Korean J. Math. **31** (2023), No. 1, pp. 35–48
http://dx.doi.org/10.11568/kjm.2023.31.1.35

ITERATIVE PROCESS FOR FINDING FIXED POINTS OF QUASI-NONEXPANSIVE MULTIMAPS IN CAT(0) SPACES

Pitchaya Kingkam* and Jamnian Nantadilok

Abstract. Let $\mathbb{E}$ be a CAT(0) space and K be a nonempty closed convex subset of $\mathbb{E}$. Let $T : K \to \mathcal{P}(K)$ be a multimap such that $F(T) \neq \emptyset$ and $\mathbb{P}_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Define sequence $\{x_n\}$ by $x_{n+1} = (1-\alpha)v_n \bigoplus \alpha w_n$, $y_n = (1-\beta)u_n \bigoplus \beta w_n$, $z_n = (1-\gamma)x_n \bigoplus \gamma u_n$ where $\alpha, \beta, \gamma \in [0; 1]; u_n \in \mathbb{P}_T(x_n); v_n \in \mathbb{P}_T(y_n)$ and $w_n \in \mathbb{P}_T(z_n)$. (1) If $\mathbb{P}_T$ is quasi-nonexpansive, then it is proved that $\{x_n\}$ converges strongly to a fixed point of T. (2) If a multimap T satisfies Condition(I) and $\mathbb{P}_T$ is quasi-nonexpansive, then $\{x_n\}$ converges strongly to a fixed point of T. (3) Finally, we establish a weak convergence result. Our results extend and unify some of the related results in the literature.

1. Introduction

Let X denotes a real Banach space. A subset K of X is called proximinal if for each $x \in X$, there exists $k \in K$ such that $d(x, k) = d(x, K) = \inf\{\|x - y\| : y \in K\}$. It is well known that weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal. We denote the family of nonempty proximinal bounded subsets of K by $\mathcal{P}(K)$ and let $CB(K)$ denote the class of all nonempty closed and bounded subsets of K. It is well known that if K is a proximinal subset of X, then K is closed. For every $A, B \in CB(X)$, let $\mathcal{H}$ be a Hausdorff metric induced by the metric d of X. Define $\mathcal{H}$ by

$$\mathcal{H}(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}.$$

A point $x \in K$ is called a fixed point of a multivlaued mapping $T : K \to CB(K)$ if $x \in Tx$. A set of all fixed points of T is denoted by $F(T)$.

Definition 1.1. Let K be a closed and convex subset of a Banach space X. A multivalued mapping $T : K \to CB(K)$ is said to be:

(a) nonexpansive if $\mathcal{H}(Tx, Ty) \leq \|x - y\|$ for all $x, y \in K$,
(b) quasi-nonexpansive if $\mathcal{H}(Tx, p) \leq \|x - p|$ for all $x \in K$ and $p \in F(T)$.

Received September 20, 2022. Revised February 11, 2023. Accepted February 24, 2023.
2010 Mathematics Subject Classification: 47H10; 54H25.
Key words and phrases: Multivalued nonexpansive mappings, Quasi-nonexpansive mapping, Weak and strong convergence, Banach space, CAT(0) space.
This research work is supported by Faculty of Science, Lampang Rajabhat University.
∗ Corresponding author.

It is known that every nonexpansive multivalued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive, but the converse is not true. The investigation of fixed points for multivalued mappings using the Hausdorff metric was initiated by Markin [20] (see also [21]). Multivalued fixed point theory has applications in control theory, convex optimization, differential inclusion, and economics (see, [11] and references therein). The theory of multivalued mappings is more difficult than the corresponding theory of single valued mappings. Different iterative processes have been used to approximate the fixed points of multivalued mappings (see [6, 14, 18, 32] and references therein). Among these iterative processes, Sastry and Babu [27] considered the following.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers. Let K be a nonempty convex subset of X, and $T : K \to \mathcal{P}(K)$ be a multivalued mapping with $p \in Tp$ for all $p \in K$.

(i) The sequences of Mann iterates is defined by $x_1 \in K$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n,\ n \in \mathbb{N} \tag{1}$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$ and $\{a_n\}$ is a sequence of numbers in $(0, 1)$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$.

(ii) Ishikawa iterative process is defined by starting with $x_1 \in K$ and

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n u_n, \\ y_n &= (1 - b_n)x_n + b_n z_n,\ n \in \mathbb{N} \end{aligned} \tag{2}$$

where $u_n \in Ty_n, z_n \in Tx_n$ are such that $\|u_n - p\| = d(p, Ty_n)$, $\|z_n - p\| = d(p, Tx_n)$ and $\{a_n\}, \{b_n\}$ are sequences of numbers with $0 < a_n, b_n < 1$ satisfying $\lim\limits_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} a_n b_n = \infty$.

Later, Panyanak [25] generalized the results proved by Sastry and Babu [27].

The following lemma is due to Nadler [21].

Lemma 1.2. *Let $A, B \in CB(X)$ and $a \in A$. If $\mu > 0$, then there exists $b \in B$ such that $d(a, b) \leq \mathcal{H}(A, B) + \mu$.*

Based on the above lemma, Song and Wang [30] modified the iterative process due to Panyanak [25] and improved the results presented there. They used (2) but $a_n, b_n \in [0, 1]$ with $\lim\limits_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} a_n b_n = \infty; z_n \in Tx_n, u_n \in Ty_n$ with $\|z_n - u_n\| \leq \mathcal{H}(Tx_n, Ty_n) + \mu_n$ and $\|z_{n+1} - u_n\| \leq \mathcal{H}(Tx_{n+1}, Ty_n) + \mu_n$ where $\mu_n \in (0, \infty)$ such that $\lim\limits_{n\to\infty} \mu_n = 0$.

It is notable that Song and Wang [30] needed the condition $Tp = \{p\}$ in order to prove their result (Theorem 1). Actually, Panyanak [25] proved some results using Ishikawa type iterative process without this condition. Song and Wang [30] showed that without this condition his process was not well-defined. They reconstructed the process using the condition $Tp = \{p\}$ which made it well-defined. Such a condition was also used by Jung [15]. Later, Shazad and Zegeye [31] got rid of this condition by using

$$\mathbb{P}_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}. \tag{3}$$

for a multivalued mapping $T : K \to \mathcal{P}(K)$. They obtaind a couple of strong convergence results using Ishikawa type iterative process.

Khan and Yildirim [18] used the following iterative process using the method of direct construction of Cauchy sequence and without using the condition $Tp = \{p\}$. Starting $x_1 \in K$, define sequence $\{x_n\}$ as follows:-

$$\begin{aligned} x_{n+1} &= (1-\lambda)v_n + \lambda u_n, \\ y_n &= (1-\mu)x_n + \mu v_n,\ n \in \mathbb{N} \end{aligned} \tag{4}$$

where $v_n \in \mathbb{P}_T(x_n)$ and $u_n \in \mathbb{P}_T(y_n)$ and $0 < \lambda, \mu < 1$.

In 2014, Khan et al. [16] introduced the following iterative process for a multivalued mapping $T : K \to \mathcal{P}(K)$ using $\mathbb{P}_T$ defined by (3) above.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)v_n + \alpha_n w_n, \\ y_n &= (1-\beta_n)u_n + \beta_n w_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n u_n \end{aligned} \tag{5}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $u_n \in \mathbb{P}_T(x_n), v_n \in \mathbb{P}_T(y_n), w_n \in \mathbb{P}_T(z_n)$. Its single-valued version was used by Abbas and Nazir [2].

For simplicity, Khan et al. [16] used the following simplified version of (5):

$$\begin{aligned} x_{n+1} &= (1-\alpha)v_n + \alpha w_n, \\ y_n &= (1-\beta)u_n + \beta w_n \\ z_n &= (1-\gamma)x_n + \gamma u_n \end{aligned} \tag{6}$$

where $\alpha, \beta, \gamma \in [0,1]$ and $u_n \in \mathbb{P}_T(x_n), v_n \in \mathbb{P}_T(y_n), w_n \in \mathbb{P}_T(z_n)$.

Note that Khan et al. [16] used α, β, and γ only for the sake of simplicity and α_n, β_n, and γ_n could be used equally well under suitable conditions. Moreover, they claimed that it is faster than all of Picard, Mann and Ishikawa iterative processes in case of contractions [2]. Their results are independent but better (in the sense of speed of convergence of iterative process) and more general (in view of more general class of mappings) than corresponding results of Khan and Yildirim [18], Shazad and Zegeye [31], and Song and Cho [28] and the related results therein.

A multivalued nonexpansive mapping $T : K \to CB(K)$ is said to satisfy **Condition (I)** if there exists a continuous nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, Tx) \geq f(d(x, F(T))$ for all $x \in K$.

We mention some of results by Khan et al. [16] as follows.

Theorem 1.3. *Let E be a real Banach space and K be a nonempty compact convex subset of E. Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in* (6) *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 1.4. *Let E be a real Banach space, K a nonempty closed and convex subset of E and $T : K \to \mathcal{P}(K)$ a multivalued mapping satisfying Condition(I) such that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is quasi-nonexpansive mapping. Then the sequence $\{x_n\}$ defined by* (6) *converges strongly to a fixed point p of T.*

The purpose of this manuscript is to extend and improve some corresponding results by Khan et al. [16] in the setting of CAT(0) spaces.

2. Preliminaries

2.1. CAT(0) spaces. Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map $\omega : [0,a] \to X, [0,a] \subset R$ such that $\omega(0) = x, \omega(a) = y$, and $d(\omega(m), \omega(n)) = |m-n|$ for all $m,n \in [0,a]$. In particular, ω is an isometry and $d(x,y) = a$. The image α of ω is called a geodesic (or metric) segment joining x and y. A unique geodesic segment from x to y is denoted by $[x,y]$. The space (X,d) is called to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. If $Y \subseteq X$ then Y is said to be convex if Y includes every geodesic segment joining any two of its points. If (X,d) is a geodesic metric space, a *geodesic triangle* $\Delta(a_1,a_2,a_3)$ consists of three points a_1,a_2,a_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(a_1,a_2,a_3)$ in (X,d) is a triangle $\bar{\Delta}(a_1,a_2,a_3) := \Delta(\bar{a_1},\bar{a_2},\bar{a_3})$ in the Euclidean plane $\mathbb{R}^2$ satisfying $d_{\mathbb{R}^2}(\bar{a_i},\bar{a_j}) = d(a_i,a_j)$ for $i,j \in 1,2,3$. Such a triangle always exists (see [3]).

Definition 2.1. A geodesic space (X,d) is said to be a CAT(0) space if for any geodesic triangle $\Delta \subset X$ and $a,b \in \Delta$ we have $d(a,b) \le d(\bar{a},\bar{b})$ where $\bar{a},\bar{b} \in \bar{\Delta}$.

Remark 2.2. Any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces, R-trees, Euclidean buildings, the complex Hilbert ball with a hyperbolic metric, (see [3, 4, 12] for example).

Definition 2.3. A geodesic triangle $\Delta(p,q,r)$ in (X,d) is said to satisfy the CAT(0) *inequality* if for any $u,v \in \Delta(p,q,r)$ and for their comparison points $\bar{u},\bar{v} \in \bar{\Delta}(\bar{p},\bar{q},\bar{r})$, one has

$$d(u,v) \le d_{\mathbb{R}^2}(\bar{u},\bar{v}).$$

For other equivalent definitions and basic properties of CAT(0) spaces, we refer the readers to standard texts such as [3].

Note that if x, a_1, a_2 are points of CAT(0) space and if a_0 is the midpoint of the segment $[a_1,a_2]$ (we write $a_0 = \frac{1}{2}a_1 \bigoplus \frac{1}{2}a_2$), then the CAT(0) inequality implies

$$d(x,a_0)^2 = d(x, \frac{1}{2}a_1 \bigoplus \frac{1}{2}a_2) \le \frac{1}{2}d(x,a_1)^2 + \frac{1}{2}d(x,a_2)^2 - \frac{1}{4}d(a_1,a_2)^2. \tag{7}$$

The inequality (7) is called the **CN inequality of Bruhat and Tits** [5]. We refer readers to some brilliant known CAT(0) space results in [1, 7, 8, 22, 23] and references therein.

We now collect some useful facts about CAT(0) spaces which will be crucial in the proofs of our main results.

Lemma 2.4. *([8]) Let (X,d) be a CAT(0) space.*

(i) For $x_1, x_2 \in X$ and $\alpha \in [0,1]$, there exists a unique point $y \in [x_1,x_2]$ such that

$$d(x_1,y) = \alpha d(x_1,x_2) \text{ and } d(x_2,y) = (1-\alpha)d(x_1,x_2). \tag{8}$$

We write $y = (1-\alpha)x_1 \bigoplus \alpha x_2$ for the unique point y satisfying (8).

(ii) For $x, y, z \in X$ and $\alpha \in [0,1]$, we have

$$d\big((1-\alpha)x \bigoplus \alpha y, z\big) \le (1-\alpha)d(x,z) + \alpha d(y,z).$$

2.2. Hyperbolic spaces. In this sectionn we recall some notions of the hyperbolic spaces. This class of spaces contains the class of CAT(0) spaces (see [19]).

Definition 2.5. [19] Let (X, d) be a metric space and $\mathcal{W} : X \times X \times [0,1] \to X$ be a mapping satisfying:-

W1. $d(z, \mathcal{W}(x, y, \alpha)) \leq (1-\alpha)d(z, x) + \alpha d(z, y)$,
W2. $d(\mathcal{W}(x, y, \alpha), \mathcal{W}(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
W3. $\mathcal{W}(x, y, \alpha) = \mathcal{W}(y, x, (1-\alpha))$,
W4. $d(\mathcal{W}(x, z, \alpha), \mathcal{W}(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X, \alpha, \beta \in [0,1]$. We call the triple $(X, d, \mathcal{W})$ a **hyperbolic space**.

It follows from (W1.) that, for each $x, y \in X$ and $\alpha \in [0,1]$,

$$d(x, \mathcal{W}(x, y, \alpha)) \leq \alpha d(x, y), \ d(y, \mathcal{W}(x, y, \alpha)) \leq (1-\alpha)d(x, y) \tag{9}$$

In fact, we can get that (see [23]),

$$d(x, \mathcal{W}(x, y, \alpha)) = \alpha d(x, y), \ d(y, \mathcal{W}(x, y, \alpha)) = (1-\alpha)d(x, y). \tag{10}$$

Similar to (8), we can also use the notation $(1-\alpha)x \bigoplus \alpha y$ for such point $\mathcal{W}(x, y, \alpha)$ in a **hyperbolic space**.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called a **modulus of uniform convexity**.

Definition 2.6. [9, 13] Let $(X, d, \mathcal{W})$ be a hyperbolic metric space. X is said to be uniformly convex whenever $\delta(r, \epsilon) > 0$, for any $r > 0$ and $\epsilon > 0$, where

$$\delta(r, \epsilon) = \inf \left\{ 1 - \frac{1}{r} d(\frac{1}{2}x \bigoplus \frac{1}{2}y, a) : d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\epsilon \right\}$$

for any $a \in X$.

Note that if X is a uniformly convex hyperbolic space, then for every $s \geq 0$ and $\epsilon > 0$, there exists $\eta(s, \epsilon) > 0$ such that $\delta(r, \epsilon) > \eta(s, \epsilon) > 0$ for any $r > s$. One can see that $\delta(r, 0) = 0$. Moreover $\delta(r, \epsilon)$ is an increasing function of ϵ.

We recall that a Banach space E is said to satisfy Opial's condition [24] if for any sequence $\{x_n\}$ in E, $\{x_n\}$ converges weakly to x implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial's condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy this condition. A multivalued mapping $T : K \to CB(K)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K converging weakly to x and $y_n \in Tx_n$ converging strongly to y, we have $y \in Tx$.

Now we state some useful lemmas.

Lemma 2.7. [28] *Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping and $\mathbb{P}_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Then the following are equivalent.*

(i) $x \in F(T)$.
(ii) $\mathbb{P}_T(x) = \{x\}$.
(iii) $x \in F(\mathbb{P}_T)$.

Moreover, $F(T) = F(\mathbb{P}_T)$.

Proposition 2.8. (**The Demiclosed principle**) *[23] Let K be a closed and convex subset of a complete CAT(0) space E, and $T : K \to K$ be an asymptotic pointwise nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, and $\{x_n\}$ conveges weakly to w. Then $Tw = w$.*

The following result is a characterization of uniformly convex hyperbolic spaces which is an analog of Shu ([26] Lemma 1.3). It can be applied in a CAT(0) space.

Lemma 2.9. *[10,23] Let $(X, d, \mathcal{W})$ be a uniformly convex hyperbolic space. Let $r \in [0, +\infty)$ and $a \in X$ be such that such that $\limsup_{n\to\infty} d(x_n, a) \leq r, \limsup_{n\to\infty} d(y_n, a) \leq r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \bigoplus \alpha_n y_n, p) = r$ for some $r \geq 0$, where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then we have*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Remark 2.10. More details on a uniformly convex hyperbolic space with modulus of convexity η, we refer readers to [19].

In this manuscript, inspired and motivated by above results and the works of Khan et al. [16], we establish strong and weak convergence theorems for fixed points of multivalued quasi-nonexpansive mappings in the setting of CAT(0) spaces. Our results extend and improve the results in [16], as well as some other related results in the literature.

3. Main results

In this section, $\mathbb{E}$ will denote a CAT(0) space. Following Khan et al. [16], we introduce the following definitions.

Definition 3.1. Let K be a nonempty convex subset of a CAT(0) space $\mathbb{E}$. Let $T : K \to \mathcal{P}(K)$, define sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha)v_n \bigoplus \alpha w_n \\ y_n &= (1-\beta)u_n \bigoplus \beta w_n \qquad (11) \\ z_n &= (1-\gamma)x_n \bigoplus \gamma u_n \end{aligned}$$

where $\alpha, \beta, \gamma \in [0; 1]; u_n \in \mathbb{P}_T(x_n); v_n \in \mathbb{P}_T(y_n);\ w_n \in \mathbb{P}_T(z_n)$, and $\mathbb{P}_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$.

Definition 3.2. Let K be a nonempty convex subset of a CAT(0) space $\mathbb{E}$. A multivalued mapping $T : K \to \mathcal{P}(K)$ is said to be:

(a) nonexpansive if $\mathcal{H}(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$,
(b) quasi-nonexpansive if $\mathcal{H}(Tx, Tp) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$.

We prove the following lemma.

Lemma 3.3. *Let $\mathbb{E}$ be a CAT(0) space and K a nonempty closed convex subset of $\mathbb{E}$. Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is a quasi-nonexpansive multimaping. Let $\{x_n\}$ be the sequence generated by (11). Then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, \mathbb{P}_T(x_n)) = 0$.*

Proof. We first prove that $\lim_{n\to\infty} d(x_n, p)$ exists, for all $p \in F(T)$. By applying Lemma 2.4 (ii), we have

$$\begin{aligned} d(x_{n+1}, p) &= d\big((1-\alpha)v_n \bigoplus \alpha w_n, p\big) \\ &\le (1-\alpha)d(v_n, p) + \alpha d(w_n, p) \\ &\le (1-\alpha)\mathcal{H}(\mathbb{P}_T(y_n), \mathbb{P}_T(p)) + \alpha\mathcal{H}(\mathbb{P}_T(z_n), \mathbb{P}_T(p)) \\ &\le (1-\alpha)d(y_n, p) + \alpha d(z_n, p). \end{aligned} \tag{12}$$

Next

$$\begin{aligned} d(y_n, p) &= d\big((1-\beta)u_n \bigoplus \beta w_n, p\big) \\ &\le (1-\beta)d(u_n, p) + \beta d(w_n, p) \\ &\le (1-\beta)\mathcal{H}(\mathbb{P}_T(x_n), \mathbb{P}_T(p)) + \beta\mathcal{H}(\mathbb{P}_T(z_n), \mathbb{P}_T(p)) \\ &\le (1-\beta)d(x_n, p) + \beta d(z_n, p). \end{aligned} \tag{13}$$

And

$$\begin{aligned} d(z_n, p) &= d\big((1-\gamma)x_n \bigoplus \gamma u_n, p\big) \\ &\le (1-\gamma)d(x_n, p) + \gamma d(u_n, p) \\ &\le (1-\beta)d(x_n, p) + \beta\mathcal{H}(\mathbb{P}_T(x_n), \mathbb{P}_T(p)) \\ &\le (1-\alpha)d(x_n, p) + \alpha d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{14}$$

Therefore, from (13) and (14) we get

$$d(y_n, p) \le d(x_n, p). \tag{15}$$

From (12), (14) and (15), we obtain

$$d(x_{n+1}, p) \le d(x_n, p). \tag{16}$$

This implies that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Suppose that

$$\lim_{n\to\infty} d(x_n, p) = r, \tag{17}$$

where $r \ge 0$.

We now show that

$$\lim_{n\to\infty} d(x_n, \mathbb{P}_T x_n) = 0.$$

The case when $r = 0$ is obvious. We thus assume that $r > 0$. In as much as $d(x_n, \mathbb{P}_T x_n) \le d(x_n, u_n)$, it suffices to prove that $\lim_{n\to\infty} d(x_n, u_n) = 0$.

Now

$$d(u_n, p) \le \mathcal{H}(\mathbb{P}_T(x_n), \mathbb{P}_T(p)) \le d(x_n, p)$$

implies that

$$\limsup_{n\to\infty} d(u_n, p) \le r. \tag{18}$$

From (14), (15) and (17), we obtain

$$\limsup_{n\to\infty} d(z_n, p) \le r, \tag{19}$$

and

$$\limsup_{n\to\infty} d(y_n, p) \le r. \tag{20}$$

Noting that

$$d(v_n, p) \le \mathcal{H}(\mathbb{P}_T(y_n), \mathbb{P}_T(p)) \le d(y_n, p) \le d(x_n, p),$$

we have

$$\limsup_{n\to\infty} d(v_n, p) \le r. \tag{21}$$

Similarly, we can get

$$\limsup_{n\to\infty} d(w_n, p) \le r. \tag{22}$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1}, p) &= \lim_{n\to\infty} d\big((1-\alpha)v_n \bigoplus \alpha w_n, p\big)\\ &\le \lim_{n\to\infty} \big((1-\alpha)d(v_n, p) + \alpha d(w_n, p)\big)\\ &= r.\end{aligned}$$

This implies

$$\lim_{n\to\infty} d\big((1-\alpha)v_n \bigoplus \alpha w_n, p\big) = r. \tag{23}$$

From (21), (22), (23) and Lemma 2.9, we have

$$\lim_{n\to\infty} d(v_n, w_n) = 0.$$

Together with this and

$$\liminf_{n\to\infty} d(x_{n+1}, p) \le \liminf_{n\to\infty} \big((1-\alpha)d(v_n, p) + \alpha d(w_n, p)\big),$$

we obtain

$$r \le \liminf_{n\to\infty} d(v_n, p). \tag{24}$$

From (21) and (24), we get

$$\lim_{n\to\infty} d(v_n, p) = r.$$

Similarly to above, it follows that

$$\lim_{n\to\infty} d(z_n, p) = r.$$

That is

$$\begin{aligned}r = \lim_{n\to\infty} d(z_n, p) &= \lim_{n\to\infty} d\big((1-\gamma)x_n \bigoplus \gamma u_n, p\big)\\ &\le \lim_{n\to\infty} \big((1-\gamma)d(x_n, p) + \gamma d(u_n, p)\big)\\ &= r.\end{aligned} \tag{25}$$

Therefore, from (17), (18), (25) and Lemma 2.9 we get

$$\lim_{n\to\infty} d(x_n, u_n) = 0 \tag{26}$$

which yields $\lim_{n\to\infty} d(x_n, \mathbb{P}_T(x_n)) = 0$ as desired. □

We now state and prove our strong convergence theorem.

THEOREM 3.4. *Let $\mathbb{E}$ be a CAT(0) space and K be a nonempty compact convex subset of $\mathbb{E}$. Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is a quasi-nonexpansive multimaping. Let $\{x_n\}$ be the sequence as generated by (11). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.3, we know that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Now, since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} d(x_{n_k}, q) = 0$ for some $q \in K$. By Lemma 3.3, we have $\lim_{n\to\infty} d(x_{n_k}, u_{n_k}) = 0$. Thus, we have

$$\begin{aligned} d(q, \mathbb{P}_T(q)) &\leq d(q, x_{n_k}) + d(x_{n_K}, \mathbb{P}_T(x_{n_k})) + \mathcal{H}(\mathbb{P}_T(x_{n_k}), \mathbb{P}_T(q)) \\ &\leq d(q, x_{n_k}) + d(x_{n_K}, u_{n_k}) + d(x_{n_k}, q) \to 0 \text{ as } n \to \infty. \end{aligned}$$

That is, $d(q, \mathbb{P}_T(q)) = 0$. Hence q is a fixed point of $\mathbb{P}_T$. Since $F(\mathbb{P}_T) = F(T)$ by Lemma 2.7, $\{x_n\}$ converges strongly to a fixed point of T. □

EXAMPLE 3.5. Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space with usual norm, and $K = [0,1] \times [0,1]$. Here $\mathbb{R}$ is the set of real numbers. Define $T : K \to \mathcal{P}(K)$ by

$$T(x,y) = \{(0, \frac{2b+1}{4}) : 0 \leq b \leq \max\{x,y\}\}.$$

Obviously, K is a compact convex subset of $\mathbb{R}^2$. Note that $F(T) = \{(x,y) : (x,y) \in T(x,y)\} = \{(0, \frac{2b+1}{4}) : 0 \leq b \leq \frac{1}{2}\}$. Let $\alpha, \beta, \gamma = \frac{1}{2}$.

Observe that $\mathbb{P}_T(x,y) = \{(x,y)\}$ whenever $(x,y) \in \{(0, \frac{2y+1}{4}) : 0 \leq y \leq \frac{1}{2}\}$. In case $(x,y) \notin \{(0, \frac{2y+1}{4}) : 0 \leq y \leq \frac{1}{2}\}$,

$$\begin{aligned} \mathbb{P}_T(x,y) &= \{(x',y') \in T(x,y) : d((x,y),(x',y')) = d\big((x,y), \{(0, \frac{2y+1}{4}) : 0 \leq y \leq \frac{1}{2}\} \\ &= \{(x',y') \in T(x,y) : d((x,y),(x',y')) = |x| = x\} \\ &= \{(x',y') = (0, \frac{2y+1}{4}), \text{ where } 0 \leq y \leq \frac{1}{2}\}. \end{aligned}$$

We next prove that $\mathbb{P}_T(x,y)$ is quasi-nonexpansive for all $(x,y) \in K$. The case of $\{(0, \frac{2b+1}{4}) : 0 \leq b \leq \frac{1}{2}\}$ is trivial. Thus we take $(x,y) \in \{(0, \frac{2y+1}{4}) : \frac{1}{2} \leq y \leq 1\}$.

$$\mathcal{H}\big(\mathbb{P}_T(x,y), \mathbb{P}_T(0,p)\big) = \mathcal{H}\big((0, \frac{2y+1}{4}), (0,p)\big) = |\frac{2y+1}{4} - p| \leq d\big((x,y),(0,p)\big)$$

for all $(x,y) \in \{(0, \frac{2y+1}{4}) : \frac{1}{2} \leq y \leq 1\}$. Finally, we generate a sequence $\{x_n\}$ as defined in (11) and show that it converges strongly to a fixed point of T.

Choose $x_1 = (x'_1, x'_2) = (0,1) \in K$. Then $\mathbb{P}_T(x_1) = \{(0, \frac{2x'_2+1}{4})\} = \{(0, \frac{2(1)+1}{4})\} = \{(0, \frac{1}{2} + \frac{1}{4})\}$ and $u_1 \in \mathbb{P}_T(x_1) = \{(0, \frac{1}{2} + \frac{1}{4})\}$. That is, $u_1 = (0, \frac{1}{2} + \frac{1}{4})$. Then

$$\begin{aligned}
z_1 = (z'_1, z'_2) &= (1-\gamma)x_1 \bigoplus \gamma u_1 = \frac{1}{2}(0,1) + \frac{1}{2}(0, \frac{1}{2} + \frac{1}{4}) \\
&= (0, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = (0, \frac{7}{8}), \\
\mathbb{P}_T(z_1) &= \{(0, \frac{2z'_2+1}{4})\} = \{(0, \frac{2(\frac{7}{8})+1)}{4})\} \\
&= \{(0, \frac{11}{16})\}.
\end{aligned}$$

Choose $w_1 \in \mathbb{P}_T(z_1) = \{(0, \frac{11}{16})\}$. That is, $w_1 = (0, \frac{11}{16})$. Then

$$\begin{aligned}
y_1 &= (1-\beta)u_1 \bigoplus \beta w_1 = \frac{1}{2}(0, \frac{3}{4}) + \frac{1}{2}(0, \frac{11}{16}) \\
&= (0, \frac{23}{32}) = (y'_1, y'_2), \\
\mathbb{P}_T(y_1) &= \{(0, \frac{2y'_2+1}{4})\} = \{(0, \frac{2(\frac{23}{32})+1)}{4}\} \\
&= \{(0, \frac{1}{2} + \frac{7}{64})\}.
\end{aligned}$$

Choose $v_1 \in \mathbb{P}_T(y_1) = \{(0, \frac{1}{2} + \frac{7}{64})\}$. That is, $v_1 = (0, \frac{1}{2} + \frac{7}{64}) = (0, \frac{39}{64})$. Then

$$\begin{aligned}
x_2 = (x''_1, x''_2) &= = (1-\alpha)v_1 \bigoplus \alpha w_1 = \frac{1}{2}(0, \frac{39}{64}) + \frac{1}{2}(0, \frac{11}{16}) \\
&= (0, \frac{1}{2} + \frac{19}{128}) = (0, \frac{83}{128}),
\end{aligned}$$

where $x''_2 = \frac{1}{2} + \frac{19}{128} < \frac{1}{2} + \frac{1}{4}$ and

$$\begin{aligned}
\mathbb{P}_T(x_2) &= \{(0, \frac{2x''_2+1}{4})\} = \{(0, \frac{2(\frac{83}{128})+1)}{4}\} \\
&= \{(0, \frac{1}{2} + \frac{19}{256})\} = \{(0, \frac{147}{256})\}.
\end{aligned}$$

Now choose $u_2 \in \mathbb{P}_T(x_2) = \{(0, \frac{1}{2} + \frac{19}{256})\}$ That is, $u_2 = (0, \frac{1}{2} + \frac{19}{256}) = (0, \frac{147}{256})$. Then

$$\begin{aligned} z_2 &= (1-\gamma)x_2 \bigoplus \gamma u_2 = \frac{1}{2}(0, \frac{83}{128}) + \frac{1}{2}(0, \frac{147}{256}) \\ &= (0, \frac{1}{2} + \frac{57}{512}) = (0, \frac{313}{512}) = (z_1'', z_2''), \\ \mathbb{P}_T(z_2) &= \{(0, \frac{2z_2'' + 1}{4})\} = \{(0, \frac{2(\frac{313}{512}) + 1)}{4}\} \\ &= \{(0, \frac{1}{2} + \frac{57}{1024})\}. \end{aligned}$$

Choose $w_2 \in \mathbb{P}_T(z_2) = \{(0, \frac{1}{2} + \frac{57}{1024})\}$. That is, $w_2 = (0, \frac{1}{2} + \frac{57}{1024}) = (0, \frac{569}{1024})$. Then

$$\begin{aligned} y_2 &= (1-\beta)u_2 \bigoplus \beta w_2 \\ &= \frac{1}{2}(0, \frac{147}{256}) + \frac{1}{2}(0, \frac{569}{1024}) \\ &= (0, \frac{1}{2} + \frac{133}{2048}) = (0, \frac{1157}{2048}) = (y_1'', y_2''), \\ \mathbb{P}_T(y_2) &= \{(0, \frac{2(y_2'') + 1}{4})\} = \{(0, \frac{2(\frac{1157}{2048}) + 1}{4})\} = \{(0, \frac{2181}{4096})\} \\ &= \{(0, \frac{1}{2} + \frac{133}{4096})\}. \end{aligned}$$

Choose $v_2 \in \mathbb{P}_T(y_2) = \{(0, \frac{1}{2} + \frac{133}{4096})\}$. That is, $v_2 = (0, \frac{1}{2} + \frac{133}{4096}) = (0, \frac{2181}{2048})$. Then

$$\begin{aligned} x_3 = (x_1''', x_2''') &= (1-\alpha)v_2 \bigoplus \alpha w_2 \\ &= \frac{1}{2}(0, \frac{2181}{4096}) + \frac{1}{2}(0, \frac{569}{1024}) = (0, \frac{1}{2} + \frac{361}{8192}) \end{aligned}$$

where $x_2''' = \frac{1}{2} + \frac{361}{8192} < \frac{1}{2} + \frac{1}{6}$.

$$\vdots$$

In a similar method, one can obtain a sequence $\{x_n\}$ which converges strongly to a point $(0, \frac{1}{2}) \in F(T)$ where $F(T) = \{(0, \frac{2b+1}{4}) : 0 \leq b \leq \frac{1}{2}\}$.

We also obtain the following strong convergence theorem in CAT(0) space via the *Condition (I)* which is originally due to Senter and Dotson [29].

Recall that a multivalued nonexpansive mapping $T : K \to CB(K)$ is said to satisfy **Condition (I)** if there exists a continuous nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $f(d(x, F(T)) \leq d(x, Tx)$ for all $x \in K$.

THEOREM 3.6. *Let $\mathbb{E}$ be a CAT(0) space, K a nonempty closed and convex subset of $\mathbb{E}$. And let $T : K \to \mathcal{P}(K)$ be a multivalued mapping satisfying Condition(I) such* *that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is a quasi-nonexpansive multimaping. Then the sequence $\{x_n\}$ defined by* (11) *converges strongly to a fixed point p of T.*

Proof. By Lemma 3.3, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T) = F(\mathbb{P}_T)$. If $\lim_{n\to\infty} d(x_n, p) = 0$, it is obvious. Thus we asuume $\lim_{n\to\infty} d(x_n, p) > 0$. Again from Lemma 3.3, we know $d(x_{n+1}, p) \leq d(x_n, p)$ so that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Hence $\lim_{n\to\infty} d(x_{n+1}, F(T))$ exists. We now prove that $\lim_{n\to\infty} d(x_{n+1}, F(T)) = 0$.

Since T satisfies Condition(I), we have $f(d(x_n, F(T))) \leq d(x_n, Tx_n) \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} d(x_{n+1}, F(T)) = 0$ Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for some $\{p_k\} \subset F(T)$ and all k. Note that in the proof of Lemma 3.3 we obtain

$$d(x_{n_k+1}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_k+1}) + d(x_{n_k+1}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges to $q \in K$. Since

$$\begin{aligned} d(p_k, Tq) &\leq \mathcal{H}(Tp_k, Tq) \\ &\leq d(p_k, q) \end{aligned}$$

and $p_k \to q$ as $k \to \infty$, it follows that $d(q, Tq) = 0$ and thus $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes our proof. □

We now prove a weak convergence theorem via the sequence as defined in (11)

THEOREM 3.7. *Let $\mathbb{E}$ be a CAT(0) space satisfying Opial's condition and K a nonempty closed convex subset of $\mathbb{E}$. Let $T : K \to \mathcal{P}(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $\mathbb{P}_T$ is a quasi-nonexpansive multimaping. Let $\{x_n\}$ be the sequence as defined in* (11). *Let $I - \mathbb{P}_T$ be demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a fixed point p of T.*

Proof. Let $p \in F(T) = F(\mathbb{P}_T)$. By Lemma 3.3, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$.

Let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$, respectively. By (26), there exists $u_n \in Tx_n$ such that $\lim_{n\to\infty} d(x_n, u_n) = 0$. Since $I - \mathbb{P}_T$ is demiclosed with respect to zero, we obtain $z_1 \in F(\mathbb{P}_T) = F(T)$. Similarly, we can prove that $z_2 \in F(T)$. To prove uniqueness, suppose that $z_1 \neq z_2$. Then by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} d(x_n, z_1) &= \lim_{n\to\infty} d(x_{n_i}, z_1)\\ &< \lim_{n\to\infty} d(x_{n_i}, z_2)\\ &= \lim_{n\to\infty} d(x_n, z_2)\\ &= \lim_{n\to\infty} d(x_{n_j}, z_2)\\ &< \lim_{n\to\infty} d(x_{n_j}, z_1)\\ &= \lim_{n\to\infty} d(x_n, z_1).\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(T)$. □

Authors' Contributions

The authors wrote, read and approved the final manuscript.

References

[1] K. Amnuaykarn, P. Kumam, and J. Nantadilok, *On the existence of best proximity points of multi-valued mappings in CAT(0) spaces*, J. Nonlinear Funct. Anal. **2021** Article ID 25 (2021), 1–14.
[2] M. Abbas, T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mathematicki Vesnik **106** (2013), 1–12.
[3] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.
[4] K.S. Brown, *Buildings*, Springer, New York (1989).
[5] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251. (in French)
[6] C.E. Chidume, C.C.O Chidume, N. Djitte, M.S. Minjibir, *Krasnoselskii-type algorithm for fixed points of multi-valued strictly pseudo-contractive mappings*, Fixed Point Theory and Applications **2013** 2013:58, (2013) .
[7] S. Dhompongsa, W.A. Kirk, and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 35–45.
[8] S. Dhompongsa, B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (2008), 2572–2579.
[9] H. Fukhar-ud-din, A.R. Khan, Z. Akhtar, *Fixed point results for a generalized nonexpansive map in uniformly convex metric spaces*, Nonlinear Anal. **75** (2012), 4747–4760.
[10] H. Fukhar-ud-din, M.A. Khamsi, *Approximating common fixed points in hyperbolic spaces*, Fixed Point Theory and Applications **2014**, 2014:113.
[11] L. Gorniewicz, *Topological fixed point theory of multivalued mappings*, Kluwer Academic Pub., Dordrecht, Netherlands, 1999.
[12] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, Inc., New York, 1984.
[13] B.A. Ibn Dehaish, M.A. Khamsi, A.R. Khan, *Mann iteration process for asymptotic pointwise nonexpansive mappings in metric spaces*, J. Math. Anal. Appl. **397** (2013), 861–868.
[14] F.O. Isiogugu, *Demiclosedness principle and approximation theorems for certain classes of multivalued mappings in Hilbert spaces*, Fixed Point Theory and Applications **2013**, 2013:61, (2013).
[15] J.S. Jung, *Strong convergence theorems for multivalued nonexpansive nonself mappings in Banach spaces*, Nonlinear Anal. **66** (2007), 2345–2354.
[16] S.H. Khan, M. Abbas, and S. Ali, *Fixed points of multivalued quasi-nonexpansive mappings using faster iterative process*, Acta Mathematica Sinica, English series **30** (7) (2014), 1231–1241.

[17] W.A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. 68 (2008), 3689–3696.
[18] S.H. Khan, I. Yildirim, *Fixed points of multivalued nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications **2012**, 2012:73, (2012).
[19] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0) spaces*, Journal of Mathematical Analysis and Applications **325** (1) (2007), 386–399.
[20] J.T. Markin, *Continuous dependence of fixed point sets*, Proc. Am. Math. Soc. **38** (1973), 545–547.
[21] S.B. Jr. Nadler, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
[22] J. Nantadilok, C. Khanpanuk, *Best proximity point results for cyclic contractions in CAT(0) spaces*, Comm. Math. Appl. **12** (2) (2021), 1–9.
[23] B. Nanjarus, B. Panyanak, *Demicloesed principle for asymptotically nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory and Applications **210** Article ID 268780, 14 pages, 2010. doi:10.1155/2010/268780
[24] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. **73** (1967), 591–597.
[25] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comp. Math. Appl. **54** (2007), 872–877.
[26] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soci. **43** (1) (1991), 153–159.
[27] K.P.R. Sastry, G.V.R. Babu, *Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point*, Czechoslovak Math. J. **55** (2005), 817–826.
[28] Y. Song, Y.J. Cho, *Some notes on Ishikawa iteration for multivalued mappings*, Bull. Korean. Math. Soc. **48** (3) (2011), 575–584. doi:10.4134/BKMS.2011.48.3.575.
[29] H.F. Senter, W.G. Dotson, *Approximating fixed points of nonexpansive mappings.*, Proc. Am. Math. Soc. **44** (2) (1974), 375–380.
[30] Y. Song, H. Wang, *Erratum to Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comp. Math. Appl. **55** (2008), 2999–3002.
[31] N. Shahzad, H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal. **71** (3-4) (2009), 838–844.
[32] K. Ullah, J. Ahmad, M. Arshad, Z. Ma, and T. Abdeljawad, *An efficient iterative procedure for proximally quasi-nonexpansive Mappings and a class of boundary value problems*, Axioms 2022, 11, 90, 15 pages, 2022.

Pitchaya Kingkam
Department of Mathematics,
Faculty of Science, Lampang Rajabhat University, Lampang, Thailand
E-mail: pitchaya@g.lpru.ac.th

Jamnian Nantadilok
Department of Mathematics,
Faculty of Science, Lampang Rajabhat University, Lampang, Thailand
E-mail: jamnian2010@gmail.com