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A GENERALIZATION OF AN INEQUALITY CONCERNING THE SMIRNOV OPERATOR

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Abstract. In this paper we establish a generalization of a result recently proved by Ganenkova and Starkov [J. Math. Anal. Appl., **476** (2019), 696-714] concerning a modified version of Smirnov operator.

1. Introduction and Summary

Let $\mathbb{P}_n$ denote the class of polynomials in $\mathbb{C}$ of degree atmost $n \in \mathbb{N}$ and $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C}; |z| < 1\}$, so that $\overline{\mathbb{D}}$ is its closure and $\delta\mathbb{D}$ denotes the boundary. For any polynomial $f \in \mathbb{P}_n$, we have the following result due to Bernstein [2].

Theorem 1.1. *Let $f \in \mathbb{P}_n$, then*

$$\max_{z\in\delta\mathbb{D}} |f'(z)| \leq n \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having zeros at the origin.

Later on Bernstein [3], proved the following result:

Theorem 1.2. *Let $F \in \mathbb{P}_n$ have all zeros in $\overline{\mathbb{D}}$ and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \leq |F(z)|$ for $z \in \delta\mathbb{D}$, then*

$$|f'(z)| \leq |F'(z)| \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{D}. \tag{1.2}$$

For $z \in \mathbb{C} \setminus \mathbb{D}$ equality holds only if $f = e^{i\gamma}F, \gamma \in \mathbb{R}$.

For $z \in \mathbb{C} \setminus \mathbb{D}$, denoting by $\Omega_{|z|}$ the image of the disc $\{t \in \mathbb{C}; |t| \leq |z|\}$ under the mapping $\psi(t) = \dfrac{t}{1+t}$, Smirnov [7] as a generalization of Theorem 1.2 proved the following:

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THEOREM 1.3. *Let f and F be polynomials possessing conditions as in Theorem 1.2. Then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\mathbb{S}_\alpha[f](z)| \leq |\mathbb{S}_\alpha[F](z)| \tag{1.3}$$

for all $\alpha \in \overline{\Omega_{|z|}}$, with $\mathbb{S}_\alpha[f](z) = zf'(z) - n\alpha f(z)$, where α is a constant.
For $\alpha \in \overline{\Omega_{|z|}}$ in inequality (1.3) equality holds at a point $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ only if $f = e^{i\gamma}F, \gamma \in \mathbb{R}$.

We note that for fixed $z \in \mathbb{C} \setminus \mathbb{D}$ inequality (1.3) can be replaced (for reference see [4]) by

$$|zf'(z) - n\frac{az}{1+az}f(z)| \leq |zF'(z) - n\frac{az}{1+az}F(z)|,$$

where $a \in \overline{\mathbb{D}}$ is not the exceptional value of f.
Or, equivalently for $z \in \mathbb{C} \setminus \mathbb{D}$

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](z)| \ z \in \mathbb{C} \setminus \mathbb{D},$$

where $\tilde{\mathbb{S}}_a[f](z) = (1+az)f'(z) - naf(z)$ is known as modified Smirnov operator. The modified Smirnov operator $\tilde{\mathbb{S}}_a$ is more preferred than Smirnov operator $\mathbb{S}_a$, in a sense that the parameter a of $\tilde{\mathbb{S}}_a$ does not depend on z unlike parameter α of $\mathbb{S}_\alpha$. Marden [5] introduced a differential operator $\mathbb{B} : \mathbb{P}_n \to \mathbb{P}_n$ of mth order. This operator carries a polynomial $f \in \mathbb{P}_n$ into

$$\mathbb{B}[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z) + ... + \lambda_m \left(\frac{mz}{2}\right)^m f^m(z),$$

where $\lambda_0, \lambda_1, ..., \lambda_m$ are constants such that

$$u(z) = \lambda_0 + {}^nC_1\lambda_1 z + ... + {}^nC_m\lambda_m z^m \neq 0 \quad \text{for} \quad Re(z) > \frac{n}{4}. \tag{1.4}$$

Rahman and Schmeisser [6] considered the Marden operator for $m = 2$ and showed that this operator preserves the inequalities between polynomials and accordingly proved the following:

THEOREM 1.4. *Let f and F be polynomials possessing conditions as in Theorem 1.2. Then*

$$|\mathbb{B}[f](z)| \leq |\mathbb{B}[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{1.5}$$

where the constants $\lambda_0, \lambda_1, \lambda_2$ possess condition (1.4). For $z \in \mathbb{C} \setminus \mathbb{D}$ in inequality (1.5), equality holds if and only if $f = e^{i\gamma}F, \gamma \in \mathbb{R}$.

In order to compare the Smirnov operator $\mathbb{S}_\alpha[f](z) = zf'(z) - n\alpha f(z)$ and the Rahman's operator (with $\lambda_2 = 0$) $\mathbb{B}[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z)$, we require $\alpha \in \overline{\Omega}_{|z|}$ in inequality (1.3) and in inequality (1.5) the root of the polynomial $u(z) = \lambda_0 + n\lambda_1 z$ should lie in the half-plane $Re(z) \leq \frac{n}{4}$, that is

$$Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}.$$

Compare the sets of parameters in Theorem 1.3 and Theorem 1.4, we see that in Theorem 1.3, this set(coefficient near $-f(z)$) is $\mathcal{A} = \{n\alpha : \alpha \in \overline{\Omega}_{|z|}\}$ and in Theorem 1.4, the set of such coefficient near $-f(z)$ is

$$\mathcal{B} = \left\{-\frac{2\lambda_0}{\lambda_1 n} : Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}\right\} = \left\{t : Re(t) \leq \frac{n}{2}\right\}.$$

Consider the differential inequalities from Theorem 1.3 and Theorem 1.4 for $z \in \delta\mathbb{D}$, we have $\mathcal{A} = \mathcal{B}$. But for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ we have $\mathcal{B} \subset \mathcal{A}$. In other words in Theorem 1.3 and Theorem 1.4 formally the same inequality was obtained but for different set of parameters. Moreover, the set of parameters in Theorem 1.3 is essentially wider than that of Theorem 1.4.
Consequently

$$\mathbb{B}[f](z) = \lambda_1 \frac{n}{2} \mathbb{S}_\alpha[f](z). \tag{1.6}$$

These facts were first observed by Ganenkova and Starkov [4].

2. Main Results

Theorem 2.1. *Let $f(z)$ and $F(z)$ be two polynomials such that $deg f(z) \leq deg F(z) = n$. If $F(z)$ has all zeros in $\overline{\mathbb{D}}$ and $|f(z)| \leq |F(z)|$ for $z \in \delta\mathbb{D}$. Then for any complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$, we have for $z \in \delta\mathbb{D}$*

$$|\tilde{\mathbb{S}}_a[f](Rz) - \beta\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](Rz) - \beta\tilde{\mathbb{S}}_a[F](z)|. \tag{2.1}$$

Equivalently

$$\begin{aligned} &|(1+az)[Rf^{'}(Rz) - \beta f^{'}(z)] - na[f(Rz) - \beta f(z)]| \\ &\leq |(1+az)[RF^{'}(Rz) - \beta F^{'}(z)] - na[F(Rz) - \beta F(z)]| \quad \text{for} \quad z \in \delta\mathbb{D}. \end{aligned} \tag{2.2}$$

The result is sharp and equality holds if $a \in \overline{\mathbb{D}}$ is not the exceptional value for the polynomial $f(z) \equiv e^{i\gamma}F(z)$, where $\gamma \in \mathbb{R}$ and $F(z)$ is any polynomial having all zeros in $\overline{\mathbb{D}}$ and strict inequality holds for $z \in \mathbb{D}$, unless $f(z) \equiv e^{i\gamma}F(z)$, $\gamma \in \mathbb{R}$.

For the proof of Theorem 2.1 we require the following lemmas. First lemma is due to Ganenkova and Starkov [4].

Lemma 2.2. *Let $F \in \mathbb{P}_n$, and has all zeros in $\overline{\mathbb{D}}$. Let $a \in \delta\mathbb{D}$ be not the exceptional value for F. Then all zeros of $\tilde{\mathbb{S}}_a[F]$ lie in $\overline{\mathbb{D}}$.*

The next lemma is due to Aziz and Zargar [1]

Lemma 2.3. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 0$, then for every $R \geq r$ and $rR \geq k^2$,*

$$|f(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |f(rz)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

Proof of Theorem 2.1. Since by hypothesis

$$|f(z)| \leq |F(z)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

Therefore for every complex number δ with $|\delta| > 1$, we have $|f(z)| < |\delta F(z)|$ for $z \in \delta\mathbb{D}$.Further all zeros of $F(z)$ lie in $\overline{\mathbb{D}}$. It follows from Rouche's theorem that all zeros of polynomial $g(z) = f(z) - \delta F(z)$ of degree n also lie in $\overline{\mathbb{D}}$. Therefore by Lemma

2.2, all zeros of

$$\begin{aligned}\tilde{\mathbb{S}}_a[g](z) &= \tilde{\mathbb{S}}_a[f-\delta F](z)\\ &= (1+az)[f-\delta F]^{'}(z) - na[f-\delta F](z)\\ &= (1+az)f^{'}(z) - \delta(1+az)F^{'}(z) - naf(z) + \delta naF(z)\\ &= \tilde{\mathbb{S}}_a[f](z) - \delta\tilde{\mathbb{S}}_a[F](z)\end{aligned}$$

lie in $\overline{\mathbb{D}}$ for every a such that $a \in \delta\mathbb{D}$ is not the exceptional value of g.
This gives

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](z)| \quad \text{for} \quad z \in \mathbb{C}\setminus\mathbb{D}.$$

If this is not true then there exists $z_0 \in \mathbb{C}\setminus\mathbb{D}$ such that

$$|\tilde{\mathbb{S}}_a[f](z)|_{z=z_0} > |\tilde{\mathbb{S}}_a[F](z)|_{z=z_0}.$$

Since all zeros of $F(z)$ lie in $\overline{\mathbb{D}}$, therefore by Lemma 2.2, for $z_0 \in \mathbb{C}\setminus\mathbb{D}$

$$\tilde{\mathbb{S}}_a[F](z) \neq 0.$$

Hence we can choose

$$\delta = \frac{\tilde{\mathbb{S}}_a[f](z_0)}{\tilde{\mathbb{S}}_a[F](z_0)}.$$

So that $\delta \in \mathbb{C}\setminus\overline{\mathbb{D}}$ and with such choice of δ, we get for $z_0 \in \mathbb{C}\setminus\mathbb{D}$

$$\tilde{\mathbb{S}}_a[g](z_0) = 0,$$

which contradicts. Hence, we get

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](z)| \quad \text{for} \quad z \in \mathbb{C}\setminus\mathbb{D}.$$

After substituting $z = Re^{i\theta}$, $R > 1$, we have for $0 \leq \theta < 2\pi$

$$|\tilde{\mathbb{S}}_a[f](Re^{i\theta})| \leq |\tilde{\mathbb{S}}_a[F](Re^{i\theta})|.$$

By using argument of continuity, the theorem is true for $R \geq 1$.
Since $g(z)$ has all zeros in $\overline{\mathbb{D}}$. Therefore $g(Re^{i\theta}) \neq 0$ for every $R > 1$, $0 \leq \theta < 2\pi$.
Hence by Lemma 2.3 with $k = 1$ and $r = 1$, we get

$$|g(Re^{i\theta})| \geq \left(\frac{R+1}{2}\right)^n |g(e^{i\theta})| \quad \text{for} \quad 0 \leq \theta < 2\pi.$$

From this we get for $R > 1$,

$$|g(z)| < |g(Rz)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

If β is any complex number with $\beta \in \overline{\mathbb{D}}$, then it follows that $|\beta g(z)| < |g(Rz)|$ for $z \in \delta\mathbb{D}$ and $R > 1$. As all zeros of $g(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, by Rouche's theorem the polynomial

$$\begin{aligned}h(z) &= g(Rz) - g(z)\\ &= f(Rz) - \beta f(z) - \delta\{F(Rz) - \beta F(z)\}\end{aligned}$$

has its zeros in $\mathbb{D}$ for every complex δ with $|\delta| > 1$ and $R > 1$. Therefore by Lemma 2.2, it follows that all zeros of

$$\begin{aligned}\tilde{\mathbb{S}}_a[h](z) &= \tilde{\mathbb{S}}_a[f(Rz) - \beta f(z) - \delta(F(Rz) - \beta F(z))] \\ &= (1 + az)\{[f(Rz)]' - \beta f'(z) - \delta[F(Rz)]' + \delta\beta F'(z)\} \\ &- na\{f(Rz) - \beta f(z) - \delta F(Rz) + \delta\beta F(z)\} \\ &= \tilde{\mathbb{S}}_a[f](Rz) - \beta\tilde{\mathbb{S}}_a[f](z) - \delta\{\tilde{\mathbb{S}}_a[F](Rz) - \beta\tilde{\mathbb{S}}_a[F](z)\}\end{aligned}$$

lie in $\mathbb{D}$.

Hence we conclude for $R > 1$ and $z \in \delta\mathbb{D}$

$$|\tilde{\mathbb{S}}_a[f](Rz) - \beta\tilde{\mathbb{S}}_a[f](z)| \le |\tilde{\mathbb{S}}_a[F](Rz) - \beta\tilde{\mathbb{S}}_a[F](z)|.$$

□

Remark 2.1. If in inequality (2.2) we choose $a = 0$, we get for $R \ge 1$ and $\beta \in \overline{\mathbb{D}}$,

$$|Rf'(Rz) - \beta f'(z)| \le |RF'(Rz) - \beta F'(z)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

If we choose $a = 0$, $\beta = 0$ and $R = 1$, we get Theorem 1.2.

Remark 2.2. If in inequality (2.1) we take $\beta = 1$ and then divide both sides by $R - 1$, we get after making $R \to 1$, the following:

Corollary 2.4. *Let $f(z)$ and $F(z)$ be two polynomials of degree n. If $F(z)$ has all its zeros in $\overline{\mathbb{D}}$ and $|f(z)| \le |F(z)|$ for $z \in \delta\mathbb{D}$, then we have for $z \in \delta\mathbb{D}$*

$$|\tilde{\mathbb{S}}'_a[f](z)| \le |\tilde{\mathbb{S}}'_a[F](z)|.$$

Equivalently

$$|(1 + az)f''(z) - (n - 1)af'(z)| \le |(1 + az)F''(z) - (n - 1)aF'(z)|$$

Remark 2.3. In inequality (2.1), if we take $F(z) = Mz^n$, where $M = \max_{z\in\delta\mathbb{D}} |f(z)|$, we get the following:

Corollary 2.5. *If $f(z)$ is a polynomial of degree n, then for any complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \ge 1$, we have*

$$|\tilde{\mathbb{S}}_a[f](Rz) - \beta\tilde{\mathbb{S}}_a[f](z)| \le |R^n - \beta||\tilde{\mathbb{S}}_a[E_n](z)| \max_{z\in\delta\mathbb{D}} |f(z)| \quad \textit{for} \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{2.3}$$

where $E_n(z) = z^n$. The result is sharp and equality holds for $f(z) \equiv \gamma z^n$, $\gamma \ne 0$

If we choose $\beta = 0$ in inequality (2.3), we get the following:

Corollary 2.6. *If $f(z)$ is a polynomial of degree n, then for $R \ge 1$, we have*

$$|\tilde{\mathbb{S}}_a[f](Rz)| \le R^n|\tilde{\mathbb{S}}_a[E_n](z)| \max_{z\in\delta\mathbb{D}} |f(z)| \quad \textit{for} \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \tag{2.4}$$

Equivalently for $R \ge 1$

$$|(1 + az)Rf'(Rz) - naf(Rz)| \le R^n n|z|^n \max_{z\in\delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

The result is sharp and equality holds for $f(z) \equiv \gamma z^n$, $\gamma \neq 0$.
For fixed $z \in \mathbb{C} \setminus \mathbb{D}$, we rewrite inequality (2.1) in the following form

$$\begin{aligned}&|z(f(Rz))' - n\frac{az}{1+az}f(Rz) - \beta\{zf'(z) - n\frac{az}{1+az}f(z)\}|\\ &\leq |z(F(Rz))' - n\frac{az}{1+az}F(Rz) - \beta\{zF'(z) - n\frac{az}{1+az}F(z)\}|.\end{aligned}$$

This after replacing $\frac{az}{1+az}$ by α, gives

$$\begin{aligned}&|z(f(Rz))' - n\alpha f(Rz) - \beta\{zf'(z) - n\alpha f(z)\}|\\ &\leq |z(F(Rz))' - n\alpha F(Rz) - \beta\{zF'(z) - n\alpha F(z)\}|.\end{aligned}$$

That is

$$|\mathbb{S}_\alpha[f](Rz) - \beta\mathbb{S}_\alpha[f](z)| \leq |\mathbb{S}_\alpha[F](Rz) - \beta\mathbb{S}_\alpha[F](z)|$$

Or equivalently

$$\left|\frac{\lambda_1 n}{2}\mathbb{S}_\alpha f(Rz) - \beta\frac{\lambda_1 n}{2}\mathbb{S}_\alpha f(z)\right| \leq \left|\frac{\lambda_1 n}{2}\mathbb{S}_\alpha[F](Rz) - \beta\frac{\lambda_1 n}{2}\mathbb{S}_\alpha[F](z)\right|. \tag{2.5}$$

Using (1.6), we have for $R \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{D}$

$$|\mathbb{B}[f](Rz) - \beta\mathbb{B}[f](z)| \leq |\mathbb{B}[F](Rz) - \beta\mathbb{B}[F](z)|.$$

Note that if $P(z)$ is a polynomial of degree n which does not vanish in $\mathbb{D}$, then the polynomial $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ has all its zeros in $\overline{\mathbb{D}}$. Hence if we replace $f(z)$ by $P(z)$ and $F(z)$ by $Q(z)$ in (2.1), we get the following:

Corollary 2.7. *If $P(z)$ is a polynomial of degree n which does not vanish in $\mathbb{D}$, then for any complex number β with $\beta \in \overline{\mathbb{D}}$, $R > 1$ and $z \in \delta\mathbb{D}$, we have*

$$|\tilde{\mathbb{S}}_a[P](Rz) - \beta\tilde{\mathbb{S}}_a[P](z)| \leq |\tilde{\mathbb{S}}_a[Q](Rz) - \beta\tilde{\mathbb{S}}_a[Q](z)|,$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

The result is sharp and equality holds for any polynomial having zeros on $\delta\mathbb{D}$.

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