Bull. Korean Math. Soc. **60** (2023), No. 2, pp. 325–338
https://doi.org/10.4134/BKMS.b220120
pISSN: 1015-8634 / eISSN: 2234-3016

SOME INTEGRAL INEQUALITIES FOR THE LAPLACIAN WITH DENSITY ON WEIGHTED MANIFOLDS WITH BOUNDARY

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Abstract. In this paper, we derive a Reilly-type inequality for the Laplacian with density on weighted manifolds with boundary. As its applications, we obtain some new Poincaré-type inequalities not only on weighted manifolds, but more interestingly, also on their boundary. Furthermore, some mean-curvature type inequalities on the boundary are also given.

1. Introduction

Let $(M^n, g, d\mu)$ be an n-dimensional compact weighted manifold with boundary ∂M. A weighted Riemannian manifold is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given compact n-dimensional Riemannian manifold (M^n, g) with the metric g, the triple $(M^n, g, d\mu)$ is called a compact weighted Riemannian manifold, where $d\mu = e^{-f}dv$ is a weighted volume form, and f is a smooth real-valued function on M, and dv is the Riemannian volume element related to g.

Let $\mathbf{n}$ be the unit outward normal of ∂M. Define the second fundamental form of ∂M by $\Pi(X,Y) = \langle \nabla_X \mathbf{n}, Y\rangle$ for any two tangent vector fields X and Y on M, and the mean curvature by $H = tr(\Pi)$. The f-mean curvature (see [16, p. 398]) at a point $x \in M$ with respect to $\mathbf{n}$ is given by $H_f(x) = H(x) - \langle \nabla f(x), \mathbf{n}(x)\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the Riemannian metric g.

Following [13], on $(M^n, g, d\mu)$, we consider the Laplacian with density as follows:

$$\mathcal{L}\cdot := \sigma^{-1} div(\sigma^{\alpha}\nabla\cdot) = e^{-f(\alpha-1)}(\Delta \cdot - \alpha\langle \nabla f, \nabla\cdot\rangle), \tag{1.1}$$

Received February 14, 2022; Revised September 7, 2022; Accepted October 28, 2022.
2010 *Mathematics Subject Classification.* 53C21, 58J32.
Key words and phrases. Reilly-type inequality, generalized Ricci curvature, generalized mean curvature.
This work was supported by NSFC (No.11971415), Natural Science Foundation of Henan (No.212300410235), and the Key Scientific Research Program in Universities of Henan Province (Nos.21A110021, 22A110021) and Nanhu Scholars Program for Young Scholars of XYNU (No.2019).

where $\alpha > 0$ is a given real constant, σ is the positive function defined by $\sigma := e^{-f}$, ∇ denotes the Levi-Civita connection, $div = tr(\nabla\cdot)$ denotes the Riemannian divergence operator, and $\Delta = div\nabla$ is the Laplace-Beltrami operator.

Notice that the Green formula (the integration by parts formula)

$$\begin{aligned}\int_M h\mathcal{L}u\,d\mu &= \int_{\partial M} e^{-f(\alpha-1)} h\partial_{\mathbf{n}}u\,d\mu_\partial - \int_M e^{-f(\alpha-1)}\langle \nabla u, \nabla h\rangle\,d\mu \\ &= \int_{\partial M} e^{-f(\alpha-1)}(h\partial_{\mathbf{n}}u - u\partial_{\mathbf{n}}h)\,d\mu_\partial + \int_M u\mathcal{L}h\,d\mu\end{aligned}$$

holds provided u or h belongs to $C^2(M)$, where $\partial_{\mathbf{n}}u = \langle \mathbf{n}, \nabla u\rangle$, and $d\mu_\partial = e^{-f}dv_\partial$ and dv_∂ is the volume form on ∂M.

Following [1], to relate $\mathcal{L}$ with geometry we consider the (α, N)-Bakry-Émery curvature $Ric_f^{\alpha,N}$ given by

$$Ric_f^{\alpha,N} = \mathrm{Ric} + \alpha\nabla^2 f - \frac{\alpha^2}{N-n}\nabla f\otimes\nabla f, \tag{1.2}$$

where $\alpha > 0$ is a given real constant and N is a constant. We note that the only case in which $N = n$ is permitted is when f is a constant. Here ∇^2 and Ric denote the Hessian operator and Ricci curvature, respectively. When $N = \infty$, (1.2) gives the tensor

$$Ric_f^{\alpha,\infty} = \mathrm{Ric} + \alpha\nabla^2 f, \tag{1.3}$$

which is called (α,∞)-Bakry-Émery curvature. The (α, f)-mean curvature of ∂M is defined by

$$H_f^\alpha(x) = H(x) - \alpha\langle\nabla f(x), \mathbf{n}(x)\rangle. \tag{1.4}$$

In the case where $\alpha = 1$, the operator $\mathcal{L}$ becomes the Witten Laplacian

$$\Delta_f\cdot = \Delta\cdot - \langle\nabla f, \nabla\cdot\rangle.$$

Meanwhile, the (α, N)-Bakry-Émery curvature $Ric_f^{\alpha,N}$, (α,∞)-Bakry-Émery curvature $Ric_f^{\alpha,\infty}$ and (α, f)-mean curvature H_f^α become the N-Bakry-Émery Ricci curvature

$$Ric_f^N = Ric_f - \frac{1}{N-n}\nabla f\otimes\nabla f,$$

∞-Bakry-Émery Ricci curvature

$$Ric_f = Ric + \nabla^2 f$$

and f-mean curvature, respectively. In recent years, the Witten Laplacian received much attention from many mathematicians (see [2, 4–7, 10–12, 17, 18] and the references therein).

Among the important formulae in differential geometry, the Reilly formula [14] is an important tool in the study of various geometric and analytical problems on a Riemannian manifold with smooth boundary. Ma and Du [12] extended the Reilly formula for the Witten Laplacian and applied it to study eigenvalue estimates for the Witten Laplacian on compact Riemannian manifolds with boundary. Kolesnikov and Milman [8,9] obtained new Poincaré-type inequalities for weighted manifolds by systematically using Ma-Du's Reilly-type formula combined with various conditions on the boundary of the manifold and boundary conditions for elliptic equations. Further more recent applications may be found in [3,19].

The purpose of this paper is to study some integral inequalities for the operator $\mathcal{L}$ and their applications on weighted manifolds with boundary. Firstly, we derive a Reilly-type inequality for the operator $\mathcal{L}$ on weighted manifolds with boundary, which is an important tool to prove our main theorems.

Theorem 1.1. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $N \in (-\infty, 0) \cup [n, +\infty)$. Then for any $u \in C^2(M)$:*

$$0 \geq \int_M e^{-f(\alpha-1)} Ric_f^{\alpha,N}(\nabla u, \nabla u) - \frac{N-1}{N} e^{f(\alpha-1)} |\mathcal{L}u|^2 \, d\mu + \int_{\partial M} e^{-f(\alpha-1)} (e^{f(\alpha-1)} \mathcal{L}_\partial u + H_f^\alpha \partial_{\mathbf{n}} u) \partial_{\mathbf{n}} u \, d\mu_\partial + \int_{\partial M} e^{-f(\alpha-1)} [\Pi(\nabla_\partial u, \nabla_\partial u) - g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)] \, d\mu_\partial, \tag{1.5}$$

where $\mathcal{L}_\partial$ denotes the operator $\mathcal{L}_\partial \cdot = e^{-f(\alpha-1)}(\Delta_\partial \cdot - \alpha \langle \nabla_\partial f, \nabla_\partial \cdot \rangle)$ on the boundary. Here, Δ_∂, ∇_∂ and $d\mu_\partial$, respectively, denote the Laplacian, gradient operators and weighted volume measure on ∂M.

Remark 1.2. In [11] (or see [8,9]), Li and Wei provide a Reilly-type inequality for the Witten Laplacian and give some applications. In particular, if $\alpha = 1$, then our (1.5) becomes the formula (9) of Li and Wei in [11].

Throughout this work we employ Einstein summation convention. By abuse of notation, $Ric_f^{\alpha,N}$ may denote the 2-covariant tensor $(Ric_f^{\alpha,N})_{pq}$, but also may denote its 1-contravariant version $(Ric_f^{\alpha,N})_p^q$, as in:

$$\begin{aligned} \langle Ric_f^{\alpha,N} \nabla u, \nabla u \rangle &= g_{ij} (Ric_f^{\alpha,N})_k^i \nabla^k u \nabla^j u \\ &= (Ric_f^{\alpha,N})_{ij} \nabla^i u \nabla^j u \\ &= Ric_f^{\alpha,N}(\nabla u, \nabla u). \end{aligned}$$

Similarly, the 2-contravariant tensor $(\Pi^{-1})^{\alpha\beta}$ and $((Ric_f^{\alpha,N})^{-1})^{pq}$ are defined by:

$$(\Pi^{-1})^{ij} \Pi_{jk} = \delta_k^i, \quad ((Ric_f^{\alpha,N})^{-1})^{ij} (Ric_f^{\alpha,N})_{jk} = \delta_k^i.$$

Given an integrable function φ on $(M^n, g, d\mu)$, the dimensional mean-value and dimensional variance of φ on $(M^n, g, d\mu)$ are defined by

$$\overline{\varphi} = \frac{\int_M e^{-f(\alpha-1)} \varphi \, d\mu}{\int_M e^{-f(\alpha-1)} \, d\mu}, \quad Var_f(\varphi) = \int_M e^{-f(\alpha-1)} (\varphi - \overline{\varphi})^2 \, d\mu.$$

Next, by applying the above Reilly-type inequality (1.5), we obtain some new Poincaré-type inequalities for the operator $\mathcal{L}$ on weighted Riemannian manifolds with boundary.

Theorem 1.3. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} > 0$, where $N \in (-\infty, 0) \cup [n, +\infty)$. Then for any $\varphi \in C^1(M)$:*

(1) *Assume that $\Pi \geq 0$ (M is locally convex). Then*

$$\frac{N}{N-1} Var_f(\varphi) \leq \int_M e^{-f(\alpha-1)} (Ric_f^{\alpha,N})^{-1}(\nabla\varphi, \nabla\varphi)\, d\mu.$$

(2) *Assume that $H_f^\alpha \geq 0$ (M is generalized mean-convex), $\varphi \equiv 0$ on ∂M. Then*

$$\frac{N}{N-1} \int_M e^{-f(\alpha-1)} \varphi^2\, d\mu \leq \int_M e^{-f(\alpha-1)} (Ric_f^{\alpha,N})^{-1}(\nabla\varphi, \nabla\varphi)\, d\mu.$$

(3) *Assume that $H_f^\alpha > 0$ (M is strictly generalized mean-convex). Then*

$$\frac{N}{N-1} \int_M e^{-f(\alpha-1)} \varphi^2\, d\mu \leq \int_M e^{-f(\alpha-1)} (Ric_f^{\alpha,N})^{-1}(\nabla\varphi, \nabla\varphi)\, d\mu + \int_{\partial M} e^{-f(\alpha-1)} \frac{\varphi^2}{H_f^\alpha}\, d\mu_\partial.$$

Remark 1.4. Particularly, when $\alpha = 1$, then Theorem 1.3 reduces to Theorem 1.2 of Kolesnikov and Milman in [8].

By using the above Reilly-type inequality (1.5), we can obtain some Poincaré-type inequalities on the boundary of weighted Riemannian manifolds $(M^n, g, d\mu)$.

Theorem 1.5. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} \geq \rho g$, where $\rho \in \mathbb{R}$ and $N \in (-\infty, 0) \cup [n, +\infty)$. Assume that $H_f^\alpha > 0$ on ∂M. Then for any $\psi \in C^2(\partial M)$*

$$\int_{\partial M} e^{-f(\alpha-1)} \Pi(\nabla_\partial\psi, \nabla_\partial\psi)\, d\mu_\partial \leq \int_{\partial M} \frac{e^{-f(\alpha-1)}}{H_f^\alpha} \left(\frac{\rho}{2}\psi + e^{f(\alpha-1)}\mathcal{L}_\partial\psi\right)^2 d\mu_\partial. \tag{1.6}$$

We also obtain a dual-version of Theorem 1.5:

Theorem 1.6. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} \geq 0$, where $N \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M. Then for any $\psi \in C^1(\partial M)$*

$$\int_{\partial M} e^{-f(\alpha-1)} \Pi^{-1}(\nabla_\partial\psi, \nabla_\partial\psi)\, d\mu_\partial \geq \int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha \psi^2\, d\mu_\partial$$

$$- \frac{N-1}{N} \frac{(\int_{\partial M} e^{-f(\alpha-1)}\psi \, d\mu_\partial)^2}{V(M)},$$

where $V(M) = \int_M e^{-f(\alpha-1)} \, d\mu$.

Remark 1.7. Particularly, when $\alpha = 1$, then Theorem 1.5 and Theorem 1.6 reduce to Theorem 1.1 and Theorem 1.2 of Kolesnikov and Milman in [9], respectively.

On the other hand, by applying Theorem 1.5, we also achieve the following geometric inequalities involving (α, f)-mean curvature.

Theorem 1.8. *Let* $(M^n, g, d\mu)$ *be a smooth compact weighted Riemannian manifold of dimension* $n \geq 2$ *with boundary and* $Ric_f^{\alpha,N} \geq \rho g$, *where* $\rho \in \mathbb{R}$ *and* $N \in (-\infty, 0) \cup [n, +\infty)$. *Assume that* $H_f^\alpha > 0$ *on* ∂M. *Then for any nontrivial* ψ, *one of the following conclusions holds*:

$$B^2(\psi) \leq A(\psi)C(\psi), \tag{1.7}$$

$$\frac{\rho}{2} \leq -\frac{B(\psi)}{A(\psi)} - \sqrt{\left(\frac{B(\psi)}{A(\psi)}\right)^2 - \frac{C(\psi)}{A(\psi)}}, \tag{1.8}$$

$$\frac{\rho}{2} \geq -\frac{B(\psi)}{A(\psi)} + \sqrt{\left(\frac{B(\psi)}{A(\psi)}\right)^2 - \frac{C(\psi)}{A(\psi)}},$$

where

$$A(\psi) = \int_{\partial M} \frac{\psi^2}{H_f^\alpha} e^{-f(\alpha-1)} \, d\mu_\partial, \quad B(\psi) = \int_{\partial M} \frac{\psi \mathcal{L}_\partial \psi}{H_f^\alpha} \, d\mu_\partial,$$

$$C(\psi) = \int_{\partial M} \left[\frac{(e^{f(\alpha-1)}\mathcal{L}_\partial \psi)^2}{H_f^\alpha} - \Pi(\nabla_\partial u, \nabla_\partial u) \right] e^{-f(\alpha-1)} \, d\mu_\partial.$$

Remark 1.9. From Theorem 1.8, we can deduce Theorem 1.1 of Tu and Huang in [15].

In the end, by applying Theorem 1.6, the following mean-curvature type inequalities can be proven.

Theorem 1.10. *Let* $(M^n, g, d\mu)$ *be a smooth compact weighted Riemannian manifold of dimension* $n \geq 2$ *with boundary and* $Ric_f^{\alpha,N} \geq 0$, *where* $N \in (-\infty, 0) \cup [n, +\infty)$. *Assume that* $H_f^\alpha > 0$ *on* ∂M. *Then*

$$\int_{\partial M} e^{-f(\alpha-1)} \frac{1}{H_f^\alpha} \, d\mu_\partial \geq \frac{N}{N-1} V(M). \tag{1.9}$$

Remark 1.11. Particularly, when $\alpha = 1$, then Theorem 1.10 reduces to some previous results in [9, Theorem 4.4] and [7, Theorem 1.1].

This paper is organized as follows. In Section 2 we prove Theorem 1.1, and Theorem 1.3 is proved in Section 3. Theorem 1.5 and Theorem 1.6 are proved in Section 4. In Section 5 we make some global estimates on the generalized mean curvature H_f^α on the boundary of weighted Riemannian manifolds.

2. Proof of Theorem 1.1

In this section, we give the proof of our main tool, Theorem 1.1 from the Introduction.

Proof of Theorem 1.1. A simple calculation gives the following Bochner-type formula (see [13, Lemma 1]) for any function $u \in C^3(M)$:

$$\begin{aligned}\frac{1}{2}\mathcal{L}|\nabla u|^2 &= e^{-f(\alpha-1)}\big(|\nabla^2 u|^2 + (\mathrm{Ric} + \alpha\nabla^2 f)(\nabla u, \nabla u)\big)\\ &\quad + g(\nabla u, \nabla \mathcal{L}u) + (\alpha-1)g(\nabla u, \nabla f)\mathcal{L}u.\end{aligned} \tag{2.1}$$

Using the Bochner-type formula (2.1) and integration by parts, we obtained the following Reilly-type formula (see [13, Theorem 2]):

$$\begin{aligned}&\int_M \Big(e^{f(\alpha-1)}|\mathcal{L}u|^2 - e^{-f(\alpha-1)}|\nabla^2 u|^2\Big)\,d\mu\\ &= \int_M e^{-f(\alpha-1)}(\mathrm{Ric} + \alpha\nabla^2 f)(\nabla u, \nabla u))d\mu\\ &\quad + \int_{\partial M} e^{-f(\alpha-1)} g(\partial_{\mathbf{n}} u, H\partial_{\mathbf{n}} u - \alpha g(\nabla u, \nabla f) + \Delta_\partial u)d\mu_\partial\\ &\quad + \int_{\partial M} e^{-f(\alpha-1)}[\Pi(\nabla_\partial u, \nabla_\partial u) - g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)]d\mu_\partial.\end{aligned} \tag{2.2}$$

Now, we can consider the Bochner-type formula (2.1) for (α, N)-Bakry-Émery curvature. Note that

$$\begin{aligned}&|\nabla^2 u|^2 + Ric_f^{\alpha,\infty}(\nabla u, \nabla u)\\ &= \left|\nabla^2 u - \frac{\Delta u}{n} g\right|^2 + \frac{1}{N}(e^{f(\alpha-1)}\mathcal{L}u)^2 + Ric_f^{\alpha,N}(\nabla u, \nabla u)\\ &\quad + \left(\sqrt{\frac{N-n}{Nn}}\Delta u + \sqrt{\frac{n}{N(N-n)}}\alpha(\nabla f\cdot\nabla u)\right)^2\\ &\geq \frac{1}{N}(e^{f(\alpha-1)}\mathcal{L}u)^2 + Ric_f^{\alpha,N}(\nabla u, \nabla u)\end{aligned} \tag{2.3}$$

provided $N \in (-\infty, 0) \cup [n, +\infty)$. Substituting this into (2.1) and (2.2), we get

$$\begin{aligned}&\frac{1}{2}\mathcal{L}|\nabla u|^2\\ &\geq e^{-f(\alpha-1)}\big(\frac{1}{N}(e^{f(\alpha-1)}\mathcal{L}u)^2 + Ric_f^{\alpha,N}(\nabla u, \nabla u)\big)\\ &\quad + g(\nabla u, \nabla\mathcal{L}u) + (\alpha-1)g(\nabla u, \nabla f)\mathcal{L}u\end{aligned} \tag{2.4}$$

$$= \frac{1}{N} e^{f(\alpha-1)} |\mathcal{L}u|^2 + e^{-f(\alpha-1)} Ric_f^{\alpha,N}(\nabla u, \nabla u) + g(\nabla u, \nabla \mathcal{L}u) + (\alpha-1) g(\nabla u, \nabla f)\mathcal{L}u$$

and

$$\begin{aligned} 0 \geq & \int_M e^{-f(\alpha-1)} Ric_f^{\alpha,N}(\nabla u, \nabla u) - \frac{N-1}{N} e^{f(\alpha-1)} |\mathcal{L}u|^2 \, d\mu \\ & + \int_{\partial M} e^{-f(\alpha-1)} (e^{f(\alpha-1)} \mathcal{L}_\partial u + H_f^\alpha \partial_{\mathbf{n}} u) \partial_{\mathbf{n}} u \, d\mu_\partial \\ & + \int_{\partial M} e^{-f(\alpha-1)} [\Pi(\nabla_\partial u, \nabla_\partial u) - g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)] \, d\mu_\partial . \end{aligned}$$

This completes the proof. □

Note that the Bochner-type formula (2.4) looks very similar to the Bochner formula for the Ricci tensor of an n-dimensional manifold. This is our motivation for the definition of the (α, N)-Bakry-Émery curvature $Ric_f^{\alpha,N}$.

3. Proof of Theorem 1.3

The idea in the proof of Theorem 1.3 is similar to the one used by Kolesnikov and Milman in [8]. We use the Reilly-type inequality to prove Theorem 1.3 below.

Proof of Theorem 1.3. (1) Since M^n is compact, by integration by parts, we have

$$\int_{\partial M} \partial_{\mathbf{n}} u \mathcal{L}_\partial u \, d\mu_\partial = -\int_{\partial M} e^{-f(\alpha-1)} g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u) \, d\mu_\partial . \tag{3.1}$$

By (1.5) and (3.1), we can get

$$\begin{aligned} 0 \geq & \int_M \left(e^{-f(\alpha-1)} Ric_f^{\alpha,N}(\nabla u, \nabla u) - \frac{N-1}{N} e^{f(\alpha-1)} |\mathcal{L}u|^2 \right) d\mu \\ & + \int_{\partial M} e^{-f(\alpha-1)} [\Pi(\nabla_\partial u, \nabla_\partial u) - 2g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)] \, d\mu_\partial \\ & + \int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial . \end{aligned} \tag{3.2}$$

Let u be a smooth solution to the Neumann problem

$$\begin{cases} e^{f(\alpha-1)} \mathcal{L}u = \varphi & \text{on } M, \\ \partial_{\mathbf{n}} u \equiv 0 & \text{on } \partial M. \end{cases} \tag{3.3}$$

Then

$$\begin{aligned} \int_M e^{-f(\alpha-1)} \varphi^2 \, d\mu = & \int_M e^{f(\alpha-1)} (\mathcal{L}u)^2 \, d\mu \\ = & -\int_M e^{-f(\alpha-1)} \langle \nabla\varphi, \nabla u \rangle \, d\mu + \int_{\partial M} e^{-f(\alpha-1)} \varphi \partial_{\mathbf{n}} u \, d\mu_\partial . \end{aligned} \tag{3.4}$$

Consequently, by the Cauchy-Schwarz inequality

$$\begin{aligned}(3.5)\quad &\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu\\ &\le \left(\int_M e^{-f(\alpha-1)}\langle Ric_f^{\alpha,N}\nabla u,\nabla u\rangle\,d\mu\right)^{\frac12}\left(\int_M e^{-f(\alpha-1)}\langle (Ric_f^{\alpha,N})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu\right)^{\frac12}\\ &\quad+\int_{\partial M} e^{-f(\alpha-1)}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial.\end{aligned}$$

Since $\partial_{\mathbf{n}}u\big|_{\partial M}\equiv 0$ and $\Pi\ge 0$, we obtain from (3.2)

$$(3.6)\qquad \int_M e^{f(\alpha-1)}|\mathcal{L}u|^2\,d\mu \ge \frac{N}{N-1}\int_M e^{-f(\alpha-1)}\langle Ric_f^{\alpha,N}\nabla u,\nabla u\rangle\,d\mu.$$

Consequently, we obtain

$$(3.7)\qquad \frac{N-1}{N}\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu \ge \int_M e^{-f(\alpha-1)}\langle Ric_f^{\alpha,N}\nabla u,\nabla u\rangle\,d\mu.$$

Plugging this back into (3.5) and using that $\partial_{\mathbf{n}}u\big|_{\partial M}\equiv 0$ yields

$$\frac{N}{N-1}\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu \le \int_M e^{-f(\alpha-1)}\langle (Ric_f^{\alpha,N})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu.$$

By the fact that $\int_M e^{-f(\alpha-1)}\varphi\,d\mu = \int_M \mathcal{L}u\,d\mu = \int_{\partial M} e^{-f(\alpha-1)}\partial_{\mathbf{n}}u\,d\mu_\partial$, we obtain the assertion of Case (1).

(2) Let φ be a smooth solution to the Dirichlet problem

$$(3.8)\qquad \begin{cases} e^{f(\alpha-1)}\mathcal{L}u=\varphi & \text{on } M,\\ u\equiv 0 & \text{on } \partial M.\end{cases}$$

Observe that (3.7) still holds since $u\equiv 0$ and $H_f^\alpha \ge 0$. Plugging (3.7) back into (3.5) and using that $\varphi\big|_{\partial M}\equiv 0$ yields the assertion of Case (2).

(3) Let φ be a smooth solution to the Dirichlet problem (3.8). If $H_f^\alpha>0$, by (3.2), we have

$$\begin{aligned}(3.9)\qquad \frac{N-1}{N}\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu &\ge \int_M e^{-f(\alpha-1)}Ric_f^{\alpha,N}(\nabla u,\nabla u)\,d\mu\\ &\quad+\int_{\partial M} e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2\,d\mu_\partial.\end{aligned}$$

On the other hand, we obtain for any $\varepsilon>0$:

$$\begin{aligned}(3.10)\qquad &\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu\\ &= -\int_M e^{-f(\alpha-1)}\langle\nabla\varphi,\nabla u\rangle\,d\mu+\int_{\partial M} e^{-f(\alpha-1)}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial\\ &\le \frac{\varepsilon}{2}\int_M e^{-f(\alpha-1)}\langle Ric_f^{\alpha,N}\nabla u,\nabla u\rangle\,d\mu\\ &\quad+\frac{1}{2\varepsilon}\int_M e^{-f(\alpha-1)}\langle (Ric_f^{\alpha,N})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu\end{aligned}$$

$$+\int_{\partial M} e^{-f(\alpha-1)}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial.$$

By (3.9) and (3.10), we can get

$$\begin{aligned}
&\left(1-\frac{\varepsilon}{2}\frac{N-1}{N}\right)\int_M e^{-f(\alpha-1)}\varphi^2\,d\mu\\
&\le \frac{1}{2\varepsilon}\int_M e^{-f(\alpha-1)}\langle (Ric_f^{\alpha,N})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu-\frac{\varepsilon}{2}\int_{\partial M} e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2\,d\mu_\partial\\
&\quad+\int_{\partial M} e^{-f(\alpha-1)}\varphi\partial_{\mathbf{n}}u\,d\mu_\partial\\
&\le \frac{1}{2\varepsilon}\int_M e^{-f(\alpha-1)}\langle (Ric_f^{\alpha,N})^{-1}\nabla\varphi,\nabla\varphi\rangle\,d\mu+\frac{1}{2\varepsilon}\int_{\partial M} e^{-f(\alpha-1)}\frac{\varphi^2}{H_f^\alpha}\,d\mu_\partial.
\end{aligned}$$

Multiplying by 2ε and using the optimal $\varepsilon=\frac{N}{N-1}$, we obtain the assertion of Case (3). This completes the proof. □

4. Proof of Theorem 1.5 and Theorem 1.6

We use the Reilly-type inequality to prove Theorem 1.5 below.

Proof of Theorem 1.5. Let u be a smooth solution to the Dirichlet problem

$$\begin{cases}\mathcal{L}u=0 & \text{on } M,\\ u\equiv\psi & \text{on } \partial M.\end{cases}\tag{4.1}$$

By (1.5), we have

$$\begin{aligned}
0\ge \rho\int_M e^{-f(\alpha-1)}|\nabla u|^2\,d\mu&+\int_{\partial M} e^{-f(\alpha-1)}(e^{f(\alpha-1)}\mathcal{L}_\partial\psi+H_f^\alpha\partial_{\mathbf{n}}u)\partial_{\mathbf{n}}u\,d\mu_\partial\\
&+\int_{\partial M} e^{-f(\alpha-1)}[\Pi(\nabla_\partial\psi,\nabla_\partial\psi)-g(\nabla_\partial\psi,\nabla_\partial\partial_{\mathbf{n}}u)]\,d\mu_\partial.
\end{aligned}$$

By (3.1), we have

$$\begin{aligned}
0\ge \rho\int_M e^{-f(\alpha-1)}|\nabla u|^2\,d\mu&+\int_{\partial M} e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2\,d\mu_\partial\\
&+\int_{\partial M} e^{-f(\alpha-1)}\Pi(\nabla_\partial\psi,\nabla_\partial\psi)\,d\mu_\partial+2\int_{\partial M}\partial_{\mathbf{n}}u\mathcal{L}_\partial\psi\,d\mu_\partial.
\end{aligned}\tag{4.2}$$

On the other hand, note that

$$\int_M e^{-f(\alpha-1)}|\nabla u|^2\,d\mu=\int_{\partial M} e^{-f(\alpha-1)}u\partial_{\mathbf{n}}u\,d\mu_\partial-\int_M u\mathcal{L}u\,d\mu.$$

It follows by (4.1) that

$$\int_M e^{-f(\alpha-1)}|\nabla u|^2\,d\mu=\int_{\partial M} e^{-f(\alpha-1)}\psi\partial_{\mathbf{n}}u\,d\mu_\partial.\tag{4.3}$$

By (4.2) and (4.3), we can get

$$\begin{aligned}
&\int_{\partial M} e^{-f(\alpha-1)}\Pi(\nabla_\partial\psi,\nabla_\partial\psi)\,d\mu_\partial\\
\le\ & -\rho\int_{\partial M} e^{-f(\alpha-1)}\psi\partial_{\mathbf{n}}u\,d\mu_\partial-\int_{\partial M} e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2\,d\mu_\partial\\
&-2\int_{\partial M}\partial_{\mathbf{n}}u\mathcal{L}_\partial\psi\,d\mu_\partial\\
=\ & -\int_{\partial M}\left(e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2+\rho e^{-f(\alpha-1)}\psi\partial_{\mathbf{n}}u+2\partial_{\mathbf{n}}u\mathcal{L}_\partial\psi\right)d\mu_\partial\\
=\ & -\int_{\partial M}\left(e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2+\left(\rho e^{-f(\alpha-1)}\psi+2\mathcal{L}_\partial\psi\right)\partial_{\mathbf{n}}u\right)d\mu_\partial\\
\le\ & \int_{\partial M}\frac{e^{-f(\alpha-1)}}{H_f^\alpha}\left(\frac{\rho}{2}\psi+e^{f(\alpha-1)}\mathcal{L}_\partial\psi\right)^2d\mu_\partial.
\end{aligned}$$

This completes the proof. □

Next we use the Reilly-type inequality to prove Theorem 1.6 below.

Proof of Theorem 1.6. By the Cauchy-Schwarz inequality and (3.2), we have

$$\begin{aligned}
&\int_M \frac{N-1}{N}e^{f(\alpha-1)}|\mathcal{L}u|^2\,d\mu \qquad (4.4)\\
\ge\ & \int_{\partial M} e^{-f(\alpha-1)}[\Pi(\nabla_\partial u,\nabla_\partial u)+H_f^\alpha(\partial_{\mathbf{n}}u)^2]\,d\mu_\partial\\
&-2\int_{\partial M} e^{-f(\alpha-1)}g(\nabla_\partial u,\nabla_\partial\partial_{\mathbf{n}}u)\,d\mu_\partial\\
\ge\ & \int_{\partial M} e^{-f(\alpha-1)}[\Pi(\nabla_\partial u,\nabla_\partial u)+H_f^\alpha(\partial_{\mathbf{n}}u)^2]\,d\mu_\partial\\
&-\int_{\partial M} e^{-f(\alpha-1)}\Pi^{-1}(\nabla_\partial\partial_{\mathbf{n}}u,\nabla_\partial\partial_{\mathbf{n}}u)\,d\mu_\partial\\
&-\int_{\partial M} e^{-f(\alpha-1)}\Pi(\nabla_\partial u,\nabla_\partial u)\,d\mu_\partial\\
=\ & \int_{\partial M} e^{-f(\alpha-1)}H_f^\alpha(\partial_{\mathbf{n}}u)^2\,d\mu_\partial-\int_{\partial M} e^{-f(\alpha-1)}\Pi^{-1}(\nabla_\partial\partial_{\mathbf{n}}u,\nabla_\partial\partial_{\mathbf{n}}u)\,d\mu_\partial.
\end{aligned}$$

Let u be a smooth solution to the Neumann problem

$$\begin{cases} e^{f(\alpha-1)}\mathcal{L}u=\frac{\int_{\partial M}e^{-f(\alpha-1)}\psi\,d\mu_\partial}{V(M)} & \text{on } M,\\ \partial_{\mathbf{n}}u\equiv\psi & \text{on } \partial M.\end{cases} \tag{4.5}$$

By (4.4) and (4.5), we have

$$\int_M \frac{N-1}{N}e^{-f(\alpha-1)}\left(\frac{\int_{\partial M}e^{-f(\alpha-1)}\psi\,d\mu_\partial}{V(\partial M)}\right)^2d\mu$$

$$\geq \int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial - \int_{\partial M} e^{-f(\alpha-1)} \Pi^{-1}(\nabla_\partial \partial_{\mathbf{n}} u, \nabla_\partial \partial_{\mathbf{n}} u) \, d\mu_\partial.$$

This completes the proof. □

5. Proof of Theorem 1.8

In this section, we will use Theorem 1.5 to prove Theorem 1.8.

Proof of Theorem 1.8. In particular, when $H_f^\alpha > 0$, then the formula (1.6) can be written as

$$\begin{aligned} &\int_{\partial M} e^{-f(\alpha-1)} \Pi(\nabla_\partial \psi, \nabla_\partial \psi) \, d\mu_\partial \\ &\leq \int_{\partial M} \frac{e^{-f(\alpha-1)}}{H_f^\alpha} \left(t\psi + e^{f(\alpha-1)} \mathcal{L}_\partial \psi \right)^2 d\mu_\partial, \end{aligned} \tag{5.1}$$

which is equivalent to

$$\begin{aligned} &\left(\int_{\partial M} \frac{\psi^2}{H_f^\alpha} e^{-f(\alpha-1)} \, d\mu_\partial \right) t^2 + 2 \left(\int_{\partial M} \frac{\psi \mathcal{L}_\partial \psi}{H_f^\alpha} \, d\mu_\partial \right) t \\ &+ \int_{\partial M} \left[\frac{(e^{f(\alpha-1)} \mathcal{L}_\partial \psi)^2}{H_f^\alpha} - \Pi(\nabla_\partial \psi, \nabla_\partial \psi) \right] e^{-f(\alpha-1)} \, d\mu_\partial \geq 0 \end{aligned}$$

holds for any $t \leq \frac{\rho}{2}$. Thus, we have

$$B^2(\psi) \leq A(\psi) C(\psi)$$

or

$$\frac{\rho}{2} \leq -\frac{B(\psi)}{A(\psi)} - \sqrt{\left(\frac{B(\psi)}{A(\psi)} \right)^2 - \frac{C(\psi)}{A(\psi)}}$$

or

$$\frac{\rho}{2} \geq -\frac{B(\psi)}{A(\psi)} + \sqrt{\left(\frac{B(\psi)}{A(\psi)} \right)^2 - \frac{C(\psi)}{A(\psi)}}.$$

This completes the proof. □

6. Mean-curvature type inequalities

In this section, we will get some global estimates for the generalized mean curvature on ∂M. Setting $\psi \equiv 1$ in Theorem 1.6, we can get:

Theorem 6.1. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} \geq 0$, where $N \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M. Then*

$$\int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha \, d\mu_\partial \leq \frac{N-1}{N} \frac{(V(\partial M))^2}{V(M)}, \tag{6.1}$$

where $V(\partial M) = \int_{\partial M} e^{-f(\alpha-1)} \, d\mu_\partial$ and $V(M) = \int_M e^{-f(\alpha-1)} \, d\mu$.

By the Cauchy-Schwarz inequality, we have

$$\int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha \, d\mu_\partial \int_{\partial M} e^{-f(\alpha-1)} \frac{1}{H_f^\alpha} \, d\mu_\partial \geq (V(\partial M))^2. \tag{6.2}$$

An immediate consequence of (6.1) and (6.2) is:

Theorem 6.2. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} \geq 0$, where $N \in (-\infty, 0) \cup [n, +\infty)$. Assume that $\Pi > 0$ on ∂M. Then*

$$\int_{\partial M} e^{-f(\alpha-1)} \frac{1}{H_f^\alpha} \, d\mu_\partial \geq \frac{N}{N-1} V(M). \tag{6.3}$$

In fact, we can replace the assumption $\Pi > 0$ in Theorem 6.2 by a weaker condition $H_f^\alpha > 0$, and obtain the following theorem on ∂M.

Theorem 6.3. *Let $(M^n, g, d\mu)$ be a smooth compact weighted Riemannian manifold of dimension $n \geq 2$ with boundary and $Ric_f^{\alpha,N} \geq 0$, where $N \in (-\infty, 0) \cup [n, +\infty)$. Assume that $H_f^\alpha > 0$ on ∂M. Then*

$$\int_{\partial M} e^{-f(\alpha-1)} \frac{1}{H_f^\alpha} \, d\mu_\partial \geq \frac{N}{N-1} V(M). \tag{6.4}$$

Proof. Let u be a smooth solution to the Dirichlet problem

$$\begin{cases} e^{f(\alpha-1)} \mathcal{L} u = 1 & \text{on } M, \\ u \equiv 0 & \text{on } \partial M. \end{cases} \tag{6.5}$$

By (3.2), we can get

$$\begin{aligned} \frac{N-1}{N} V(M) &= \int_M \frac{N-1}{N} e^{f(\alpha-1)} |\mathcal{L} u|^2 \, d\mu \\ &\geq \int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial. \end{aligned} \tag{6.6}$$

On the other hand, note that

$$\begin{aligned} (V(M))^2 &= \left(\int_M e^{f(\alpha-1)} |\mathcal{L} u|^2 \, d\mu \right)^2 \\ &= \left(\int_{\partial M} e^{-f(\alpha-1)} \partial_{\mathbf{n}} u \, d\mu_\partial \right)^2 \\ &\leq \int_{\partial M} e^{-f(\alpha-1)} H_f^\alpha (\partial_{\mathbf{n}} u)^2 \, d\mu_\partial \int_{\partial M} e^{-f(\alpha-1)} \frac{1}{H_f^\alpha} \, d\mu_\partial. \end{aligned} \tag{6.7}$$

By (6.6) and (6.7), the assertion follows. □

Acknowledgements. The author would like to thank the referee for several useful comments and suggestions.

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