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NEW EXTENSION FOR REVERSE OF THE OPERATOR CHOI-DAVIS-JENSEN INEQUALITY

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Abstract. In this paper, we introduce the reverse of the operator Davis-Choi-Jensen's inequality. Our results are employed to establish a new bound for the Furuta inequality. More precisely, we prove that, if $A, B \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators with the spectra contained in the interval $[m, M]$ with $m < M$ and $A \leq B$, then for any $r \geq \frac{1}{t} > 1$, $t \in (0,1)$

$$A^r \leq \left(\frac{M\mathbf{1}_{\mathcal{H}} - A}{M-m} m^{rt} + \frac{A - m\mathbf{1}_{\mathcal{H}}}{M-m} M^{rt} \right)^{\frac{1}{t}} \leq K\left(m, M, r\right) B^r,$$

where $K\left(m, M, r\right)$ is the generalized Kantorovich constant.

1. Introduction and Preliminaries

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces, and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all (bounded linear) operators on $\mathcal{H}$. Recall that an operator A on $\mathcal{H}$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A > 0$ if A is positive and invertible. For self-adjoint operators A and B, we write $A \geq B$ if $A - B$ is positive, i.e., $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for all $x \in \mathcal{H}$.

In particular, for some scalars m and M, we write $m \leq A \leq M$ if

$$m\langle x, x\rangle \leq \langle Ax, x\rangle \leq M\langle x, x\rangle, \qquad \forall\, x \in \mathcal{H}.$$

We extensively use the continuous functional calculus for self-adjoint operators, e.g., see [4, p. 3].

Definition 1.1. *A continuous function f defined on the interval J is called an operator convex function if*

$$f\big((1-\lambda)A + \lambda B\big) \leq (1-\lambda)f(A) + \lambda f(B)$$

for every $0 < \lambda < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J.

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Definition 1.2. *A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A)$ is positive for all positive A in $\mathcal{B}(\mathcal{H})$. It is said to be unital (or, normalized) if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$.*

We recall the Davis-Choi-Jensen inequality [1, 2] for operator convex functions, which is regarded as a noncommutative version of Jensen's inequality:

Theorem A. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with the spectra contained in the interval J and Φ be a normalized positive linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. If f is operator convex function on an interval J, then*

$$f(\Phi(A)) \le \Phi(f(A)). \tag{1}$$

Though in the case of convex function the inequality (1) does not hold in general, we have the following estimate from [7, Remark 4.14].

Theorem B. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and Φ be a unital positive linear map from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$. If f is non-negative convex function, then*

$$\frac{1}{\alpha}\Phi\big(f(A)\big) \le f\big(\Phi(A)\big) \le \alpha\Phi\big(f(A)\big), \tag{2}$$

where α is defined by

$$\alpha = \max_{m \le t \le M}\left\{\frac{1}{f(t)}\left(\frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)\right)\right\}.$$

For other generalizations of inequality, we refer the interested readers to see [5, 6] and [8]. Remind that the function $f(t) = t^r$ for $r > 1$ is not operator monotone on $[0, \infty)$. In the sense that $A \le B$ does not always ensure $A^r \le B^r$. Related to this problem, Furuta [3] proved: Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators such that their spectrums contained in the interval $[m, M]$, for some scalars $0 < m < M$. If $A \le B$, then

$$A^r \le K(m, M, r)\, B^r \quad \text{for } r \ge 1, \tag{3}$$

where the generalized Kantorovich constant $K(m, M, r)$ ([4, Definition 2.2]) is defined by

$$K(m, M, r) = \frac{(mM^r - Mm^r)}{(r-1)(M-m)}\left(\frac{r-1}{r}\frac{M^r - m^r}{mM^r - Mm^r}\right)^r.$$

In Section 2, we establish a considerable improvement of the first inequality in the (2). Some applications of these inequalities are considered as well. Particularly, we obtain an improvement of inequality (3).

2. Main Results

We introduce two notations that will be used in the sequel:

$$a_f \equiv \frac{f(M)-f(m)}{M-m} \quad \& \quad b_f \equiv \frac{Mf(m)-mf(M)}{M-m}.$$

The main result of the paper reads as follows.

Theorem 2.1. *Let $i = 1, ..., n$ and $A_i, B_i \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators with the spectra contained in the interval $[m, M]$ with $m < M$, and let $\Phi_1, ..., \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings such that $\sum_{i=1}^n \Phi_i(1_{\mathcal{H}}) = 1_{\mathcal{K}}$. If $\sum_{i=1}^n A_i \leq \sum_{i=1}^n B_i$ and $f : [m, M] \to (0, \infty)$ is a continuous increasing function such that $f^t, t \in (0, 1)$ is convex, then*

$$\begin{aligned}\sum_{i=1}^n \Phi_i(f(A_i)) &\leq \sum_{i=1}^n \Phi_i\left(\left(\frac{M\mathbf{1}_{\mathcal{H}}-A_i}{M-m}f^t(m)+\frac{A_i-m\mathbf{1}_{\mathcal{H}}}{M-m}f^t(M)\right)^{\frac{1}{t}}\right)\\ &\leq \alpha\sum_{i=1}^n f(\Phi_i(B_i)),\end{aligned}$$

where

$$\alpha = \max_{m\leq t\leq M}\left\{\frac{1}{f(t)}\left(\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\right)\right\}.$$

Proof. From the hypothesis $f^t, t \in (0, 1)$ is convex, it follows that for every $x, y \in [m, M]$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned}f((1-\lambda)x+\lambda y) &= (f^t((1-\lambda)x+\lambda y))^{\frac{1}{t}}\\ &\leq ((1-\lambda)f^t(x)+\lambda f^t(y))^{\frac{1}{t}}\\ &\leq ((1-\lambda)f(x)+\lambda f(y)).\end{aligned}$$

Therefore,

$$f((1-\lambda)x+\lambda y) \leq ((1-\lambda)f^t(x)+\lambda f^t(y))^{\frac{1}{t}} \leq (1-\lambda)f(x)+\lambda f(y). \tag{4}$$

Let $x = m$, $y = M$ and $\lambda = \frac{z-m}{M-m}$ in (4) to give

$$\begin{aligned}f(z) &\leq \left(\frac{M-z}{M-m}f^t(m)+\frac{z-m}{M-m}f^t(M)\right)^{\frac{1}{t}}\\ &\leq \frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M),\end{aligned}$$

for any $m \leq z \leq M$. For any $1 \leq i \leq n$, applying the continuous functional calculus for the operator A_i whose spectrum is contained in the interval $[m, M]$,

$$\begin{aligned} f(A_i) &\leq \left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M) \right)^{\frac{1}{t}} \\ &\leq \frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f(M) \\ &= a_f A_i + b_f I_H. \end{aligned}$$

Then, for any unit vector $x \in \mathcal{H}$

$$\begin{aligned} \langle f(A_i)x, x\rangle &\leq \left\langle \left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M) \right)^{\frac{1}{t}} x, x \right\rangle \\ &\leq a_f \langle A_i x, x\rangle + b_f. \end{aligned}$$

Since f is increasing and convex, then

$$\begin{aligned} \langle f(A_i)x, x\rangle &\leq \left\langle \left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M) \right)^{\frac{1}{t}} x, x \right\rangle \\ &\leq a_f \langle A_i x, x\rangle + b_f \\ &\leq \max_{m \leq t \leq M} \{a_f t + b_f\}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(A_i) - \alpha f\left(\sum_{i=1}^{n} \Phi_i(B_i)\right) &\leq \left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M) \right)^{\frac{1}{t}} \\ &\quad - \alpha f\left(\sum_{i=1}^{n} \Phi_i(B_i)\right) \\ &\leq \max_{m \leq t \leq M} \{a_f t + b_f\} - \alpha f\left(\sum_{i=1}^{n} \Phi_i(B_i)\right). \end{aligned}$$

Since $\Phi_i(\cdot)$ is order preserving and $\sum_{i=1}^{n} A_i \le \sum_{i=1}^{n} B_i$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right) - \alpha f\left(\sum_{i=1}^{n} \Phi_i\left(B_i\right)\right) \\
&\le \sum_{i=1}^{n} \Phi_i\left(\left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M)\right)^{\frac{1}{t}}\right) - \alpha f\left(\sum_{i=1}^{n} \Phi\left(B_i\right)\right) \\
&\le \max_{m \le t \le M} \left\{a_f t + b_f\right\} - \alpha f\left(\sum_{i=1}^{n} \Phi_i\left(B_i\right)\right) \\
&\le \max_{m \le t \le M} \left\{a_f t + b_f\right\} - \alpha f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right) \\
&\le \max_{m \le t \le M} \left\{a_f t + b_f - \alpha f(t)\right\} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right) &\le \sum_{i=1}^{n} \Phi_i\left(\left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M)\right)^{\frac{1}{t}}\right) \\
&\le \sum_{i=1}^{n} \alpha f\left(\Phi_i\left(B_i\right)\right),
\end{aligned}$$

which is exactly the desired result. □

Remark 2.2. *Let the hypothesis of Theorem 2.1 be satisfied. It follows from Theorem 2.1 that*

$$\begin{aligned}
\sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right) &\le \sum_{i=1}^{n} \Phi_i\left(\left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M)\right)^{\frac{1}{t}}\right) \\
&\le \alpha \sum_{i=1}^{n} f\left(\Phi_i\left(B_i\right)\right).
\end{aligned}$$

Integrating in the last inequality over $t \in [0, 1]$, we gives

$$\begin{aligned}
\sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right) &\le \int_0^1 \left(\sum_{i=1}^{n} \Phi_i\left(\left(\frac{M\mathbf{1}_{\mathcal{H}} - A_i}{M-m} f^t(m) + \frac{A_i - m\mathbf{1}_{\mathcal{H}}}{M-m} f^t(M)\right)^{\frac{1}{t}}\right)\right) dt \\
&\le \alpha \sum_{i=1}^{n} f\left(\Phi_i\left(B_i\right)\right).
\end{aligned}$$

Since the mapping Φ_i *is linear and continuous, then*

$$\sum_{i=1}^{n}\Phi_i(f(A_i)) \leq \sum_{i=1}^{n}\Phi_i\left(\int_0^1\left(\frac{M\mathbf{1}_{\mathcal{H}}-A_i}{M-m}f^t(m)+\frac{A_i-m\mathbf{1}_{\mathcal{H}}}{M-mf^t(M)}\right)^{\frac{1}{t}}dt\right)$$
$$\leq \alpha\sum_{i=1}^{n}f(\Phi_i(B_i)),$$

which is new inequality concerning the inequality in Theorem 2.1.

As a direct consequence of Theorem 2.1, we have:

Corollary 2.3. *Let* $A,B\in\mathcal{B}(\mathcal{H})$ *be self-adjoint operators with the spectra contained in the interval* $[m,M]$ *with* $m<M$, *and let* $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ *be a unital positive linear map. If* $A\leq B$ *and* $f:[m,M]\to(0,\infty)$ *is a continuous increasing function such that* $f^t, t\in(0,1)$ *is convex, then*

$$\Phi(f(A)) \leq \Phi\left(\left(\frac{M\mathbf{1}_{\mathcal{H}}-A}{M-m}f^t(m)+\frac{A-m\mathbf{1}_{\mathcal{H}}}{M-m}f^t(M)\right)^{\frac{1}{t}}\right)$$
$$\leq \alpha f(\Phi(B)).$$

In particular,

$$\Phi(f(A)) \leq \Phi\left(\left(\frac{M\mathbf{1}_{\mathcal{H}}-A}{M-m}f^t(m)+\frac{A-m\mathbf{1}_{\mathcal{H}}}{M-m}f^t(M)\right)^{\frac{1}{t}}\right)$$
$$\leq \alpha f(\Phi(A)).$$

The following inequality is improvement of the inequality (2).

Remark 2.4. *Let* $A,B\in\mathcal{B}(\mathcal{H})$ *be self-adjoint operators with the spectra contained in the interval* $[m,M]$ *with* $m<M$ *such that* $A\leq B$, *and let* $f:[m,M]\to(0,\infty)$ *is a continuous increasing function such that* $f^t, t\in(0,1)$ *is convex. Replacing* $\Phi(\cdot)$ *by* I *in Corollary 2.3 gives*

$$f(A) \leq \left(\frac{M\mathbf{1}_{\mathcal{H}}-A}{M-m}f^t(m)+\frac{A-m\mathbf{1}_{\mathcal{H}}}{M-m}f^t(M)\right)^{\frac{1}{t}}$$
$$\leq \alpha f(B).$$

Let $0<t<1$ and $1<\frac{1}{t}\leq r$. Consider the function $f(t)=t^r$ and by Corollary 2.3 we have the following Corollary.

Corollary 2.5. *Let* $A,B\in\mathcal{B}(\mathcal{H})$ *are self-adjoint operators with the spectra contained in the interval* $[m,M]$ *with* $m<M$, *and let* $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ *be a unital positive linear map. If* $A\leq B$, *then for any* $1<\frac{1}{t}\leq r$, $t\in(0,1)$

$$\Phi(A^r) \leq \Phi\left(\left(\frac{M\mathbf{1}_{\mathcal{H}}-A}{M-m}m^{rt}+\frac{A-m\mathbf{1}_{\mathcal{H}}}{M-m}M^{rt}\right)^{\frac{1}{t}}\right) \leq K(m,M,r)\,\Phi^r(B).$$

In particular,

$$A^r \leq \left(\frac{M\mathbf{1}_{\mathcal{H}} - A}{M-m} m^{rt} + \frac{A - m\mathbf{1}_{\mathcal{H}}}{M-m} M^{rt}\right)^{\frac{1}{t}} \leq K(m, M, r) B^r,$$

which is a considerable improvement of the Furuta inequality (3).

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References

[1] M. D. Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math. **18** (1974), 565–574.

[2] C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), 42–44.

[3] T. Furuta, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl. **2** (1998), 137–148.

[4] T. Furuta, J. Mićić-Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić method in operator inequalities*, Element, Zagreb, 2005.

[5] A. W. Marshall and I. Olkin, *Matrix versions of Cauchy and Kantorovich inequalities*, Aequationes Math. **40** (1990), 89–93.

[6] J. Mićić, H. R. Moradi, and S. Furuichi, *Choi-Davis-Jensen's inequality without convexity*, J. Math. Inequal. **12** (2018), no. 4, 1075–1085.

[7] J. Mićić, J. Pečarić, Y. Seo, and M. Tominaga, *Inequalities for positive linear maps on Hermitian matrices*, Math. Inequal. Appl. **3** (2000), 559–591.

[8] M. Sababheh, H. R. Moradi, and S. Furuichi, *Integrals refining convex inequalities*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 3, 2817–2833.

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