp$, we have

$$\mathcal{T}_W V = \mathcal{T}_V W, \tag{3}$$

$$\mathcal{A}_{X'}Y^{'} = -\mathcal{A}_{Y'}X^{'} = \frac{1}{2}\mathcal{V}[X^{'}, Y^{'}]. \tag{4}$$

From (1) and (2), we have the following equations [14]:

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \tag{5}$$

$$\nabla_W Y = \mathcal{T}_W Y + \mathcal{H}\nabla_W Y \tag{6}$$

$$\nabla_Y W = \mathcal{A}_Y W + \mathcal{V}\nabla_Y W \tag{7}$$

$$\nabla_Y Z = \mathcal{A}_Y Z + \mathcal{H}\nabla_Y Z \tag{8}$$

for $V, W \in \Gamma(Ker\varphi_*)$ and $Y, Z \in \Gamma(Ker\varphi_*)^\perp$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If Y is basic, then $\mathcal{H}\nabla_W Y = \mathcal{A}_Y W$.

An almost Hermitian manifold is an almost complex manifold $\bar{M}$ with almost hermitian structure $(J, g_{\bar{M}})$ which, for all $Y, Z \in \Gamma(T\bar{M})$, satisfies

$$g(JY, JZ) = g(Y, Z). \tag{9}$$

Definition 2.2. [5] *Let $(\bar{M}, g, J)$ be an almost Hermitian manifold and ∇ be the metric connection on $\bar{M}$ with respect " $g_{\bar{M}} = g$. Then $\bar{M}$ is said to be a nearly Kähler manifold if*

$$(\nabla_Y J)Z + (\nabla_Z J)Y = 0 \tag{10}$$

for any $Y, Z \in \Gamma(T\bar{M})$

3. Pointwise Slant Riemannian Submersion

Definition 3.1. *Suppose $(\bar{M}, g_{\bar{M}}, J)$ be a nearly Kähler manifold and (N, g_N) be a Riemannian manifold. Then, the Riemannian submersion $\varphi : \bar{M} \longrightarrow N$ is said to be pointwise slant Riemannian submersion If, at each point $q \in \bar{M}$, the $\theta(Y_q)$ (known as Wirtinger angle) between JY_q and the space $ker(\varphi_*)_q$ is independent of the choice of the non-zero vector $Y_q \in ker(\varphi_*)_q$.*

For any $U \in ker(\varphi_*)_p$, we define

$$JU = fU + wU \tag{11}$$

where $fU \in ker(\varphi_*)_p$ and $wU \in (ker\varphi_*)^{\perp}$.

For any $Y \in (ker(\varphi_*))^{\perp}$, we define

$$JY = \mathfrak{B}Y + \mathfrak{C}Y \tag{12}$$

where $\mathfrak{B}Y \in Ker\varphi_*$ and $\mathfrak{C}Y \in (Ker\varphi_*)^{\perp}$.
Also,

$$(Ker\varphi_*)^{\perp} = w(Ker\varphi_*) \perp \mu$$

Using (10), (11), (12) and (5)-(8), we get

$$(\nabla_U f)V + (\nabla_V f)U = \mathfrak{B}(\mathcal{T}_V U + \mathcal{T}_U V) - \mathcal{T}_V wU - \mathcal{T}_U wV \tag{13}$$

and

$$(\nabla_U w)V + (\nabla_V w)U = \mathfrak{C}(\mathcal{T}_V U + \mathcal{T}_U V) - \mathcal{T}_V fU - \mathcal{T}_U fV. \tag{14}$$

Remark 1. A point p with respect to pointwise slant submersion φ is said to be totally real and complex point if its slant function $\theta = \frac{\pi}{2}$ and $\theta = 0$ at p, respectively.
2. φ is said to be proper if there does not exist any totally real and complex point.
3. A pointwise slant submersion is said to be slant submersion if its slant function θ is globally constant or independent of the choice of the point p on $\bar{M}$.

Definition 3.2. [14] *Suppose $\bar{M}$ and N are Riemannian manifolds, $\varphi : \bar{M} \rightarrow N$ be a smooth map then φ is said to be totally geodesic if $\varphi(\alpha)$ is a geodesic in N for any given geodesic $\alpha \in \bar{M}$. Equivalently, $\nabla\varphi_* = 0$.*

Theorem 3.3. [11] *Suppose φ is a Riemannian submersion from an almost Hermitian manifold $(\bar{M}, g_{\bar{M}}, J)$ onto Riemannian manifold (N, g_N). Then, the Riemannian submersion $\bar{M}$ is said to be a pointwise slant submersion if and only if there exist a real valued function θ defined on $ker\varphi_*$ such that*

$$f^2 = -(\cos^2\theta)I. \tag{15}$$

Theorem 3.4. *Suppose φ is a pointwise slant submersion defined from a nearly Kähler manifold $(\bar{M}, g_{\bar{M}}, J)$ to Riemannian manifold (N, g_N) then the fibers are totally geodesic submanifold in $\bar{M}$ if and only if, for $U, V \in Ker\varphi_*$ and $Y \in Ker\varphi_*^{\perp}$, we have*

$$\begin{aligned} g([U,Y],V) &= 2\cot\theta Y(\theta)g(U,V) + \sec^2\theta\{g(\mathcal{A}_Y wfU, V) \\ &\quad + g(\mathcal{V}\nabla_Y \mathfrak{B}wU, V) + g(\mathcal{A}_Y \mathfrak{C}wU, V)\}. \end{aligned} \tag{16}$$

Proof. For $U, V \in Ker\varphi_*$ and $Y \in Ker\varphi_*^{\perp}$, using (5) in $g(\mathcal{T}_U V, Y)$, we obtain

$$g(\mathcal{T}_U V, Y) = g(\nabla_U V, Y) = -g(\nabla_U Y, V). \tag{17}$$

Since metric connection ∇ is torsion free, (17) can be written as

$$g(\mathcal{T}_U V, Y) = -g([U,Y],V) - g(\nabla_Y U, V). \tag{18}$$

Using (9) and (10) in (18), we obtain

$$g(\mathcal{T}_U V, Y) = -g([U,Y],V) - g(\nabla_Y JU + (\nabla_U J)Y, JV). \tag{19}$$

Now, using (11) in (19), we get

$$g(\mathcal{T}_U V, Y) = -g([U,Y],V) - g(\nabla_Y fU, JV) - g(\nabla_Y wU, JV) - g((\nabla_U J)Y, JV). \tag{20}$$

Further, using $g(JX', Y') = -g(Y', JY')$ and (10) in (20) , we get

$$\begin{aligned} g(\mathcal{T}_U V, Y) &= -g([U,Y],V) + g(\nabla_Y f^2 U + \nabla_Y wfU, V) + g(\nabla_Y JwU, V) \\ &\quad + g((\nabla_{fU} J)Y, V) + g((\nabla_{wU} J)Y, V) - g((\nabla_U J)Y, JV). \end{aligned} \tag{21}$$

Using (15) in (21), we get

$$\begin{aligned} g(\mathcal{T}_U V, Y) &= -g([U,Y],V) + \sin 2\theta Y(\theta)g(U,V) - \cos^2\theta g(\nabla_Y U, V) \\ &\quad + g(\nabla_Y wfU, V) + g(\nabla_Y JwU, V) + g((\nabla_{fU} J)Y, V) \\ &\quad + g((\nabla_{wU} J)Y, V) + g(J(\nabla_U J)Y, V). \end{aligned} \tag{22}$$

On the other hand, $(\nabla_{fU} J)Y + (\nabla_{wU} J)Y + J(\nabla_U J)Y = (\nabla_{JU} J)Y + J(\nabla_U J)Y$. Using (10) in above expression, we get

$$(\nabla_{fU} J)Y + (\nabla_{wU} J)Y + J(\nabla_U J)Y = -(\nabla_Y J)JU - J(\nabla_Y J)U = 0. \tag{23}$$

Using (23) in (22), we obtain

$$\begin{aligned} g(\mathcal{T}_U V, Y) &= -g([U,Y],V) + \sin 2\theta Y(\theta)g(U,V) - \cos^2\theta g(\nabla_Y U, V) \\ &\quad + g(\nabla_Y wfU, V) + g(\nabla_Y JwU, V). \end{aligned} \tag{24}$$

Further, (24) can be reduced

$$\begin{aligned} g(\mathcal{T}_U V, Y) &= -\sin^2\theta g([U,Y],V) + \sin 2\theta Y(\theta)g(U,V) + \cos^2\theta g(\nabla_U V, Y) \\ &\quad + g(\nabla_Y wfU, V) + g(\nabla_Y JwU, V). \end{aligned} \tag{25}$$

Using (5)-(8) in (25), we obtain

$$g(\mathcal{T}_U V, Y) = -g([U,Y],V) + 2\cot\theta Y(\theta) g(U,V) + \sec^2\theta\{g(\mathcal{A}_Y wfU, V) \quad +g(\mathcal{V}\nabla_Y \mathfrak{B}wU, V) + g(\mathcal{A}_Y \mathfrak{C}wU, V)\}. \tag{26}$$

□

Theorem 3.5. *Let φ be a pointwise slant submersion from nearly Kähler manifold $(\bar{M}, g_{\bar{M}}, J)$ to Riemannian manifold (N, g_N). Then φ is harmonic map if and only if*

$$trace({}^*\varphi_*)(\nabla\varphi_*((.), wf(.)) + trace w\mathcal{T}_{(.)}w(.) + trace\mathfrak{C}\mathcal{H}\nabla_{(.)}w(.) \quad +trace\mathcal{A}_{f(.)}f(.) + trace({}^*\varphi_*)(\nabla\varphi_*(f(.), w(.))) \quad -trace\ w\hat{\nabla}_{f(.)}(.) - trace\mathfrak{C}\mathcal{T}_{f(.)}(.) = 0, \tag{27}$$

where $(.)$ represents the place of $V \in \mathcal{V}$.

Proof. For vector field $V \in \mathcal{V}$ and $Y \in \mathcal{H}$, from (5), we get

$$g(\mathcal{T}_V V, Y) = g(\nabla_V V, Y). \tag{28}$$

Since $(\nabla_V J)V = 0$, using (9) in (28), we get

$$g(\mathcal{T}_V V, Y) = g(\nabla_V JV, JY). \tag{29}$$

Using (11) in (29), we get

$$g(\mathcal{T}_V V, Y) = -g(J\nabla_V fV, Y) - g(J\nabla_V wV, Y). \tag{30}$$

Using (10) and (11) in (30), we obtain

$$g(\mathcal{T}_V V, Y) = -g(\nabla_V f^2 V, Y) - g(\nabla_V wfV, Y) + g(\nabla_V wV, JY) - g((\nabla_{fV} J)V, Y). \tag{31}$$

Using (15) and (5)-(8) in (31), we get

$$g(\mathcal{T}_V V, Y) = -\sin 2\theta V(\theta) g(V,Y) + \cos^2\theta g(\mathcal{T}_V V, Y) - g(\nabla_V wfV, Y) \quad +g(\mathcal{T}_V wV, \mathfrak{B}Y) + g(\nabla_V wV, \mathfrak{C}Y) - g(\nabla_{fV} fV, Y) - g(\nabla_{fV} wV, Y) \quad -g(\mathcal{V}\nabla_{fV} V, \mathfrak{B}Y) - g(\mathcal{T}_{fV} V, \mathfrak{C}Y). \tag{32}$$

Since φ is Riemannian submersion, for $Y^{'}, Y^{'} \in \Gamma(T\bar{M})$, then we have the expression

$$\nabla\varphi_*(Y^{'}, Y^{'}) = \nabla^{\varphi}_{Y^{'}}\varphi_* Y^{'} - \varphi(\nabla_{Y^{'}} Y^{'}). \tag{33}$$

Therefore, using (33) and $g(V,Y) = 0$ in (32), we get

$$\sin^2\theta g(\mathcal{T}_V V, Y) = g_N(\nabla\varphi_*(V, wfV), \varphi_* Y) + g(\mathcal{T}_V wV, \mathfrak{B}Y) + g(\mathcal{H}\nabla_V wV, \mathfrak{C}Y) \quad -g(\mathcal{A}_{fV} fV, Y) + g_N(\nabla\varphi_*(fV, wV), \varphi_* Y) + g(\mathcal{V}\nabla_{fV} V, \mathfrak{B}Y) + g(\mathcal{T}_{fV} V, \mathfrak{C}Y). \tag{34}$$

Further, (34) reduces to

$$\begin{aligned}\sin^2\theta g(\mathcal{T}_V V,Y) &= -g({}^*\varphi_*(\nabla\varphi_*(V,wfV)),Y) - g(V\mathcal{T}_V wV,Y)\\ &-g(\mathfrak{C}\mathcal{H}\nabla_V wV,Y) - g(\mathcal{A}_{fV} fV,Y) - g({}^*\varphi_*(\nabla\varphi_*(fV,wV)),Y)\\ &+g(V\hat{\nabla}_{fV}V,Y) + g(\mathfrak{C}\mathcal{T}_{fV}V,Y).\end{aligned} \tag{35}$$

This implies that φ is harmonic if and only if

$$\begin{aligned}&trace({}^*\varphi_*)(\nabla\varphi_*((.),wf(.)) + traceV\mathcal{T}_{(.)}V(.) + trace\mathfrak{C}\mathcal{H}\nabla_{(.)}V(.)\\ &+trace\mathcal{A}_{f(.)}f(.) + trace({}^*\varphi_*)(\nabla\varphi_*(f(.),V(.)))\\ &-trace\ V\hat{\nabla}_{f(.)}(.) - trace\mathfrak{C}\mathcal{T}_{f(.)}(.) = 0.\end{aligned} \tag{36}$$

□

Theorem 3.6. *Let φ be a pointwise slant submersion from nearly Kähler manifold $(\bar{M}, g_{\bar{M}}, J)$ to Riemannian manifold (N, g_N). Then φ is totally geodesic map if and only if*

$$\begin{aligned}-\sin^2\theta g_N(\nabla\varphi_*(X,U),\varphi_*Y) &= g(\nabla_X wfU,Y) + g(\mathcal{H}\nabla_X U,Y)\\ &+g(A_X\mathfrak{B}(fU),Y) + g(\mathcal{H}\nabla_X\mathfrak{C}(fU),Y)\end{aligned} \tag{37}$$

and

$$\begin{aligned}g([U,X],V) &= 2\cot\theta X(\theta)g(U,V) + \sec^2\theta\{g(\mathcal{A}_X wfU,V)\\ &+g(\mathcal{V}\nabla_X\mathfrak{B}wU,V) + g(\mathcal{A}_X\mathfrak{C}wU,V)\}.\end{aligned} \tag{38}$$

Proof. Since $(\nabla\varphi_*)(X,Y) = 0$ and from (27), we obtain $(\nabla\varphi_*)(X,Y) = 0$ if and only if

$$\begin{aligned}g([U,X],V) &= 2\cot\theta X(\theta)g(U,V) + \sec^2\theta\{g(\mathcal{A}_X wfU,V)\\ &+g(\mathcal{V}\nabla_X\mathfrak{B}wU,V) + g(\mathcal{A}_X\mathfrak{C}wU,V)\}.\end{aligned} \tag{39}$$

Therefore, it is enough to derive the conditions such that $\nabla\varphi_*(X,U) = 0$. Since φ is Riemannian submersion,

$$g_N(\nabla\varphi_*(X,U),\varphi_*Y) = -g(\nabla_X U,Y). \tag{40}$$

Using (9), (10) and (11) in (40), we get

$$g_N(\nabla\varphi_*(X,U),\varphi_*Y) = g(J\nabla_X fU,Y) - g(\nabla_X wU,JY) - g((\nabla_U J)X,JY). \tag{41}$$

Again using (10) and (11) in $g(J\nabla_X fU,Y)$, (41) reduces to

$$\begin{aligned}g_N(\nabla\varphi_*(X,U),\varphi_*Y) &= g(\nabla_X f^2U,Y) + g(\nabla_X wfU,Y) - g(\nabla_X wU,JY)\\ &-g((\nabla_U J)X,JY) + g((\nabla_{fU}J)X,Y).\end{aligned} \tag{42}$$

Using (15) in (42), we get

$$\begin{aligned}g_N(\nabla\varphi_*(X,U),\varphi_*Y) &= \sin 2\theta X(\theta)g(U,Y) - \cos^2\theta g(\nabla_X U,Y) + g(\nabla_X wfU,Y)\\ &-g(\mathcal{A}_X wU,\mathfrak{B}Y) - g(\nabla_X wU,\mathfrak{C}Y) - g((\nabla_U J)X,JY) + g((\nabla_{fU}J)X,Y).\end{aligned} \tag{43}$$

On the other hand,

$$g(J(\nabla_U J)X, Y) + g((\nabla_{fU} J)X, Y) = -g(J(\nabla_X J)U, Y) - g((\nabla_X J)fU, Y). \tag{44}$$

Further, this can be reduced to

$$g(J(\nabla_U J)X, Y) + g((\nabla_{fU} J)X, Y) = -g(J(\nabla_X wU) + \nabla_X U + \nabla_X J(fU), Y). \tag{45}$$

Using (5)-(8) in (45), we get

$$\begin{aligned} g(J(\nabla_U J)X, Y) + g((\nabla_{fU} J)X, Y) &= g(\nabla_X wU, \mathfrak{C}Y) + g(\mathcal{A}_X wU, \mathfrak{B}Y) \\ &\quad -g(\mathcal{A}_X U, Y) - g(A_X \mathfrak{B}(fU), Y) - g(\mathcal{H}\nabla_X \mathfrak{C}(fU), Y). \end{aligned} \tag{46}$$

Using (46) in (43), we get

$$\begin{aligned} g_N(\nabla\varphi_*(X, U), \varphi_* Y) &= -\cos^2\theta g(\nabla_X U, Y) + g(\nabla_X wfU, Y) \\ &\quad +g(\mathcal{H}\nabla_X U, Y) + g(A_X \mathfrak{B}(fU), Y) + g(\mathcal{H}\nabla_X \mathfrak{C}(fU), Y). \end{aligned} \tag{47}$$

Above equation reduces to

$$\begin{aligned} -\sin^2\theta g_N(\nabla\varphi_*(X, U), \varphi_* Y) &= g(\nabla_X wfU, Y) + g(\mathcal{H}\nabla_X U, Y) \\ &\quad +g(A_X \mathfrak{B}(fU), Y) + g(\mathcal{H}\nabla_X \mathfrak{C}(fU), Y). \end{aligned} \tag{48}$$

Therefore from (16) and (48), we obtain that

$$\begin{aligned} -\sin^2\theta g_N(\nabla\varphi_*(X, U), \varphi_* Y) &= g(\nabla_X wfU, Y) + g(\mathcal{H}\nabla_X U, Y) \\ &\quad +g(A_X \mathfrak{B}(fU), Y) + g(\mathcal{H}\nabla_X \mathfrak{C}(fU), Y) \end{aligned} \tag{49}$$

and

$$\begin{aligned} g([U, X], V) &= 2\cot\theta X(\theta) g(U, V) + \sec^2\theta\{g(\mathcal{A}_X wfU, V) \\ &\quad +g(\mathcal{V}\nabla_X \mathfrak{B}wU, V) + g(\mathcal{A}_X \mathfrak{C}wU, V)\}. \end{aligned} \tag{50}$$

□

4. Clairaut Pointwise Slant Submersion

Suppose μ is a geodesic on a S(surface of revolution). Let λ be the distance of a point of S from the axis of rotation, and Θ be the angle between $\dot{\mu}$ and the meridians of S. Then, according to Clairaut's theorem, $\lambda sin\Theta$ is constant along μ. Conversely, if $rsin\Theta$ is constant along some curve μ in the surface of revolution S, and if no part of μ is part of some parallel of S, then μ is geodesic [14].

Definition 4.1. [14, 16] *Suppose $\varphi : \bar{M} \longrightarrow N$ be a Riemannian submersion with connected fibers. Then φ is said to be a clairaut submersion with $r = e^{\mathfrak{g}}$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -grad\ \mathfrak{g}$.*

Theorem 4.2. *Let φ be a clairaut pointwise slant Riemannian submersion from nearly Kähler manifold $(\bar{M}, g_{\bar{M}}, J)$ to Riemannian manifold (N, g_N). If $\alpha : J \subset \mathbb{R} \longrightarrow M$ is a regular curve $X(t)$ and $V(t)$ represents the horizontal and vertical components of $\dot{\alpha}$, respectively. Then $\alpha(s)$ is geodesic if and only if,*

$$\begin{aligned}
&-\sin 2\theta T(\theta)V + \cos^2\theta(\hat{\nabla}_V V + \mathcal{V}\nabla_X V) + (T_U + A_X)(wfV) + (\hat{\nabla}_V \\
&\qquad + \mathcal{V}\nabla_X)\mathfrak{B}wV + (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) - \nabla_X X \\
&\qquad + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}(Q) = 0 \qquad (51)
\end{aligned}$$

and

$$\begin{aligned}
&\cos^2\theta(\mathcal{T}_V V + \mathcal{A}_X V) + \mathcal{H}(\nabla_V + \nabla_X)(wfV + \mathfrak{C}wV) + \mathcal{T}_V + \mathcal{A}_X(\mathfrak{B}wV) \\
&\qquad - \mathcal{H}\nabla_X X + \mathcal{A}_X V + w(P) + \mathfrak{C}(Q) = 0. \qquad (52)
\end{aligned}$$

Proof. Suppose $\alpha : I \subset \mathbb{R} \longrightarrow M$ is a regular curve and $X(t)$ and $V(t)$ represents the horizontal and vertical components of $\dot{\alpha}$. From (10), we have

$$(\nabla_{\dot{\alpha}(s)} J)\dot{\alpha}(s) = 0. \qquad (53)$$

Therefore, from (53), we have

$$-\nabla_{\dot{\alpha}(s)}\dot{\alpha}(s) = J\nabla_{\dot{\alpha}(s)} J\dot{\alpha}(s). \qquad (54)$$

Using (11)

$$-\nabla_{\dot{\alpha}(s)}\dot{\alpha}(s) = J\nabla_{\dot{\alpha}(s)}(fV(s) + wV(s)) + J\nabla_{\dot{\alpha}(s)}(JX(s)). \qquad (55)$$

Using (11) and (10) in (54), (54) can be expanded to

$$-\nabla_{\dot{\alpha}(s)}\dot{\alpha}(s) = \nabla_{\dot{\alpha}} f^2 V + \nabla_{\dot{\alpha}} wfV + (\nabla_{fV} J)\dot{\alpha} + J\nabla_{\dot{\alpha}} wV + J\nabla_{\dot{\alpha}} JX. \qquad (56)$$

Further, using (10) for $J\nabla_{\dot{\alpha}(s)}(wV(s)) = \nabla_{\dot{\alpha}(s)} JwV(s) + (\nabla_{wV(s)} J)\dot{\alpha}(s)$ in (55), we get

$$-\nabla_{\dot{\alpha}}\dot{\alpha} = \nabla_{\dot{\alpha}} f^2 V + \nabla_{\dot{\alpha}} wfV + \nabla_{\dot{\alpha}} JwV + J\nabla_{\dot{\alpha}} JX + (\nabla_{fV} J)\dot{\alpha} + (\nabla_{wV} J)\dot{\alpha}. \qquad (57)$$

Also,

$$(\nabla_{JV} J)\dot{\alpha} = (\nabla_{fV} J)\dot{\alpha} + (\nabla_{wV} J)\dot{\alpha} = (\nabla_{JV} J)V + (\nabla_{JV} J)X. \qquad (58)$$

Since $(\nabla_{JV} J)V = 0$, (58) reduces to

$$(\nabla_{fV} J)\dot{\alpha} + (\nabla_{wV} J)\dot{\alpha} = (\nabla_{JV} J)X \qquad (59)$$

Using (59) in (57), we obtain

$$-\nabla_{\dot{\alpha}}\dot{\alpha} = \nabla_{\dot{\alpha}} f^2 V + \nabla_{\dot{\alpha}} wfV + \nabla_{\dot{\alpha}} JwV - \nabla_X X + J\nabla_V JX - (\nabla_X J)JV \qquad (60)$$

Using (12) and (15) in (60), we obtain

$$\begin{aligned}
-\nabla_{\dot{\alpha}}\dot{\alpha} &= 2\sin\theta\cos\theta T(\theta)V - \cos^2\theta(\nabla_V V + \nabla_X V) + \nabla_{\dot{\alpha}} wfV \\
&\quad + \nabla_{\dot{\alpha}}\mathfrak{B}wV + \nabla_{\dot{\alpha}}\mathfrak{C}wV - \nabla_X X + \nabla_X V + J\nabla_V JX + J\nabla_X JV \qquad (61)
\end{aligned}$$

On the other hand, using (11) and (12) in $J\nabla_V JX + J\nabla_X JV$, we obtain

$$J\nabla_V JX + J\nabla_X JV = J(\nabla_V \mathcal{B}X + \nabla_V \mathcal{C}X + \nabla_X fU + \nabla_X wU) \qquad (62)$$

Using (5), (6), (7) and (8) in (62), we get

(63)
$$J\nabla_V JX + J\nabla_X JV = J(\hat{\nabla}_V \mathfrak{B}X + \mathcal{T}_V X + \mathcal{V}\nabla_X V + \mathcal{A}_X wV) + J(\mathcal{T}_V \mathfrak{B}X + \mathcal{H}\nabla_V X + \mathcal{A}_X fV + \mathcal{H}\nabla_X wV)$$

Above equation can be written as

$$J\nabla_V JX + J\nabla_X JV = f(P) + w(P) + \mathcal{B}(Q) + \mathcal{C}(Q) \tag{64}$$

where $P = \hat{\nabla}_V \mathfrak{B}X + \mathcal{T}_V X + \mathcal{V}\nabla_X V + \mathcal{A}_X wV \in Ker(\varphi_*)$ and $Q = \mathcal{T}_V \mathfrak{B}X + \mathcal{H}\nabla_V X + \mathcal{A}_X fV + \mathcal{H}\nabla_X wV \in Ker(\varphi_*)^{\perp}$.
Using (64) in (61), we obtain

$$\begin{aligned} -\nabla_{\dot{\alpha}}\dot{\alpha} &= \sin 2\theta T(\theta)V - \cos^2\theta(\hat{\nabla}_V V + \mathcal{V}\nabla_X V + \mathcal{T}_V V + \mathcal{A}_X V) \\ &+ \mathcal{H}\nabla_V wfV + \mathcal{T}_V wfV + \mathcal{H}\nabla_X wfV + \mathcal{A}_X wfV + T_V \mathfrak{B}wV + \hat{\nabla}_U \mathfrak{B}wV \\ &+ \mathcal{A}_X \mathfrak{B}wV + \mathcal{V}\nabla_X \mathfrak{B}wV + \mathcal{H}\nabla_V \mathfrak{c}wV + \mathcal{T}_V \mathfrak{C}wV + \mathcal{H}\nabla_X \mathfrak{C}wV + \mathcal{A}_X \mathfrak{C}wV \\ &- \mathcal{H}\nabla_X X - \mathcal{A}_X X + \mathcal{A}_X V + \mathcal{V}\nabla_X V + f(P) + w(P) + \mathcal{B}(Q) + \mathcal{C}(Q). \end{aligned} \tag{65}$$

Comparing the horizontal and vertical parts in above equation, we get

$$\begin{aligned} -\mathcal{V}\nabla_{\dot{\alpha}}\dot{\alpha} &= \sin 2\theta T(\theta)V - \cos^2\theta(\hat{\nabla}_V V + \mathcal{V}\nabla_X V) + (T_V + A_X)(wfV) + (\hat{\nabla}_V \\ &+ \mathcal{V}\nabla_X)\mathfrak{B}wV + (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) - \mathcal{A}_X X + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}Q \end{aligned} \tag{66}$$

and

$$\begin{aligned} -\mathcal{H}\nabla_{\dot{\alpha}}\dot{\alpha} &= \cos^2\theta(\mathcal{T}_V V + \mathcal{A}_X V) + \mathcal{H}(\nabla_V + \nabla_X)(wfV + \mathfrak{C}wV) + \mathcal{T}_V \\ &+ \mathcal{A}_X(\mathfrak{B}wV) - \mathcal{H}\nabla_X X + \mathcal{A}_X V + w(P) + \mathfrak{C}(Q). \end{aligned} \tag{67}$$

□

Theorem 4.3. *Let φ be a pointwise slant submersion from a nearly Kähler manifold $(\bar{M}, g_{\bar{M}}, J)$ onto a Riemannian manifold (N, g_N) admitting horizontally characteristic vector field. If $\alpha : J \subset \mathbb{R} \longrightarrow M$ is a regular curve and V and X are vertical and horizontal parts of the tangent vector field $\dot{\alpha}(s) = T$. Then, φ is a clairaut submersion with $r = e^{\mathfrak{g}}$ if and only if along α, we have*

$$\begin{aligned} (\cos^2\theta g(grad(\mathfrak{g}), \dot{\alpha}) + sin2\theta T(\theta))g(V, V) &= g((T_V + A_X)(wfV) + (\mathcal{V}\nabla_X \\ &- \hat{\nabla}_V)\mathfrak{B}wV + (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) - \mathcal{A}_X X + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}Q, V). \end{aligned} \tag{68}$$

Proof. Let $\alpha(s) : J \subset \mathbb{R} \longrightarrow M$ be a geodesic with $||\dot{\alpha}(s)|| = c$ and $\Theta(s)$ be the angle between $\dot{\alpha}(s)$ and the horizontal space $\alpha(s)$. Then

$$g(V(s), V(s)) = c\sin^2\Theta(s) \tag{69}$$

and

$$g(X(s), X(s)) = c\cos^2\Theta(s). \tag{70}$$

Differentiating (69), we obtain

$$\frac{d}{ds}g(V(s),V(s)) = 2c\sin\Theta(s)\cos\Theta(s)\frac{d\Theta}{ds}. \tag{71}$$

This is reduces to

$$g(\nabla_{\dot{\alpha(s)}}V(s),V(s)) = c\sin\Theta(s)\cos\Theta(s)\frac{d\Theta}{ds}. \tag{72}$$

Using (5) and (7) in (72), we get

$$g(\hat{\nabla}_V V + \mathcal{V}\nabla_X V, V) = c\sin\Theta(s)\cos\Theta(s)\frac{d\Theta}{ds}. \tag{73}$$

Multiplying both side by $\cos^2\theta$ in (73), we get

$$g(\cos^2\theta(\hat{\nabla}_V V + \mathcal{V}\nabla_X V), V) = c\cos^2\theta\sin\Theta(s)\cos\Theta(s)\frac{d\Theta}{ds}. \tag{74}$$

Using (51) in (74), we get

$$\begin{aligned} &g(\sin 2\theta T(\theta)V - (T_V + A_X)(wfV) + (\hat{\nabla}_V - \mathcal{V}\nabla_X)\mathfrak{B}wV - (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) \\ &+\mathcal{A}_X X - \mathcal{V}\nabla_X V - f(P) - \mathfrak{B}Q, V) = c\cos^2\theta\sin\Theta(s)\cos\Theta(s)\frac{d\Theta}{ds}. \end{aligned} \tag{75}$$

On the other hand, φ is clairaut Riemannian submersion with $r = e^{\mathfrak{g}}$ if and only if,

$$\frac{d}{ds}(e^{\mathfrak{g}}\sin\Theta(s)) = 0.$$

This reduces to

$$\frac{d(\mathfrak{g}o\alpha)}{ds}\sin\Theta(s) + \cos\Theta(s)\frac{\Theta(s)}{ds} = 0. \tag{76}$$

Multiplying (76) both side with the non zero factors $c\sin\Theta(s)$ and $\cos^2\theta$, we get

$$c\cos^2\theta\sin^2\Theta(s)\frac{d(\mathfrak{g}o\alpha)}{ds} = -c\cos^2\theta\sin\Theta(s)\cos\Theta(s)\frac{d\Theta(s)}{ds} = 0. \tag{77}$$

Using (77) in (75), we get

$$\begin{aligned} \cos^2\theta\frac{d(\mathfrak{g}o\alpha)}{ds}c\sin^2\Theta(s) &= g(-\sin 2\theta T(\theta)V + (T_V + A_X)(wfV) + (\mathcal{V}\nabla_X \\ -\hat{\nabla}_V)\mathfrak{B}wV &+ (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) - \mathcal{A}_X X + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}Q, V). \end{aligned} \tag{78}$$

Since $g(V,V) = c\sin^2\Theta(s)$, (78) reduces to

$$\begin{aligned} \cos^2\theta\frac{d(\mathfrak{g}o\alpha)}{ds}g(V,V) &= g(-\sin 2\theta T(\theta)V + (T_V + A_X)(wfV) + (\mathcal{V}\nabla_X \\ -\hat{\nabla}_V)\mathfrak{B}wV &+ (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) + \mathcal{A}_X X + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}Q, V). \end{aligned} \tag{79}$$

Also,

$$\frac{d\mathfrak{g}}{ds}(\alpha(s)) = \dot{\alpha}[\mathfrak{g}](s) = g(grad(\mathfrak{g}), \dot{\alpha}). \tag{80}$$

Using (80) in (79), we obtain

$$(\cos^2\theta g(grad(\mathfrak{g}),\dot{\alpha}) + sin2\theta T(\theta))g(V,V) = g((T_V + A_X)(wfV) + (\mathcal{V}\nabla_X - \hat{\nabla}_V)\mathfrak{B}wV + (\mathcal{T}_V + \mathcal{A}_X)(\mathfrak{C}wV) - \mathcal{A}_X X + \mathcal{V}\nabla_X V + f(P) + \mathfrak{B}Q, V).$$

□

Example 4.4. *Let $\bar{M} = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), x_i \in \mathbb{R}\}$ be a nearly Kähler manifold with complex structure $J_\Theta = \cos\Theta J_1 + \sin\Theta J_2$ and metric*

$$g = (g_{ij}) = \begin{cases} g_{ii} = \dfrac{1}{x_7^2 + x_8^2} & i = 1, ..., 6 \\ g_{ii} = 1 & i = 7, 8 \\ g_{ij} = 0 & i \neq j, i, j = 1, ..., 8 \end{cases} \tag{81}$$

where $x_7^2 + x_8^2 \neq 0$ and J_1, J_2 are complex structures such that
$J_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_3, -x_4, x_1, x_2, -x_7, -x_8, x_5, x_6)$,
$J_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, x_4, -x_3, -x_6, x_5, x_8, -x_7)$
which satisfy $J_1J_2 = -J_2J_1$.
Let us define a map from nearly Kähler manifold $(\bar{M}, g_{\bar{M}})$ to Riemannian manifold (N, g_N) i.e., $\varphi : \bar{M} \longrightarrow N$ such that

$$\varphi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_2 - x_3}{\sqrt{2}}, x_5, x_6, x_7, x_8).$$

Clearly, $Ker\varphi_* = \Big(e_1 = \dfrac{1}{\sqrt{2}\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}), e_2 = \dfrac{1}{\sqrt{2}\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_3} + \dfrac{\partial}{\partial x_4})\Big)$ *and*
$(Ker\varphi_*)^\perp = \Big(e_3 = \dfrac{1}{\sqrt{2}\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_1} - \dfrac{\partial}{\partial x_2}), e_4 = \dfrac{1}{\sqrt{2}\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_3} - \dfrac{\partial}{\partial x_4}), e_5 = \dfrac{1}{\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_5}), e_6 = \dfrac{1}{\sqrt{x_7^2 + x_8^2}}(\dfrac{\partial}{\partial x_6}), e_7 = \dfrac{\partial}{\partial x_7}, e_8 = \dfrac{\partial}{\partial x_8}\Big)$.
Now, for $V = ae_1 + be_2 \in Ker\varphi_*$, we will show existence of a smooth function h on $\bar{M}$ satisfying

$$\mathcal{T}_V V = -g(V, V)\nabla^{\bar{M}} h.$$

Also, for $X, Y \in T\bar{M}$, we have

$$\nabla^{\bar{M}}_X Y = X_i Y_j \nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + E_i \frac{\partial F}{x_i} \frac{\partial}{x_j},\ \nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$

Since $\Gamma^k_{ij} = \dfrac{1}{2} g^{kl}_M (\dfrac{\partial}{\partial x_i} g_{jl} + \dfrac{\partial}{\partial x_j} g_{il} - \dfrac{\partial}{\partial x_l} g_{ij})$ *therefore* $\Gamma^7_{11} = \Gamma^7_{22} = -\dfrac{x_7}{x_7^2 + x_8^2}$, $\Gamma^8_{11} = \Gamma^8_{22} = -\dfrac{x_7}{x_7^2 + x_8^2}$, $\Gamma^k_{ij} = 0,\ i, j(i \neq j) = 1, 2$.
Hence, $\nabla_{e_1} e_1 = \dfrac{1}{2(x_7^2 + x_8^2)}\Big(\nabla_{\frac{\partial}{\partial x_7}} \dfrac{\partial}{\partial x_7} + \nabla_{\frac{\partial}{\partial x_8}} \dfrac{\partial}{\partial x_8}\Big) = -\Big(\dfrac{x_7}{(x_7^2 + x_8^2)^2} \dfrac{\partial}{\partial x_7} +$

$\displaystyle \frac{x_8}{(x_7^2+x_8^2)^2}\frac{\partial}{\partial x_8}\Big).$

Similarly, $\nabla_{e_2}e_2 = -\Big(\frac{x_7}{(x_7^2+x_8^2)^2}\frac{\partial}{\partial x_7}+\frac{x_8}{(x_7^2+x_8^2)^2}\frac{\partial}{\partial x_8}\Big).$

Therefore, $\mathcal{T}_V V = -2(a^2+b^2)\Big(\frac{x_7}{(x_7^2+x_8^2)^2}\frac{\partial}{\partial x_7}+\frac{x_8}{(x_7^2+x_8^2)^2}\frac{\partial}{\partial x_8}\Big).$

This is equivalent to

$$\mathcal{T}_V V = -g(V,V)\nabla h = \frac{2}{3}(a^2+b^2)\nabla\frac{1}{(x_7^2+x_8^2)^3}.$$

From above expression, we obtain $h = -\frac{2}{3}\frac{1}{(x_7^2+x_8^2)^3}$*. Therefore,* φ *is a pointwise clairaut submersion.*

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