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Δ-TRANSITIVITY FOR SEMIGROUP ACTIONS

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Abstract. In this paper, we study Δ-transitivity, Δ-weak mixing and Δ-mixing for semigroup actions and give several characterizations of them, which generalize related results in the literature.

1. Introduction

By a (topological) dynamical system, we mean a pair (X, g), where X is a compact metric space and $g : X \to X$ is a continuous map.

The study of transitive systems and its classification plays a big role in topological dynamics. Many authors have done much work in classifying transitive systems by their recurrence properties. A dynamical system (X, g) is Δ-transitive if for every $d \geqslant 2$ there exists a residual subset X_0 of X such that for every $x \in X_0$ the diagonal d-tuple $x^{(d)} =: (x, x, \ldots, x)$ has a dense orbit under the action $g \times g^2 \times \cdots \times g^d$. In [9], Moothathu showed that Δ-transitivity implies weakly mixing, but there exists some strongly mixing systems which are not Δ-transitive. He also pointed out that multi-transitivity, weakly mixing and Δ-transitivity were equivalent for a minimal homeomorphism. In [7], Kwietniak and Oprocha extended this result to a non-invertible case. Using a class of Furstenberg families introduced in [3], Chen, Li and Lü in [4] characterized the entering time sets of transitive points into open sets in multi-transitive and Δ-transitive systems, answering several problems proposed in [7].

Recall that a dynamical system (X, g) is Δ-weakly mixing if the product system $(X \times X, g \times g)$ is Δ-transitive. In [5], the authors showed that this kind of Δ-weakly mixing is in fact equivalent to Δ-transitivity, and then Δ-weakly mixing shares similar properties of Δ-transitivity.

A dynamical system (X, g) is Δ-mixing if for every $m \in \mathbb{N}$ and infinite subset $A \subset \mathbb{N}$, there exists a residual subset X_0 of X such that for every $x \in X_0$, $\{(g^n x, g^{2n} x, \ldots, g^{mn} x) : n \in A\}$ is dense in X^m. In [9], Moothathu gave some relations among multi-transitivity, Δ-transitivity and Δ-mixing.

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In the past years, much attention has been paid to the research of dynamical systems under general semigroup actions, see [2,6,13–15] and references therein. In [16], we studied multi-transitivity and Δ-transitivity for semigroup actions, while giving several characterizations of them. In [16], by a *G-system*, we mean a triple (X, G, π), where G is a discrete semigroup, X is a Polish space (i.e., a complete metrizable space) and

$$\pi : G \times X \to X, \quad (g, x) \mapsto gx$$

is a continuous action on $G \times X$ with the property that $\pi(g_1, \pi(g_2, x)) = \pi(g_1 g_2, x)$ for all $x \in X$ and $g_1, g_2 \in G$. Usually, we write the G-system as a pair (X, G). We define Δ-transitivity for the system (X, G) in [16] as follows: the system (X, G) is Δ-transitive if for every $n \in \mathbb{N}$, there exists a residual subset X_0 of X such that for each $x \in X_0$, $\{(gx, g^2x, \ldots, g^nx) : g \in G\}$ is dense in X^n.

In this paper we define a more powerful definition of Δ-transitivity under semigroup actions and attain stronger consequences.

In [1], Blanchard and Huang defined a local version of weak mixing, so called weakly mixing set, and proved that positive topological entropy implies the existence of weakly mixing sets. In [10–12], Oprocha and Zhang also discussed local versions of weak mixing extensively. In [5], the authors studied the property of Δ-weakly mixing and showed that a topological dynamical system with positive topological entropy has many Δ-weakly mixing subsets. Recently in [8], Liu extended the results in [5] to countable torsion-free discrete nilpotent group actions.

Recall that a group is *torsion-free* if any element has infinite order except the identity element. Liu defined Δ-transitivity in [8] as follows: Let (X, G) be a G-system, where G is a countable torsion-free discrete group with the unit e and $|X| \geq 2$. We say that (X, G) is Δ-*transitive* provided that there is a residual subset A of X such that for any $x \in A$, $d \geq 1$ and pairwise distinct $T_1, T_2, \ldots, T_d \in G \setminus \{e\}$, the orbit closure of the d-tuple $(x, x, \ldots, x)$ under the action $T_1 \times T_2 \times \cdots \times T_d$ contains X^d, i.e.,

$$\overline{\{(T_1^n x, T_2^n x, \ldots, T_d^n x) : n \in \mathbb{N}\}} = X^d.$$

Through the above ideas and results, we find that the definition of Δ-transitivity in [8] is a more general characterization of this property. There is a close connection between this definition of Δ-transitivity and other properties (such as topological entropy) in dynamical systems. In this paper, we define Δ-transitivity in a similar way as in [8] for semigroup actions. Our aim is to give characterizations of Δ-transitivity under semigroup actions of this notion and find some relations among Δ-transitivity, Δ-weakly mixing and Δ-mixing. Besides, we study local version of Δ-transitivity, and show that if G is abelian, a G-system is Δ-transitive if and only if it is Δ-weakly mixing while it is no longer true for Δ-transitive subsets and Δ-weakly mixing sets.

This paper is organized as follows. In Section 2, we introduce some notions and results which will be used later. In Section 3, we study Δ-transitivity and Δ-mixing for semigroup actions. We give some characterizations of Δ-transitivity and discuss some properties of Δ-mixing for general semigroup actions.

2. Preliminaries

In this paper, let $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote the set of all positive integers, non-negative integers and integers, respectively. The cardinality of a set B is denoted by $|B|$.

A subset $P \subset \mathbb{N}$ is *thick* if it contains arbitrarily long blocks of consecutive positive integers, that is, for every $n \geq 1$ there is $m_n \in \mathbb{N}$ such that $\{m_n, m_n + 1, \dots, m_n + n\} \subset P$. A subset P of $\mathbb{N}$ is syndetic if it has a bounded gap, that is, there is $N \in \mathbb{N}$ such that $\{n, n+1, \dots, n+N\} \cap P \neq \emptyset$ for every $n \in \mathbb{N}$. Let $\mathcal{A}$ be a collection of $\mathbb{N}$ with $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$. If for every $A_1, A_2 \in \mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \subset A_1 \cap A_2$, then we say $\mathcal{A}$ is a filter base.

Let G be a countable discrete semigroup with the identity e. By a G-system, we mean a pair (X, G), where X is a compact metric space and there exists a continuous map $\phi : G \times X \to X$, $(g, x) \mapsto gx$ such that $\phi(e, x) = x$ and $\phi(h, \phi(gx)) = \phi(hg, x)$ for all $g, h \in G$ and $x \in X$. For a G-system and $m \in \mathbb{N}$, (X^m, G) is also a G-system, where $g(x_1, x_2, \dots, x_m) := (gx_1, gx_2, \dots, gx_m)$ for any $(x_1, x_2, \dots, x_m) \in X^m$ and $g \in G$. A semigroup G is *abelian* if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. For a point $x \in X$, the *orbit* of x is the set $Gx := \{gx : g \in G\}$. For a point $x \in X$ and a subset U of X, we define the *hitting time set* of x into U by

$$N(x, U) = \{g \in G : gx \in U\}.$$

For two subsets U and V of X, we define the *hitting time set* of U and V by

$$N(U, V) = \{g \in G : U \cap g^{-1}V \neq \emptyset\}.$$

When G is the semigroup $(\mathbb{Z}_+, +)$, let $g : X \to X$, $x \to \phi(1, x)$. Then ϕ can be generated by g. In this case we denote the dynamical system by (X, g).

A dynamical system (X, G) is called *transitive* if for every non-empty open subsets U and V of X, there exists $g \in G$ such that

$$U \cap g^{-1}V \neq \emptyset;$$

weakly mixing if $(X \times X, G)$ is transitive; *strongly mixing* if for every non-empty open subsets U and V of X, there exists a finite subset F of G such that

$$U \cap g^{-1}V \neq \emptyset, \quad \forall g \in G \setminus F.$$

Let G be a semigroup and $\mathcal{P}$ denote the collection of all subsets of G. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a *Furstenberg family* over G (or just a *family* over G), if it is hereditary upward, i.e., $F_1 \subset F_2 \subset G$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

Let (X, G) be a dynamical system and $\mathcal{F}$ be a Furstenberg family over G. The system (X, G) is called $\mathcal{F}$-transitive if for every two non-empty open subsets U and V of X, the hitting time set $N(U,V) \in \mathcal{F}$. We say that a point $x \in X$ is *$\mathcal{F}$-transitive* if for every non-empty open subset $U \subset X$, $N(x,U) \in \mathcal{F}$. The collection of $\mathcal{F}$-transitive points is denoted by $\mathrm{Trans}_{\mathcal{F}}(X,G)$.

Convention. Unless otherwise specified, in the statement of our results we will assume that X has no isolated point and G is infinite.

3. Δ-transitivity

In this section, we define Δ-transitivity for general semigroup actions and derive some properties of Δ-transitivity.

Definition 3.1. Let A be a subset of $\mathbb{N}$. We say that (X,G) is Δ-A-transitive if for any $d \geq 1$ and pairwise distinct $g_1,\dots,g_d \in G \setminus \{e\}$, there exists a dense subset Y of X such that for any $x \in Y$

$$\{(g_1^n x, g_2^n x, \dots, g_d^n x) : n \in A\}$$

is dense in X^d. If $A = \mathbb{N}$, we say that (X,G) is Δ-transitive briefly.

Let (X,G) be a G-system. For any $d \in \mathbb{N}$, pairwise distinct $g_1,\dots,g_d \in G \setminus \{e\}$, non-empty open subsets $U_0,U_1,\dots,U_d$, we set

$$N(U_0;U_1,\dots,U_d \mid g_1,\dots,g_d) = \{n \in \mathbb{N} : U_0 \cap g_1^{-n}U_1 \cap \cdots \cap g_d^{-n}U_d \neq \emptyset\}.$$

The following proposition give some equivalent conditions of Δ-A-transitivity, which extends the result in [9] for semigroup actions.

Proposition 3.2. *Let (X,G) be a G-system and $A \subset \mathbb{N}$. Then the following conditions are equivalent*:

(1) *(X,G) is Δ-A-transitive*;
(2) *for any $d \in \mathbb{N}$, pairwise distinct $g_1,\dots,g_d \in G \setminus \{e\}$ and non-empty open subsets $U_0,U_1,\dots,U_d$,*

$$N(U_0;U_1,\dots,U_d \mid g_1,\dots,g_d) \cap A \neq \emptyset.$$

(3) *for any $d \in \mathbb{N}$, pairwise distinct $g_1,\dots,g_d \in G \setminus \{e\}$, there exists a residual subset Y of X such that for any $x \in Y$*

$$\{(g_1^n x, g_2^n x, \dots, g_d^n x) : n \in A\}$$

is dense in X^d.

Proof. (1) ⇒ (2) Let $d \in \mathbb{N}$ and $g_1,\dots,g_d$ be pairwise distinct elements of $G \setminus \{e\}$. Let $U_0,U_1,\dots,U_d$ be non-empty open subsets of X. Choose $x \in U_0$ satisfying that

$$\{(g_1^n x, g_2^n x, \dots, g_d^n x) : n \in A\}$$

is dense in X^d. Then there exists some $n \in A$ such that $(g_1^n x, g_2^n x, \dots, g_d^n x) \in U_1 \times U_2 \times \cdots \times U_d$, that is

$$x \in U_0 \cap \bigcap_{i=1}^{d} g_i^{-n}(U_i),$$

and thus $n \in N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap A$.

(2) ⇒ (3) Let $d \in \mathbb{N}$ and $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. Let $\{B_k : k \in \mathbb{N}\}$ be a countable base of open balls of X. Set

$$Y = \bigcap_{(k_1,k_2,\ldots,k_d)\in\mathbb{N}^d} \bigcup_{n\in A} \bigcap_{i=1}^{d} g_i^{-n}(B_{k_i}).$$

The set $\bigcup_{n\in A} \bigcap_{i=1}^{d} g_i^{-n}(B_{k_i})$ is clearly open, and it is dense by (2). So by Baire category theorem, Y is a dense G_δ subset of X. It is easy to see that for a point $x \in X$ the set $\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in A\}$ is dense in X^d if and only if $x \in Y$.

(3) ⇒ (1) It is obvious. □

When G is abelian, the equivalent condition for Δ-transitivity can be simpler, see the following lemma.

Lemma 3.3. *Let (X, G) be a G-system with G abelian. Then (X, G) is Δ-transitive if and only if for any $d \in \mathbb{N}$ and pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, there exists a point $x \in X$ such that*

$$\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in \mathbb{N}\}$$

is dense in X^d.

Proof. The necessity is clear. Now we need to show the sufficiency. Fix $d \in \mathbb{N}$. Let $U_0, U_1, \ldots, U_d$ be non-empty open subsets of X. Choose $h \in G$ such that $y = hx \in U_0$. Let $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. As G is abelian, the set

$$\{(g_1^n y, g_2^n y, \ldots, g_d^n y) : n \in \mathbb{N}\}$$

is dense in X^d. Hence $(g_1^n y, g_2^n y, \ldots, g_d^n y) \in U_1 \times \cdots \times U_d$ for some $n \in \mathbb{N}$. Thus, $y \in \bigcap_{i=1}^{d} g_i^{-n}(U_i)$. Then by Proposition 3.2, (X, G) is Δ-transitive. □

Example 3.4. Let $X = \{0, 1\}^{\mathbb{Z}}$ and $G = \{\sigma^n : n = 0, 1, 2, \ldots\}$. Then (X, G) is the full shift $(\{0, 1\}^{\mathbb{Z}}, \sigma)$. Since there is a point $x \in \{0, 1\}^{\mathbb{Z}}$ such that for every $d \geq 2$, the diagonal d-tuple $(x, x, \ldots, x)$ has a dense orbit under the action $\sigma \times \sigma^2 \times \cdots \times \sigma^d$. Hence (X, G) is Δ-transitive by Lemma 3.3.

We say that (X, G) is Δ-A-weakly mixing if for every $n \in \mathbb{N}$, (X^n, G) is Δ-A-transitive. If $A = \mathbb{N}$, then we say that (X, G) is Δ-weakly mixing. Now we show that for a G-system and $A \subset \mathbb{N}$, Δ-A-transitive and Δ-A-weakly mixing are equivalent with G abelian by proving the following proposition.

Proposition 3.5. *Let (X, G) be a G-system with G abelian and $A \subset \mathbb{N}$. Then the following conditions are equivalent*:

(1) *(X, G) is Δ-A-transitive*;

(2) *the collection of hitting time sets*

$$\begin{aligned} W := \{ & N(U_0; U_1, \dots, U_d \mid g_1, \dots, g_d) \cap A : d \in \mathbb{N}, \\ & U_0, U_1, \dots, U_d \text{ are non-empty open subsets of } X \text{ and} \\ & g_1, \dots, g_d \text{ are pairwise distinct elements in } G \setminus \{e\}\} \end{aligned}$$

is a filter base;

(3) (X, G) *is* Δ-A*-weakly mixing.*

Proof. (1) $\Rightarrow$ (2) By Proposition 3.2, for any $d \in \mathbb{N}$, pairwise distinct $g_1, \dots, g_d \in G \setminus \{e\}$ and non-empty open subsets $U_0, U_1, \dots, U_d$, the set

$$N(U_0; U_1, \dots, U_d \mid g_1, \dots, g_d) \cap A \neq \emptyset.$$

For any $N(U_0; U_1, \dots, U_d \mid g_1, \dots, g_d) \cap A$ and $N(V_0; V_1, \dots, V_r | g_{d+1}, \dots, g_{d+r}) \cap A$ in W, choose $g_0 \in G \setminus \{e\}$ such that $g_1, \dots, g_d, g_0, g_{d+1}g_0, \dots, g_{d+r}g_0$ are pairwise distinct elements in $G \setminus \{e\}$.

Then for any $n \in N(U_0; U_1, \dots, U_d, V_0, V_1, \dots, V_r \mid g_1, \dots, g_d, g_0, g_{d+1}g_0, \dots, g_{d+r}g_0) \cap A$,

$$U_0 \cap g_1^{-n} U_1 \cap \dots \cap g_d^{-n} U_d \cap g_0^{-n} V_0 \cap g_0^{-n} g_{d+1}^{-n} V_1 \cap \dots \cap g_0^{-n} g_{d+r}^{-n} V_r \neq \emptyset.$$

Thus $U_0 \cap g_1^{-n} U_1 \cap \dots \cap g_d^{-n} U_d \neq \emptyset$ and $V_0 \cap g_{d+1}^{-n} V_1 \cap \dots \cap g_{d+r}^{-n} V_r \neq \emptyset$, which implies that $n \in N(U_0; U_1, \dots, U_d | g_1, \dots, g_d) \cap N(V_0; V_1, \dots, V_r | g_{d+1}, \dots, g_{d+r}) \cap A$. Then we have

$$\begin{aligned} & N(U_0; U_1, \dots, U_d, V_0, V_1, \dots, V_r \mid g_1, \dots, g_d, g_0, g_{d+1}g_0, \dots, g_{d+r}g_0) \cap A \\ \subset\ & N(U_0; U_1, \dots, U_d \mid g_1, \dots, g_d) \cap N(V_0; V_1, \dots, V_r | g_{d+1}, \dots, g_{d+r}) \cap A. \end{aligned}$$

(2) $\Rightarrow$ (3) Let $m, n \geq 1$, $g_1, g_2, \dots, g_m$ be pairwise distinct elements of $G \setminus \{e\}$ and $U_{1,0}, U_{2,0}, \dots, U_{n,0}, U_{1,1}, U_{2,1}, \dots, U_{n,1}, \dots, U_{1,m}, U_{2,m}, \dots, U_{n,m}$ be non-empty open subsets of X. It is easy to check that

$$\begin{aligned} & N(U_{1,0} \times \dots \times U_{n,0}; U_{1,1} \times \dots \times U_{n,1}, \dots, U_{1,m} \times \dots \times U_{n,m} | g_1, g_2, \dots, g_m) \\ & \cap A \\ = \ & N(U_{1,0}; U_{1,1}, \dots, U_{1,m} \mid g_1, g_2, \dots, g_m) \cap \cdots \\ & \cap N(U_{n,0}; U_{n,1}, \dots, U_{n,m} \mid g_1, g_2, \dots, g_m) \cap A. \end{aligned}$$

Since the collection of hitting time sets is a filter base, we know that the intersection above is not empty. By Proposition 3.2, (X^n, G) is Δ-A-transitive.

(3) $\Rightarrow$ (1) It is obvious. □

We immediately have the following.

Corollary 3.6. *Let* (X, G) *be a* G*-system with* G *abelian and* $A \subset \mathbb{N}$. (X, G) *is* Δ-A*-weakly mixing if and only if* (X^n, G) *is* Δ-A*-weakly mixing for all* $n \in \mathbb{N}$.

We have the following observations about the hitting time sets of Δ-transitive systems for abelian semigroup actions.

Proposition 3.7. *Let (X, G) be a G-system with G abelian. Then the following conditions are equivalent*:

(1) *(X, G) is Δ-transitive*;
(2) *for any $d \in \mathbb{N}$, non-empty open subsets $U_0, U_1, \ldots, U_d$ of X and pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, $N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)$ is thick of* $\mathbb{N}$;
(3) *for any syndetic subset A of $\mathbb{N}$, (X, G) is Δ-A-transitive.*

Proof. (3) $\Rightarrow$ (1) It is obvious.

(1) $\Rightarrow$ (2) Let $d \in \mathbb{N}$. Let $U_0, U_1, U_2, \ldots, U_d$ be non-empty open subsets of X and $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. By Proposition 3.5(2)with $A = \mathbb{N}$, for any $n \in \mathbb{N}$

$$N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap N(U_0; g_1^{-1}U_1, \ldots, g_d^{-1}U_d \mid g_1, \ldots, g_d)$$
$$\cap \cdots \cap N(U_0; g_1^{-n}U_1, \ldots, g_d^{-n}U_d \mid g_1, \ldots, g_d) \neq \emptyset.$$

Thus $N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)$ is thick of $\mathbb{N}$.

(2) $\Rightarrow$ (3) Let A be a syndetic subset of $\mathbb{N}$. Let $d \in \mathbb{N}$, $U_0, U_1, \ldots, U_d$ be non-empty open subsets of X and $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. Since the intersection of any thick set and a syndetic set is not empty, then $N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap A \neq \emptyset$. By Proposition 3.2, this implies that (X, G) is Δ-A-transitive. □

For $n \geq 2$, we say that a non-empty subset $F \subset G$ is an *independent set for* $(U_1, U_2, \ldots, U_n)$ if for every non-empty finite subset $J \subset F$ and $\sigma \in \{1, 2, \ldots, n\}^J$,

$$\bigcap_{g \in J} g^{-1}U_{\sigma(g)} \neq \emptyset.$$

We have the following characterization of Δ-transitivity by independent sets of open sets.

Proposition 3.8. *Let (X, G) be a G-system with G abelian. If (X, G) is Δ-transitive, then for any $d \geq 2$, non-empty open subsets $U_1, \ldots, U_d$ of X and pairwise distinct $g_1, \ldots, g_d$ of $G \setminus \{e\}$, there exists $n \in \mathbb{N}$ such that $J := \{g_1^n, \ldots, g_d^n\}$ is an independent set for $(U_1, \ldots, U_d)$.*

Proof. Fix $d \geq 2$, non-empty open subsets $V, U_1, \ldots, U_d$ of X and pairwise distinct $g_1, \ldots, g_d$ of $G \setminus \{e\}$, by Proposition 3.5(2) with $A = \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$n \in \bigcap \{N(V; U_{s_1}, \ldots, U_{s_d} \mid g_1, \ldots, g_d) : s_i \in \{1, \ldots, d\}, i = 1, \ldots, d\}.$$

Now we show that $J := \{g_1^n, \ldots, g_d^n\}$ is an independent set for $(U_1, \ldots, U_d)$. It is clear that for any $\sigma \in \{1, 2, \ldots, d\}^J$,

$$\bigcap_{i \in \{1, \ldots, d\}} (g_i^n)^{-1} U_{\sigma(g_i^n)} \neq \emptyset,$$

which implies that J is an independent set for $(U_1, \ldots, U_d)$. □

Let (X, G) be a G-system and $\mathbf{a} = (a_1, \ldots, a_r)$ be a vector in $\mathbb{N}^r$. We say that the system (X, G) is Δ-$\mathbf{a}$-transitive if there exists a dense subset X_0 of X such that for each $x \in X_0$, $\{(g^{a_1}x, \ldots, g^{a_r}x) : g \in G\}$ is dense in X^r. In [16], we defined a different Δ-transitivity and characterized Δ-$\mathbf{a}$-transitivity by $\mathcal{F}[\mathbf{a}]$-transitive points under semigroup actions. Here we obtain a similar result.

Let $d \in \mathbb{N}$ and $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. We define the family generated by $\mathbf{g} = \{g_1, \ldots, g_d\}$, denoted by $\mathcal{F}[\mathbf{g}]$ as

$$\{F \subset G : \forall h_1, h_2, \ldots, h_d \in G, \exists n \in \mathbb{N} \text{ s.t. } h_1 g_1^n, h_2 g_2^n, \ldots, h_d g_d^n \in F\}.$$

Theorem 3.9. *Let (X, G) be a G-system with G a group. Then (X, G) is Δ-transitive if and only if for any $d \in \mathbb{N}$ and pairwise distinct elements $g_1, \ldots, g_d$ of $G \setminus \{e\}$, $\mathrm{Trans}_{\mathcal{F}[\mathbf{g}]}(X, G)$ is residual in X, where $\mathbf{g} := \{g_1, \ldots, g_d\}$.*

Proof. Necessity. Let $\{B_k : k \in \mathbb{N}\}$ be a countable base of open balls of X. For any $d \in \mathbb{N}$ and pairwise distinct elements $g_1, \ldots, g_d$ of $G \setminus \{e\}$, set

$$Y = \bigcap_{(k_1, k_2, \ldots, k_d) \in \mathbb{N}^d} \bigcup_{n \in \mathbb{N}} \bigcap_{i=1}^{d} g_i^{-n}(B_{k_i}).$$

Since (X, G) is Δ-transitive, by Proposition 3.2 we obtain that Y is a dense G_δ subset of X. Now we only need to show that $Y \subset \mathrm{Trans}_{\mathcal{F}[\mathbf{g}]}(X, G)$. Choose $x \in Y$ and a non-empty open subset U of X. Let $H = \{h_1, h_2, \ldots, h_d\}$ be a finite subset of G. Then there exists $(k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ such that

$$B_{k_1} \times B_{k_2} \times \cdots \times B_{k_d} \subset h_1^{-1}U \times h_2^{-1}U \times \cdots \times h_d^{-1}U.$$

By the construction of Y, there exists $n \in \mathbb{N}$ such that $x \in \bigcap_{i=1}^{d} g_i^{-n}(B_{k_i})$, then $g_i^n x \in B_{k_i} \subset h_i^{-1}U$ for $i = 1, 2, \ldots, d$. Hence we obtain that $\{h_1 g_1^n, h_2 g_2^n, \ldots, h_d g_d^n\} \subset N(x, U)$ and $x \in \mathrm{Trans}_{\mathcal{F}[\mathbf{g}]}(X, G)$.

Sufficiency. Let $d \in \mathbb{N}$. For any non-empty open subsets $U_0, U_1, U_2, \ldots, U_d$ of X and pairwise distinct elements $g_1, \ldots, g_d$ of $G \setminus \{e\}$, there exists $x \in U_0$ which is a $\mathcal{F}[\mathbf{g}]$-transitive point where $\mathbf{g} = \{g_1, \ldots, g_d\}$. Thus there exist $h_1, \ldots, h_d$ such that $h_i x \in U_i$, $i = 1, \ldots, d$. By the continuity of h_i, there is a neighborhood U of x such that $h_i U \subset U_i$, $i = 1, \ldots, d$. By the definition of $\mathcal{F}[\mathbf{g}]$-transitive point, there is $n \in \mathbb{N}$ such that

$$(h_1^{-1} g_1^n x, h_2^{-1} g_1^n x, \ldots, h_d^{-1} g_d^n x) \in U \times \cdots \times U.$$

Thus $g_i^n x \subset U_i$, $i = 1, \ldots, d$. Therefore

$$N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \neq \emptyset$$

and by Proposition 3.2 (X, G) is Δ-transitive. □

A factor map $\pi : (X, G) \to (Y, G)$ between two G-systems is a continuous onto map satisfying that $g\pi = \pi g$ for every $g \in G$. We have the following.

Proposition 3.10. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. If (X, G) is Δ-transitive, then so is (Y, G).*

Proof. Assume that (X,G) is Δ-transitive. Let $d\in\mathbb{N}$. Let $U_0,U_1,\dots,U_d$ be non-empty open subsets of Y and $g_1,\dots,g_d$ be pairwise distinct elements of $G\setminus\{e\}$. Since (X,G) is Δ-transitive, $N(\pi^{-1}(U_0);\pi^{-1}(U_1),\dots,\pi^{-1}(U_d)\mid g_1,\dots,g_d)\neq\emptyset$. Hence $N(U_0;U_1,\dots,U_d\mid g_1,\dots,g_d)\neq\emptyset$. That is, (Y,G) is Δ-transitive. □

In [8], the author studied some properties of Δ-transitive subsets and Δ-weakly mixing subsets for countable torsion-free discrete group actions. If G is abelian, then by Proposition 3.5(2) with $A=\mathbb{N}$ we know that a G-system is Δ-transitive if and only if it is Δ-weakly mixing. But this is no longer true for Δ-transitive subsets and Δ-weakly mixing sets, as the authors proved that they are not equivalent for $\mathbb{Z}$-system in [5]. Here we define the local notions of Δ-transitivity and Δ-weakly mixing for G-systems and obtain some similar results.

Definition 3.11. Let (X,G) be a G-system and B be a closed subset of X with $|B|\geq 2$.

(1) we say that B is a Δ-transitive subset of (X,G) if there is a residual subset B_0 of B such that for any $x\in B_0$, $d\geq 1$ and pairwise distinct $g_1,\dots,g_d\in G\setminus\{e\}$, the orbit closure of the d-tuple $(x,x,\dots,x)$ under the action $g_1\times g_2\times\cdots\times g_d$ contains B^d, i.e.,
$$\overline{\{(g_1^n x, g_2^n x,\dots,g_d^n x): n\in\mathbb{N}\}}\supseteq B^d.$$
(2) we say that B is a Δ-weakly mixing subset of (X,G) if B^m is a Δ-transitive subset of (X^m,G) for any $m\in\mathbb{N}$.

We can easily see that X is a Δ-transitive (resp. Δ-weakly mixing) subset of (X,G) if and only if the G-system (X,G) is Δ-transitive (resp. Δ-weakly mixing).

The proof of the following two lemmas are similar with Lemma 3.3 and Proposition 3.4 of [8], we present the results without proof here.

Lemma 3.12. *Let (X,G) be a G-system and B be a closed subset of X with $|B|\geq 2$. Then B is Δ-transitive if and only if for any $d\in\mathbb{N}$, pairwise distinct $g_1,\dots,g_d\in G\setminus\{e\}$ and non-empty open subsets $U_0,U_1,\dots,U_d$ of X with $U_i\cap B\neq\emptyset$, $i=0,\dots,d$,*
$$N(U_0\cap B;U_1,\dots,U_d\mid g_1,\dots,g_d)\neq\emptyset.$$

Lemma 3.13. *Let (X,G) be a G-system and B be a closed subset of X with $|B|\geq 2$. Then B is a Δ-weakly mixing subset of X if and only if for any $d\in\mathbb{N}$, pairwise distinct $g_1,\dots,g_d\in G\setminus\{e\}$, non-empty open subsets $U_1,\dots,U_d$ and $V_1,\dots,V_d$ of X with $U_i\cap B\neq\emptyset$ and $V_i\cap B\neq\emptyset$ for $i=1,\dots,d$, we have*
$$\bigcap_{\sigma\in\{1,\dots,d\}^{d+1}} N(V_{\sigma(1)}\cap B;U_{\sigma(2)},\dots,U_{\sigma(d+1)}\mid g_1,\dots,g_d)\neq\emptyset.$$

We say that (X, G) is Δ-mixing if it is Δ-A-transitive for any infinite subset A of $\mathbb{N}$. We have the following equivalent condition about Δ-mixing.

Corollary 3.14. *Let (X, G) be a G-system. Then (X, G) is Δ-mixing if and only if for any $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, non-empty open subsets $U_0, U_1, \ldots, U_d$,*

$$\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)$$

is finite.

Proof. Necessity. Assume that there exist $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$ and non-empty open subsets $U_0, U_1, \ldots, U_d$ such that the set $\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)$ is infinite. Then there exists an infinite subset $A := \mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)$ such that $A \cap N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) = \emptyset$, which implies that (X, G) is not Δ-A-transitive by Proposition 3.2, contradiction.

Sufficiency. If there exists an infinite subset A such that (X, G) is not Δ-A-transitive, then by Proposition 3.2, there exist $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$ and non-empty open subsets $U_0, U_1, \ldots, U_d$ such that $A \cap N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) = \emptyset$, then $\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \supset A$ is an infinite set, contradiction. $\square$

We immediately have the following statements.

Corollary 3.15. *If (X, G) is Δ-mixing, then it is strongly mixing.*

Example 3.16. Let $X = \mathbb{R}^n / \mathbb{Z}^n$ $(n \geq 2)$ and $G = SL(n, \mathbb{Z})$. Then it is the linear action of $SL(n, \mathbb{Z})$ on the torus $\mathbb{R}^n / \mathbb{Z}^n$, see Example 8 in [2]. Since the action is not strongly mixing, we obtain that (X, G) is not Δ-mixing by Corollary 3.15.

Corollary 3.17. *(X, G) is Δ-mixing if and only if (X^n, G) is Δ-mixing for all $n \in \mathbb{N}$.*

Corollary 3.18. *If (X, G) is Δ-mixing, then it is Δ-A-weakly mixing for any infinite subset A.*

Proof. Suppose (X, G) is Δ-mixing, then (X^n, G) is Δ-mixing for all $n \in \mathbb{N}$ by Corollary 3.17. Hence (X^n, G) is Δ-A-transitive for any infinite subset A of $\mathbb{N}$. Then (X, G) is Δ-A-weakly mixing for any infinite subset A. $\square$

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