Bull. Korean Math. Soc. **60** (2023), No. 2, pp. 293–305
https://doi.org/10.4134/BKMS.b210802
pISSN: 1015-8634 / eISSN: 2234-3016

THE RECURRENT HYPERCYCLICITY CRITERION FOR TRANSLATION C_0-SEMIGROUPS ON COMPLEX SECTORS

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Abstract. Let $\{T_t\}_{t\in\Delta}$ be the translation semigroup with a sector $\Delta \subset \mathbb{C}$ as index set. The recurrent hypercyclicity criterion (RHCC) for the C_0-semigroup $\{T_t\}_{t\in\Delta}$ is established, and then the equivalent conditions ensuring $\{T_t\}_{t\in\Delta}$ satisfying the RHCC on weighted spaces of p-integrable and of continuous functions are presented. Especially, every chaotic semigroup $\{T_t\}_{t\in\Delta}$ satisfies the RHCC.

1. Introduction

Let X be a separable infinite dimensional Banach space and $L(X)$ denote the space of linear continuous operators on X. As usual, $\mathbb{R}$ ($\mathbb{R}_0^+$) is the set of all (non-negative) real numbers, $\mathbb{N}$ is the set of all natural numbers and $\mathbb{C}$ is the complex plane. Throughout the paper, our semigroups have an index set, a sector Δ in the complex plane of the form

$$\Delta = \Delta(\alpha) := \{re^{i\theta} : \ r \geq 0, \ |\theta| \leq \alpha\} \subset \mathbb{C}$$

for some $0 < \alpha \leq \pi/2$, or $\Delta = \mathbb{C}$. For $\Delta = \Delta(\alpha)$ with $0 < \alpha \leq \pi/2$, let $\partial\Delta$ denote the boundary of Δ. Given $\tau \in \Delta$ and $r > 0$, we define $\Delta_r := \{t \in \Delta : |t| \leq r\}$ and $\Delta_r^{-1}(\tau) := \{t \in \Delta : \text{there is } s \in \Delta_r \text{ such that } \tau = t + s\}$. Moreover, we set

$$\Delta_r \setminus \Delta_{r'} := \{z \in \Delta_r : z \notin \Delta_{r'}\}$$

for given $0 \leq r' < r$.

We say an operator $T \in L(X)$ is *hypercyclic* if there exists a vector $x \in X$ such that the orbit $Orb(T, x) := \{T^n x : n \in \mathbb{N}\}$ is dense in X, and x is the hypercyclic vector for T. The notion of hypercyclic vectors arises in

Received November 3, 2021; Revised December 29, 2022; Accepted January 26, 2023.
2020 *Mathematics Subject Classification.* Primary 37B20, 47A16, 47D60.
Key words and phrases. Hypercyclicity, translation semigroup, recurrent hypercyclicity criterion.
The first author was supported in part by the National Natural Science Foundation of China Grant #11701422. The second author was supported in part by the National Natural Science Foundation of China Grant #12271396. The third author was supported in part by the National Natural Science Foundation of China Grant #12171353.

the study of invariant subsets. Recall that an operator $T \in L(X)$ admits no non-trivial closed invariant subset if and only if every non-zero vector in X is hypercyclic for T. Hypercyclicity is one of most studied notions in linear dynamics, which has closed relation with topologically transitivity, mixing and chaos (cf. [15–17]). In recent years, the notation recurrence, posed by Poincaré and Birkhoff, attracted lots of attention. More specially, an operator $T \in L(X)$ is called *recurrent* if for every open set $U \subset X$, there exists some $k \in \mathbb{N}$ such that $U \cap T^{-k}(U) \neq \emptyset$. More details on recurrent operators can be found in the recent articles [3, 4, 9, 10, 34]. For motivation, more examples and background about linear dynamics we refer the readers to the excellent books [2, 18], recent articles [14, 20, 32] and their references therein.

Later, different phenomena related with hypercyclicity have been developed into dynamical systems defined by C_0-semigroups of operators, which are also observed from the first order partial differential equations and the mathematical models of cell population dynamics (cf. [21, 23, 24, 31]). Recall that a one-parameter family $\mathcal{T} = \{T_t\}_{t\in\Delta}$ of linear continuous operators in $L(X)$ is a *strongly continuous semigroup* (or C_0*-semigroup*) of operators in $L(X)$ provided that $T_0 = I$, $T_tT_s = T_{t+s}$ for all $t, s \in \Delta$, and $\lim_{t\to s} T_tx = T_sx$ for all $s \in \Delta$, $x \in X$. Within the context, we always mean a semigroup is a C_0-semigroup and any semigroup $\{T_t\}_{t\in\Delta}$ is *locally equicontinuous*, i.e., for any $t_0 > 0$ the family of linear continuous operators $\{T_t : |t| \leq t_0\}$ is *equicontinuous*. Moreover, to avoid degenerated cases, we assume that all the operators in the C_0-semigroup have *dense range*.

Recall that $\mathcal{T} = \{T_t\}_{t\in\Delta}$ of operators in $L(X)$ is said to be *topologically transitive* if for any pair of nonempty open sets U, V, there exists some $t_0 \in \Delta$ such that $T_{t_0}(U) \cap V \neq \emptyset$. The orbit of x under $\mathcal{T}$ is defined as $Orb(\mathcal{T}, x) := \{T_tx : t \in \Delta\}$. If there exists some element with dense orbit, $\mathcal{T}$ is *hypercyclic*, which is equivalent to $\mathcal{T}$ is *topologically transitive* when X is a separable infinite dimensional Banach space (cf. [6]). Furthermore, $\mathcal{T}$ is called *weakly mixing* if the 2-fold product system $\mathcal{T} \oplus \mathcal{T}$ is *topologically transitive* on $X \times X$. Given $x \in X$, if there exists nonzero $t \in \Delta$ such that $T_tx = x$, then x is said to be a *periodic point* for $\mathcal{T}$. A *hypercyclic* semigroup $\mathcal{T}$ with a dense set of periodic points is said to be *chaotic* (in the sense of Devaney).

In the linear function spaces, the first well-known example of a hypercyclic semigroup was the translation semigroup on the space of entire functions (see, e.g. [13, 30]), which is also mixing and chaotic. The translation semigroup on the weighted function spaces $L^p_\rho(I)$ and $C_{0,\rho}(I)$ is characterized to be hypercyclic, chaotic, supercyclic according to the property of the admissible weight function ρ, where $I = \mathbb{R}^+$ or $I = \mathbb{R}$ (see, e.g. [25–27, 33]). The situation for semigroup $\{T_t\}_{t\in\Delta}$ with index set Δ is much more complicated, and their behavior is much richer. As far as we are concerned, the chaos, hypercyclicity and supercyclicity of the translation semigroup on the weighted function spaces $L^p_\rho(\Delta)$ and $C_{0,\rho}(\Delta)$ have been investigated in the recent papers [7, 8, 22]. Very recently, some broader characterizations for a general $\mathcal{F}$-transitive translation semigroup on a complex sector were presented in [19] with a *Furstenberg family* $\mathcal{F}$, which are equivalent to recurrence property in a very weak sense. As an extension, we will offer more technical methods to describe recurrent translation semigroup defined on $X := L^p_\rho(\Delta)$ or $C_{0,\rho}(\Delta)$, where ρ is an admissible weight function on the sector Δ. This work could also be seen as a generalization of the work in [12]. Here we will use more general syndetic subsets (see, e.g. [29]) of the complex sector Δ to consider the recurrence.

For completeness, an admissible weight function ρ on Δ is introduced to define the weighted function spaces $L^p_\rho(\Delta)$, $C_{0,\rho}(\Delta)$, and the translation semigroup.

Definition 1.1 ([8, Definition 4.1])**.** Let Δ be a complex sector. A measurable function $\rho : \Delta \to \mathbb{R}$ is said to be an admissible weight function if it satisfies $\rho(t) > 0$ for every $t \in \Delta$, and there exist constants $M \geq 1$ and $w \in \mathbb{R}$ such that

$$\rho(t_1) \leq Me^{w|t_2|}\rho(t_1 + t_2) \text{ for all } t_1, t_2 \in \Delta.$$

Hereafter, ρ always denotes an admissible weight function on Δ, and M, w are the constants given in Definition 1.1. Especially, we can suppose $w \geq 0$ for convenience. Let $[t, s]$ denote the segment in Δ whose endpoints are t and s. The following lemma is a technical result for admissible weight function ρ.

Lemma 1.1 ([8, Lemma 4.2])**.** *For every $r > 0$, there exist constants $0 < A < B$, depending on ρ and r, such that for every $t \in \Delta$, $t' \in \Delta_r$, $s \in [t, t + t']$, we have that $A\rho(t) \leq \rho(s) \leq B\rho(t + t')$.*

Definition 1.2 ([7, Definition 4.5])**.** For $1 \leq p < \infty$, we define the space

$$L^p_\rho(\Delta) = \{u : \Delta \to \mathbb{C} : u \text{ measurable and } \|u\|_p < \infty\},$$

with $\|u\|_p := (\int_\Delta |u(\tau)|^p \rho(\tau) d\tau)^{1/p}$, and the space

$$C_{0,\rho}(\Delta) = \{u : \Delta \to \mathbb{C} : u \text{ continuous and } \lim_{\tau\to\infty} u(\tau)\rho(\tau) = 0\},$$

with $\|u\|_\infty := \sup_{\tau\in\Delta} |u(\tau)|\rho(\tau)$.

Definition 1.3. Let X be one of the spaces $L^p_\rho(\Delta)$ or $C_{0,\rho}(\Delta)$. For $t \in \Delta$ and $u \in X$ we define $T_t u$ as $T_t u(s) := u(s + t)$ for every $s \in \Delta$. We call $\{T_t\}_{t\in\Delta}$ the translation semigroup on X.

Next we present the definition for the discretization of the C_0-semigroup $\{T_t\}_{t\in\Delta}$ to state the hypercyclicity criterion (HCC) for the semigroup $\{T_t\}_{t\in\Delta}$.

Definition 1.4 ([18, p. 192])**.** A discretization of $\{T_t\}_{t\in\Delta}$ is a sequence of operators $\{T_{t_n}\}_n$ in the semigroup, where $\{t_n\}_n \subset \Delta$ is an arbitrary sequence such that $\lim_{n\to\infty} t_n = \infty$. If there is $t \neq 0$ such that $t_n = nt$ for each $n \in \mathbb{N}$, then $\{T_{t_n}\}_n = \{T^n_t\}_n$ is called an *autonomous discretization* of $\{T_t\}_{t\in\Delta}$.

Hypercyclicity criterion (HCC) for $\{T_t\}_{t\in\Delta}$ (cf. [8, Criterion 3.1])

Let $\{T_t\}_{t\in\Delta}$ be a semigroup in $L(X)$. If there exist a sequence $\{t_n\}_n \subset \Delta$ with $\lim_{n\to\infty} t_n = \infty$, dense subsets $Y, Z \subset X$, and maps $S_{t_n} : Z \to X$, $n \in \mathbb{N}$, such that

(1) $\lim_{n\to\infty} T_{t_n} y = 0$ for every $y \in Y$,
(2) $\lim_{n\to\infty} S_{t_n} z = 0$ for every $z \in Z$,
(3) $\lim_{n\to\infty} T_{t_n} S_{t_n} z = z$ for every $z \in Z$,

then $\{T_t\}_{t\in\Delta}$ is weakly mixing (in particular, hypercyclic). We say that $\{T_t\}_{t\in\Delta}$ satisfies the HCC.

Using the fact $\{T_t\}_{t\in\Delta}$ is weakly mixing, we obtain an equivalent definition.

Definition 1.5. A semigroup $\{T_t\}_{t\in\Delta}$ on X satisfies the HCC if and only if for all nonempty open sets $U, V \subset X$ and all neighborhoods $W \subset X$ of 0 there exists some $t \in \Delta$ such that

$$T_t U \cap W \neq \emptyset \ \text{ and } \ T_t W \cap V \neq \emptyset.$$

(Notice that the same t satisfies both conditions.)

2. Main results

In this section, we aim to characterize the recurrent translation C_0-semigroup $\{T_t\}_{t\in\Delta}$ acting on $L^p_\rho(\Delta)$ or $C_{0,\rho}(\Delta)$. Here we explore the equivalent characterizations for $\{T_t\}_{t\in\Delta}$ satisfying the recurrent hypercyclicity criterion (RHCC), which is a sufficient condition for recurrence. We first prove Definition 1.5 is true when the translation semigroup $\{T_t\}_{t\in\Delta}$ is hypercyclic.

Proposition 2.1 ([8, Theorem 4.8])**.** *Let $\{T_t\}_{t\in\Delta}$ be the translation semigroup.*

(1) *If $\Delta \neq \mathbb{R}_0^+$, $\mathbb{R}$ or $\mathbb{C}$, then $\{T_t\}_{t\in\Delta}$ is hypercyclic if and only if there is a sequence $\{s_k\}_k \subset \Delta$ such that $\lim_{k\to\infty} d(s_k, \partial\Delta) = \infty$ and $\lim_{k\to\infty} \rho(s_k) = 0$.*

(2) *If $\Delta = \mathbb{C}$, then $\{T_t\}_{t\in\Delta}$ is hypercyclic if and only if there exist $\theta \in \mathbb{C}$ and a sequence $\{t_k\}_k \subset \mathbb{C}$ tending to ∞ and verifying*

$$\lim_{k\to\infty} \rho(\theta + t_k) = \lim_{k\to\infty} \rho(\theta - t_k) = 0.$$

Furthermore, the semigroup $\{T_t\}_{t\in\Delta}$ satisfies Definition 1.5 in both cases.

Proof. Let $U, V \subset X$ be nonempty open sets and $W = \{f \in X : \|f\| < \epsilon\}$. Choose two functions $u \in U$, $v \in V$ satisfying $\operatorname{supp} u \subset \Delta_\tau$ and $\operatorname{supp} v \subset \Delta_\tau$ for some $\tau > 0$.

(1) If $\Delta \neq \mathbb{R}_0^+$, $\mathbb{R}$ or $\mathbb{C}$, there exists a sequence $\{s_k\}_k \subset \Delta$ satisfying

$$d(s_k, \partial\Delta) > 3k \ \text{ and } \ \rho(s_k) \leq \frac{1}{kMe^{3kw}}$$

with M, w given in Definition 1.1. Taking $t_k := s_k - 2k \in \Delta$, $k \in \mathbb{N}$, it holds that

$$\lim_{k\to\infty} T_{t_k} u = 0 \text{ for } u \in U. \tag{2.1}$$

Using $v \in V$, define

$$S_k v(t) := \begin{cases} v(t-t_k), & \text{if } t \in t_k + \Delta_\tau, \\ 0, & \text{else.} \end{cases}$$

It yields that $T_{t_k} S_k v = v$ for every $k \in \mathbb{N}$. Hereafter, we only consider our estimates on $X = L^p_\rho(\Delta)$, and for the case $X = C_{0,\rho}(\Delta)$ the same estimates work with $p = 1$ and all integrals replaced by suprema. Lemma 1.1 implies there is $A > 0$ such that

$$\|v\|^p = \int_{\Delta_\tau} |v(t)|^p \rho(t) dt \geq A\rho(0) \int_{\Delta_\tau} |v(t)|^p dt. \tag{2.2}$$

Then for $k > \tau$, it follows $t_k + \Delta_\tau \subset \Delta^{-1}_{3k}(s_k)$ and $\rho(t) \leq Me^{w|s_k - t|}\rho(s_k) \leq 1/k$ for $t \in t_k + \Delta_\tau$. So (2.2) entails that

$$\|S_k v\|^p = \int_{t_k + \Delta_\tau} |v(t-t_k)|^p \rho(t) dt \leq \frac{1}{k} \int_{\Delta_\tau} |v(t)|^p dt \leq \frac{\|v\|^p}{kA\rho(0)}.$$

This implies that

$$\lim_{k\to\infty} S_k v = 0 \quad \text{for} \quad v \in V. \tag{2.3}$$

For the set $W = \{f \in X : \|f\| < \epsilon\}$, equations (2.1) and (2.3) ensure there exits $t_k \in \Delta$ such that $T_{t_k} u \in W$ and $S_k v \in W$. Since $T_{t_k} S_k v = v \in V$, it yields that

$$T_{t_k} U \cap W \neq \emptyset \text{ and } T_{t_k} W \cap V \neq \emptyset. \tag{2.4}$$

This means the semigroup $\{T_t\}_{t\in\Delta}$ of case (1) satisfies Definition 1.5.

(2) If $\Delta = \mathbb{C}$. Suppose that there exist $\theta \in \mathbb{C}$ and $\{t_k\}_k \subset \mathbb{C}$ such that

$$\rho(\theta + t_k) \leq \frac{1}{kMe^{2kw}} \text{ and } \rho(\theta - t_k) \leq \frac{1}{kMe^{2kw}}$$

with $M \geq 1, w > 0$ given in Definition 1.1. Let $\tau > |\theta|$, Definition 1.1 implies

$$\rho(s) \leq Me^{w2\tau}\rho(\theta + t_k) \text{ for } s \in t_k + \Delta_\tau; \tag{2.5}$$

$$\rho(s) \leq Me^{w2\tau}\rho(\theta - t_k) \text{ for } s \in -t_k + \Delta_\tau. \tag{2.6}$$

On the space $L^p_\rho(\Delta)$, by (2.6) we obtain that

$$\|T_{t_k} u\|^p = \int_{-t_k + \Delta_\tau} |u(s+t_k)|^p \rho(s) ds \leq \frac{1}{k} \int_{\Delta_\tau} |u(s)|^p ds$$

for every $k \geq \tau$. Replacing v by u in equation (2.2), we deduce that

$$\|T_{t_k} u\| \leq \left(\frac{1}{A\rho(0)k}\right)^{1/p} \|u\|,$$

implying $\lim_{k\to\infty} T_{t_k} u = 0$ for $u \in U$. Similarly, by (2.5), it yields $\lim_{k\to\infty} T_{-t_k} v = 0$ for $v \in V$. Hence there exists $t_k \in \mathbb{C}$ such that (2.4) holds for the nonempty sets U, V, W, implying $\{T_t\}_{t\in\Delta}$ of case (2) satisfies Definition 1.5. $\square$

There follows the recurrent hypercyclicity criterion for the semigroup $\{T_t\}_{t\in\Delta}$, which can be seen as an extension of [12, Definition 2.1].

Definition 2.1 (Recurrent hypercyclicity criterion (RHCC) for $\{T_t\}_{t\in\Delta}$)**.** A semigroup $\{T_t\}_{t\in\Delta}$ satisfies the RHCC if and only if

(a) $\forall$ nonempty open set $U \subset X$, and $\forall$ $W \subset X$, neighborhood of 0, there exists $L_1 > 0$ such that for any $t \in \Delta$, there exists $s \in t + \Delta_{L_1}$, $s \neq t$, such that $T_s U \cap W \neq \emptyset$;

(b) $\forall$ nonempty open set $V \subset X$, and $\forall$ $W \subset X$, neighborhood of 0, there exists $L_2 > 0$ such that for any $t \in \Delta$, there exists $s \in t + \Delta_{L_2}$, $s \neq t$, such that $T_s W \cap V \neq \emptyset$.

Following [8, Proposition 2.2] and [32, Definition 1.4], an equivalent description for the semigroup $\{T_t\}_{t\in\Delta}$ fulfilling the RHCC is obtained.

Proposition 2.2. *A semigroup $\{T_t\}_{t\in\Delta}$ satisfies the RHCC if and only if for all nonempty open subsets $U, V, W \subset X$ with $0 \in W$, there exists $L > 0$ such that for any $t \in \Delta$, there exists $s \in t + \Delta_L$, $s \neq t$, such that $T_s U \cap W \neq \emptyset$ and $T_s W \cap V \neq \emptyset$.*

Proof. Sufficiency. The semigroup $\{T_t\}_{t\in\Delta}$ satisfies the RHCC with $L_1 = L_2 = L$.

Necessity. Let U, V, W be nonempty open subsets of X with $0 \in W$.

(i) Since $\{T_t\}_{t\in\Delta}$ satisfies the RHCC, there exists $L_1 > 0$ such that for any $t \in \Delta$, there exists $\lambda \in t + \Delta_{L_1}$, $\lambda \neq t$, such that $T_\lambda W \cap V \neq \emptyset$.

(ii) Choose a neighborhood $\widehat{W}$ of 0 such that $T_t\widehat{W} \subset W$ for all $t \in \Delta_{L_1}$.

(iii) At the same time, there exists $L_2 > 0$ such that for any $t \in \Delta$ there are $\gamma \in t + \Delta_{L_2}$, $\gamma \neq t$ and $u \in U$ such that $T_\gamma u \in \widehat{W}$.

Denote $L := L_1 + L_2$. For any $t \in \Delta$, choose γ and u as stated in (iii). For $\gamma \in \Delta$, (i) implies there exists $s \in \gamma + \Delta_{L_1} \subset t + \Delta_L$, $s \neq \gamma$, $s \neq t$ such that $T_s W \cap V \neq \emptyset$. The fact $\{T_t\}_{t\in\Delta}$ is a C_0-semigroup together with (ii)-(iii) yields that $T_s u = T_{s-\gamma} T_\gamma u \in T_{s-\gamma}\widehat{W} \subset W$, this means $T_s U \cap W \neq \emptyset$, ending the proof. □

It is obvious that the RHCC is *a strengthened version* of the HCC. It has been proved that a semigroup $\{T_t\}$ satisfies the HCC if and only if the product semigroup $\{T_t \oplus T_t\}$ on $X \times X$ is hypercyclic. [12, Theorem 5.1] shows that the RHCC is a necessary and sufficient condition on a semigroup $\{T_t\}_{t\in J}$ such that its product with any semigroup $\{S_t\}_{t\in J}$ satisfying the HCC. That is, $\{T_t \oplus S_t\}_{t\in J}$ on $X \times X$ is again hypercyclic, where $J = \mathbb{N}$ (see [5]) or $J = [0, \infty)$ (see [28]). Especially, Desch and Schappacher obtained the equivalent conditions for $\{T_t\}_{t\in I}$ satisfying the RHCC, where $I = [0, \infty)$ and $I = (-\infty, \infty)$. We summarize them as below.

Theorem 2.3 ([12, Theorem 4.6])**.** *Let $I = [0, \infty)$, let ρ be an admissible weight function on I, and let X be one of the spaces $C_{0,\rho}(I, \mathbb{R})$ or $L^p_\rho(I, \mathbb{R})$. Then the following assertions are equivalent*:

(a) *The translation semigroup* $\{T_t\}_{t\in I}$ *on* X *satisfies the RHCC.*

(b) *For each* $\epsilon > 0$ *there exist a constant* $L > 0$ *and an increasing sequence* $t_k \to \infty$ *with* $\rho(t_k) \le \epsilon$ *and* $t_{k+1} - t_k \le L$.

Theorem 2.4 ([12, Theorem 4.7])**.** *Let* $I = (-\infty, \infty)$*, let* ρ *be an admissible weight function on* I*, and let* $\rho_1(t) = \rho(-t)$ *be also admissible. Let* X *be one of the spaces* $C_{0,\rho}(I, \mathbb{R})$ *or* $L^p_\rho(I, \mathbb{R})$*. The translation semigroup* $\{T_t\}_{t\in I}$ *on* X *satisfies the RHCC if and only if for each* $\epsilon > 0$ *there exist a constant* $L > 0$ *and two increasing sequences* $s_k, t_k \to \infty$ *with* $\rho(t_k) \le \epsilon$*,* $\rho(-s_k) \le \epsilon$*,* $t_{k+1} - t_k \le L$ *and* $s_{k+1} - s_k \le L$*.*

Inspired by the above results, we continue to explore the characterizations of $\{T_t\}_{t\in\Delta}$ satisfying the RHCC on the spaces $L^p_\rho(\Delta)$ and $C_{0,\rho}(\Delta)$ with a complex sector $\Delta \subset \mathbb{C}$. The first theorem settles the case $\{T(t)\}_{t\in\Delta}$ with $\Delta \ne \mathbb{R}^+_0,\ \mathbb{R},\ \mathbb{C}$.

Theorem 2.5. *Let* $\Delta \ne \mathbb{R}^+_0,\ \mathbb{R},\ \mathbb{C}$ *and* $\{T_t\}_{t\in\Delta}$ *be the translation semigroup on* $X := L^p_\rho(\Delta)$ *or* $C_{0,\rho}(\Delta)$*, where* ρ *is an admissible weight function on the sector* Δ*. Then the following statements are equivalent*:

(a) *The translation semigroup* $\{T_t\}_{t\in\Delta}$ *on* X *satisfies the RHCC.*

(b) *There exist a constant* $L > 0$ *and a sequence* $\{t_k\}_k \subset \Delta$ *such that* $\lim_{k\to\infty} d(t_k, \partial\Delta) = \infty$ *and* $\lim_{k\to\infty} \rho(t_k) = 0$ *with* $t_{k+1} \in t_k + \Delta_L,\ t_{k+1} \ne t_k$*.*

Proof. (a)$\Rightarrow$(b) Since $\rho(t) > 0$ for all $t \in \Delta$, we define $D_k := \sup\{\rho(s) : s \in k + \Delta_1\} > 0$, $k \in \mathbb{N} \setminus \{0\}$, and choose $\{u_k\}_k \subset X$ such that $\operatorname{supp} u_k \subset k + \Delta_1$ satisfying $\inf_k \|u_k\| := \delta > 0$. Since $\{T_t\}_{t\in\Delta}$ satisfies the RHCC, there exist a constant $L > 1$, a sequence $\{v_k\}_k \subset X$ and an increasing sequence in modulus $\{s_k\}_k \subset \Delta$ such that

$$\|v_k\| < \frac{1}{(kD_k)^{1/p}} \quad \text{and} \quad \|T_{s_k} v_k - u_k\| < \frac{1}{k}$$

with $s_{k+1} \in s_k + \Delta_{L-1},\ s_{k+1} \ne s_k$. Define $t_k := s_k + k \in \Delta,\ k \in \mathbb{N} \setminus \{0\}$, it yields that $\lim_{k\to\infty} d(t_k, \partial\Delta) = \infty$ and $t_{k+1} \in t_k + \Delta_L,\ t_{k+1} \ne t_k$. Take the functions $w_k = v_k \chi_{t_k+\Delta_1}$, where $\chi_{t_k+\Delta_1}$ is the characteristic function of the set $t_k + \Delta_1,\ \ k \in \mathbb{N} \setminus \{0\}$. Thus it holds that

$$\|T_{s_k} w_k - u_k\| < \frac{1}{k}.$$

On the space $X = L^p_\rho(\Delta)$ (the same estimates work with $p = 1$ and all integrals replaced by supremum can imply the case $X = C_{0,\rho}(\Delta)$), Lemma 1.1 entails that

$$\begin{aligned}\|T_{s_k} w_k\|^p = \int_{k+\Delta_1} |w_k(s + s_k)|^p \rho(s) ds &\le D_k \int_{t_k+\Delta_1} |v_k(s)|^p ds \\ &\le \frac{D_k}{A\rho(t_k)} \|v_k\|^p.\end{aligned}$$

Hence we deduce that

$$0 < \delta \le \|u_k\| \le \frac{1}{k} + \left(\frac{1}{Ak\rho(t_k)}\right)^{1/p},$$

implying $\lim_{k\to\infty} \rho(t_k) = 0$ with $\lim_{k\to\infty} d(t_k, \partial\Delta) = \infty$ and $t_{k+1} \in t_k + \Delta_L,\ t_{k+1} \neq t_k$.

(b)⇒(a) Let U, V, W be nonempty open sets with $0 \in W$. Take $u \in U$ and $v \in V$, respectively, with $\operatorname{supp} u \subset \Delta_\tau$ and $\operatorname{supp} v \subset \Delta_\tau$ for some $\tau > 0$. By the condition (b), there exist a constant $L > 2$ and a sequence $\{t_k\}_k \subset \Delta$ verifying

$$\rho(t_k) \le 1/(kMe^{3kw}) \ \text{ and } \ d(t_k, \partial\Delta) > 3k$$

with $t_{k+1} \in t_k + \Delta_{L-2},\ t_{k+1} \neq t_k$ and $M,\ w$ given in Definition 1.1. Choosing $s_k := t_k - 2k \in \Delta,\ k \in \mathbb{N}$, it follows that $\lim_{k\to\infty} T_{s_k} u = 0$ for $u \in U$ with $s_{k+1} \in s_k + \Delta_L$. For $k > \tau$, using Definition 1.1, we have

$$\rho(s) \le Me^{w|t_k - s|}\rho(t_k) \le \frac{1}{k} \ \text{ for } \ s \in s_k + \Delta_\tau. \tag{2.7}$$

Setting

$$w_k(s) = \begin{cases} v(s - s_k), & s \in s_k + \Delta_\tau, \\ 0, & \text{else,} \end{cases}$$

it follows $\operatorname{supp} w_k \subset s_k + \Delta_\tau$. On the space $X = L^p_\rho(\Delta)$, by (2.7) we deduce that

$$\|w_k\|^p = \int_{s_k+\Delta_\tau} |v(s - s_k)|^p \rho(s)ds \le \frac{1}{k}\int_{\Delta_\tau} |v(s)|^p ds. \tag{2.8}$$

At the same time, (2.2) is also true. Combining (2.8) with (2.2) we obtain that

$$\|w_k\|^p < \frac{\|v\|^p}{kA\rho(0)}.$$

Let k be large enough so that $w_k \in W$, while $T_{s_k} w_k = v \in V$. So $T_{s_k} U \cap W \neq \emptyset$ and $T_{s_k} W \cap V \neq \emptyset$ with $s_{k+1} \in s_k + \Delta_L,\ s_{k+1} \neq s_k$. In sum, there exists a constant $L > 0$ such that for any $t \in \Delta$, there exists $s \in t + \Delta_L,\ s \neq t$ such that $T_s U \cap W \neq \emptyset$ and $T_s W \cap V \neq \emptyset$. This means (a) holds, ending the proof. □

Next theorem concerns the case $\{T(t)\}_{t\in\Delta}$ with $\Delta = \mathbb{C}$.

Theorem 2.6. *Let $\Delta = \mathbb{C}$ and $\{T_t\}_{t\in\Delta}$ be the translation semigroup on $X := L^p_\rho(\Delta)$ or $C_{0,\rho}(\Delta)$, where ρ is an admissible weight function on Δ. Then the following statements are equivalent*:

(a) *The translation semigroup $\{T_t\}_{t\in\Delta}$ on X satisfies the RHCC.*

(b) *There exist a constant $L > 0$ and two sequences $\{t_k\}_k,\ \{s_k\}_k \subset \mathbb{C}$ tending to ∞ and verifying $\lim_{k\to\infty} \rho(t_k) = \lim_{k\to\infty} \rho(-s_k) = 0$ with $t_{k+1} \in t_k + \Delta_L$, $t_{k+1} \neq t_k$ and $s_{k+1} \in s_k + \Delta_L,\ s_{k+1} \neq s_k$.*

Proof. (a)⇒(b) Let $\chi : \Delta \to [0, 1]$ be a continuous function with support contained in Δ_1 and denote the set

$$V = \{f \in X : \|\chi f\| > 1\}.$$

Given any $\eta > 0$, let $W = \{f \in X : \|f\| < \eta\}$. Since $\{T_t\}_{t\in\Delta}$ satisfies the RHCC, there exist a constant $L > 0$ and a sequence $\{t_k\}_k \subset \Delta$ as well as $w_k \in W$ such that $T_{t_k} w_k \in V$ with $t_{k+1} \in t_k + \Delta_L$, $t_{k+1} \neq t_k$. Then it yields that

$$\frac{\|\chi \cdot T_{t_k} w_k\|}{\|w_k\|} > \frac{1}{\eta}.$$

In particular, we have that

$$\begin{aligned}\frac{\|\chi \cdot T_{t_k} w_k\|^p}{\|w_k\|^p} &= \frac{\int_\Delta |\chi(s) \cdot T_{t_k} w_k(s)|^p \rho(s) ds}{\int_\Delta |w_k(s)|^p \rho(s) ds} \\ &\leq \sup_{s\in\Delta_1} \left[|\chi(s)|^p \frac{\rho(s)}{\rho(s+t_k)}\right] \frac{\int_\Delta |w_k(s+t_k)|^p \rho(s+t_k) ds}{\int_\Delta |w_k(s)|^p \rho(s) ds} \\ &= \sup_{s\in\Delta_1} |\chi(s)|^p \frac{\rho(s)}{\rho(s+t_k)}.\end{aligned}$$

So it follows that

$$\frac{1}{\eta^p} < \sup_{s\in\Delta_1} |\chi(s)|^p \frac{\rho(s)}{\rho(s+t_k)} \leq 1^p \sup_{s\in\Delta_1} \frac{\rho(s)}{\rho(s+t_k)}.$$

This means

$$\sup_{s\in\Delta_1} \frac{\rho(s)}{\rho(s+t_k)} > \frac{1}{\eta^p}. \tag{2.9}$$

By Definition 1.1, it yields that

$$\rho(t_k) \leq \rho(s+t_k) M e^{w|s|} \leq \rho(s+t_k) M e^{w} \tag{2.10}$$

for any $s \in \Delta_1$. By (2.9) and (2.10), we get that

$$\frac{Me^w}{\rho(t_k)} \sup_{s\in\Delta_1} \rho(s) > \frac{1}{\eta^p},$$

which can be changed into

$$\rho(t_k) < \eta^p M e^w \sup_{s\in\Delta_1} \rho(s).$$

For any $\epsilon > 0$, we choose η sufficiently small such that

$$\eta^p M e^w \sup_{s\in\Delta_1} \rho(s) \leq \epsilon,$$

then $\rho(t_k) < \epsilon$ with $t_{k+1} \in t_k + \Delta_L$, $t_{k+1} \neq t_k$. Analogously, there exists $\{s_k\}_k \subset \mathbb{C}$ tending to ∞ such that $\rho(-s_k) < \epsilon$ with $s_{k+1} \in s_k + \Delta_L$, $s_{k+1} \neq s_k$. That is, (b) holds.

(b)⇒(a) Let U, V, W be nonempty open sets with $0 \in W$. Let $u \in U$ and $v \in V$, respectively, satisfying $\operatorname{supp} u \subset \Delta_\tau$ and $\operatorname{supp} v \subset \Delta_\tau$ for some $\tau > 0$. By (b), there exist $L > 0$ and two sequences $\{t_k\}_k$, $\{s_k\}_k \subset \mathbb{C}$ tending to ∞

and satisfying $t_{k+1} \in t_k + \Delta_L,\ t_{k+1} \neq t_k$ and $s_{k+1} \in s_k + \Delta_L,\ s_{k+1} \neq s_k$, such that

$$\rho(t_k) < \frac{1}{kMe^{2w\tau}} \quad \text{and} \quad \rho(-s_k) < \frac{1}{kMe^{2w\tau}},$$

with $M \geq 1, w > 0$ given in Definition 1.1. And then it is true that

$$\rho(s) \leq Me^{w\tau}\rho(-s_k) \ \text{ for } s \in -s_k + \Delta_\tau.$$

We deduce that

$$\begin{aligned} \|T_{s_k}u\|^p &= \int_{-s_k+\Delta_\tau} |u(s+s_k)|^p \rho(s) ds \\ &\leq \int_{-s_k+\Delta_\tau} |u(s+s_k)|^p ds Me^{w\tau}\rho(-s_k) \\ &< \frac{1}{k}\int_{\Delta_\tau} |u(s)|^p ds. \end{aligned} \tag{2.11}$$

On the other side, Lemma 1.1 entails that

$$\|u\|^p \geq A\rho(0)\int_{\Delta_\tau} |u(s)|^p ds. \tag{2.12}$$

Combining (2.11) with (2.12) we conclude that

$$\|T_{s_k}u\| \leq \left(\frac{1}{A\rho(0)k}\right)^{\frac{1}{p}} \|u\|.$$

So $T_{s_k}U \cap W \neq \emptyset$ holds for sufficiently large k. Similarly, we can deduce $T_{-t_k}v \in W$ for sufficiently large k. Since $T_{t_k}T_{-t_k}v = v \in V$, it yields that $T_{t_k}W \cap V \neq \emptyset$ for sufficiently large k. Since $t_{k+1} \in t_k + \Delta_L,\ t_{k+1} \neq t_k$ and $s_{k+1} \in s_k + \Delta_L,\ s_{k+1} \neq s_k$, it follows that for any $t \in \Delta$, there exist $t_1, t_2 \in t + \Delta_L,\ t_i \neq t$ with $i = 1, 2$, such that $T_{t_1}U \cap W \neq \emptyset$ and $T_{t_2}W \cap V \neq \emptyset$. Therefore the semigroup $\{T_t\}_{t\in\mathbb{C}}$ satisfies the RHCC, ending the proof. □

There are several types of chaotic behaviours for discrete or continuous dynamical systems (see, e.g. [1, 11]). We continue to present the relationship between *chaotic* (in the sense of Devaney) semigroup and the semigroup satisfying the RHCC on the sector Δ analogous to [12, Section 6]. Since the proof is a minor modification of [12, Lemma 6.1], we omit the details.

Lemma 2.7. *Let $\{T_t\}_{t\in\Delta}$ be a semigroup on a Banach space X. The following assertions are equivalent*:

(a) $\{T_t\}_{t\in\Delta}$ *satisfies the RHCC.*

(b) (b1) $\forall$ *nonempty open set $U \subset X$, and $\forall$ $W \subset X$, neighborhood of* 0, *there exists $t \in \Delta$ such that $T_tU \cap W \neq \emptyset$*;

(b2) $\forall$ *nonempty open set $V \subset X$ and $\forall$ $W \subset X$, neighborhood of* 0, *there exists $t \in \Delta$ such that $T_tW \cap V \neq \emptyset$*;

(b3) $\forall$ *nonempty open set $U \subset X$, there exists $L > 0$ such that for $\forall$ $t \in \Delta$, $\exists$ $s \in t + \Delta_L,\ s \neq t$, such that $T_sU \cap U \neq \emptyset$.*

Lemma 2.7 immediately implies the corollary below.

Corollary 2.8. *Any chaotic semigroup* $\{T_t\}_{t\in\Delta}$ *satisfies the RHCC.*

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