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MODEL STRUCTURES AND RECOLLEMENTS INDUCED BY DUALITY PAIRS

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Abstract. Let $(\mathcal{L}, \mathcal{A})$ be a complete duality pair. We give some equivalent characterizations of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules and construct some model structures associated to duality pairs and Frobenius pairs. Some rings are described by Frobenius pairs. In addition, we investigate special Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules and construct some model structures and recollements associated to them.

1. Introduction

The notion of duality pairs of R-modules was introduced by Holm and Jørgensen in [16]. Duality pairs exist extensively (see for example [13, 14, 16, 19]). Duality pairs are closely related to purity and the existence of covers and envelopes (see [16, Theorem 3.1]), which implies that duality pairs are very useful in relative homological algebra. Gillespie investigated Gorenstein homological algebra with respect to a complete duality pair and constructed some relevant model structures in [13]. It was found that complete duality pairs induce abelian model structures for stable module categories. Motivated by this, we continue to study Gorenstein homological algebra with respect to a duality pair and we construct model structures from an R-module that is strongly Gorenstein projective with respect to a given complete duality pair $(\mathcal{L}, \mathcal{A})$.

In [10], Gillespie introduced exact model structures in exact categories and gave a correspondence between the exact model structure and two complete cotorsion pairs, called Hovey-Gillespie correspondence in [1]. Based on this fact, Becerril and coauthors showed how to construct an exact model structure from a Frobenius pair in [1], which tells us that Frobenius pairs also have a great importance in constructing model structures. In this direction, the first author of this paper and coauthors have tried to find Frobenius pairs by Gorenstein objects with respect to cotorsion pairs and obtained some interesting results in [5]. Wang and coauthors introduced and studied Gorenstein flat modules with

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respect to duality pairs in [19], which enriches Gorenstein homological algebra with respect to a duality pair. It is well known that a perfect duality pair can induce a perfect cotorsion pair, which builds a bridge between duality pairs and cotorsion pairs. In this paper, we get some model structures associated to duality pairs and Frobenius pairs by applying some known results in [5] and [19], and give some characterizations of rings by Frobenius pairs.

The recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne in a geometric setting in [2], which plays an important role in algebraic geometry and in representation theory. Gillespie described a general correspondence between projective (injective) recollements of triangulated categories and projective (injective) cotorsion pairs in [12]. This provides a model category description of these recollement situations. In this paper, we investigate special Gorenstein $(\mathcal{L},\mathcal{A})$-projective modules, called strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective modules. We prove that for a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G, $({}^{\perp}(G^{\perp}), G^{\perp})$ is a projective cotorsion pair, cogenerated by a set. Based on this result, we construct some recollements. These recollements involve complexes built from $G^{\perp}$ or ${}^{\perp}(G^{\perp})$. In particular, when $(\mathcal{L},\mathcal{A})$ is the level duality pair, one can get some specific examples.

This paper is organized as follows. In Section 2, we give some notions and basic facts. In Section 3, we give some equivalent characterizations of Gorenstein $(\mathcal{L},\mathcal{A})$-projective modules and construct some model structures. Moreover, we describe some rings by Frobenius pairs. In Section 4, we investigate strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective modules and construct some model structures and recollements associated to them.

2. Preliminaries

We recall some notions and basic facts which we need in the later sections.

Frobenius pairs. Let $\mathcal{C}$ be an abelian category, $\mathcal{X}$, $\mathcal{Y} \subseteq \mathcal{C}$ two classes of objects of $\mathcal{C}$ which can be also regarded as full subcategories of $\mathcal{C}$, and M, N objects in $\mathcal{C}$. The relative projective dimension of M with respect to $\mathcal{X}$ is defined as $\mathrm{pd}_{\mathcal{X}}(M) = \min\{n \geqslant 0 \,|\, \mathrm{Ext}^j_{\mathcal{C}}(M,\mathcal{X}) = 0 \text{ for every } j > n\}$. The relative injective dimension of N with respect to $\mathcal{Y}$ is defined as $\mathrm{id}_{\mathcal{Y}}(N) = \min\{n \geqslant 0 \,|\, \mathrm{Ext}^j_{\mathcal{C}}(\mathcal{Y},N) = 0 \text{ for every } j > n\}$. Furthermore, we set $\mathrm{pd}_{\mathcal{X}}\mathcal{Y} = \sup\{\mathrm{pd}_{\mathcal{X}}(Y) \,|\, Y \in \mathcal{Y}\}$ and $\mathrm{id}_{\mathcal{X}}\mathcal{Y} = \sup\{\mathrm{id}_{\mathcal{X}}(Y) \,|\, Y \in \mathcal{Y}\}$.

The class $\mathcal{X}$ is left thick if it is closed under direct summands, extensions and kernels of epimorphisms in $\mathcal{C}$. $\mathcal{X}$ is thick if it is left thick and closed under cokernels of monomorphisms in $\mathcal{C}$. Let $(\mathcal{X},\omega)$ be a pair of classes of objects in $\mathcal{C}$. It is said that ω is $\mathcal{X}$-injective if $\mathrm{id}_{\mathcal{X}}\omega = 0$. ω is called a relative cogenerator in $\mathcal{X}$ if $\omega \subseteq \mathcal{X}$ and for any $X \in \mathcal{X}$, there exists a short exact sequence $0 \to X \to W \to X^{'} \to 0$ with $W \in \omega$ and $X^{'} \in \mathcal{X}$. Definitions of $\mathcal{X}$-projective and a relative generator in $\mathcal{X}$ are dual. $(\mathcal{X},\omega)$ is called a left Frobenius pair if $\mathcal{X}$ is a left thick class, ω is an $\mathcal{X}$-injective relative cogenerator in $\mathcal{X}$ and ω is closed under direct summands in $\mathcal{C}$. A left Frobenius pair $(\mathcal{X},\omega)$ is strong if ω is an $\mathcal{X}$-projective relative generator in $\mathcal{X}$.

The $\mathcal{X}$-resolution dimension of M is the smallest non-negative integer n such that there is an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$ with each $X_i \in \mathcal{X}$. If such n doesn't exist, we say that the $\mathcal{X}$-resolution dimension of M is infinite. We denote by $\mathcal{X}^\wedge$ the class of objects in $\mathcal{C}$ having a finite $\mathcal{X}$-resolution dimension.

Let $(\mathcal{X},\mathcal{Y})$ be a pair of classes of objects in $\mathcal{C}$ and $\omega = \mathcal{X} \cap \mathcal{Y}$. We say that $(\mathcal{X},\mathcal{Y})$ is a left Auslander-Buchweitz-context (left AB-context for short) if the pair $(\mathcal{X},\omega)$ is a left Frobenius pair, $\mathcal{Y}$ is thick and $\mathcal{Y} \subseteq \mathcal{X}^\wedge$.

Exact categories. An exact category is a pair $(\mathcal{B},\tau)$ consisting of an additive category $\mathcal{B}$ and an exact structure τ on $\mathcal{B}$. Elements of τ are called short exact sequences.

An exact category $(\mathcal{B},\tau)$ is a Frobenius category if $(\mathcal{B},\tau)$ has enough projectives and enough injectives such that the projectives coincide with the injectives. For any objects $M,\ N \in \mathcal{B}$, let $\mathcal{P}(M,N)$ denote the abelian group of morphisms from M to N factoring through some projective object. Furthermore, the stable category of $\mathcal{B}$ denotes $\underline{\mathcal{B}} := \mathcal{B}/\mathcal{P}$, where the objects of $\underline{\mathcal{B}}$ are the same as that of $\mathcal{B}$ and $\mathrm{Hom}_{\underline{\mathcal{B}}}(M,N) := \mathrm{Hom}_{\mathcal{B}}(M,N)/\mathcal{P}(M,N)$. It is well known that $\underline{\mathcal{B}}$ is a triangulated category.

Recollements. Let $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ be triangulated categories. A recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram of triangulated functors

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

satisfying the following conditions:

(R1) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,

(R2) $j^* i_* = 0$,

(R3) i_*, $j_!$ and j_* are full embeddings,

(R4) any object X in $\mathcal{T}$ determines distinguished triangles $i_*i^!X \to X \to j_*j^*X \to (i_*i^!X)[1]$ and $j_!j^*X \to X \to i_*i^*X \to (j_!j^*X)[1]$ (see [2]).

Cotorsion pairs. Let $\mathcal{C}$ be an abelian category. A cotorsion pair is a pair $(\mathcal{X},\mathcal{Y})$ of classes of objects in $\mathcal{C}$ such that $\mathcal{X}^\perp = \mathcal{Y}$ and $\mathcal{X} = {}^\perp\mathcal{Y}$, where $\mathcal{X}^\perp = \{C \in \mathcal{C} \,|\, \mathrm{Ext}^1_{\mathcal{C}}(X,C) = 0,\ \forall\ X \in \mathcal{X}\}$ and ${}^\perp\mathcal{Y} = \{B \in \mathcal{C} \,|\, \mathrm{Ext}^1_{\mathcal{C}}(B,Y) = 0,\ \forall\ Y \in \mathcal{Y}\}$. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is said to be complete if it has enough projectives and injectives, i.e., for any object $C \in \mathcal{C}$, there are exact sequences $0 \to Y \to X \to C \to 0$ and $0 \to C \to Y' \to X' \to 0$, respectively, with $Y,\ Y' \in \mathcal{Y}$ and $X,\ X' \in \mathcal{X}$. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is said to be hereditary if $\mathrm{Ext}^i_{\mathcal{C}}(X,Y) = 0$ for all $X \in \mathcal{X}, Y \in \mathcal{Y}$ and all $i \geqslant 1$. It follows from [12, Lemma 2.3] that for a hereditary cotorsion pair $(\mathcal{X},\mathcal{Y})$, $\mathcal{X}$ is closed under kernels of epimorphisms and $\mathcal{Y}$ is closed under cokernels of monomorphisms. In addition, assume that $\mathcal{C}$ has enough projectives. We call a complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ a projective cotorsion pair if $\mathcal{Y}$ is thick and $\mathcal{X} \cap \mathcal{Y}$ coincides with the class of projective objects.

Unless stated to the contrary, we assume in the following that R is an associative ring with an identity, and all modules are left R-modules. R-Mod denotes the category of all left R-modules. $\mathcal{P}$ denotes the class of all projective left R-modules. $\mathcal{F}$ denotes the class of all flat left R-modules. For an R-module M, $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denotes the character module of M and the module M^+ is a right R-module. For some unexplained results, we refer the reader to [4, 8, 9, 15, 17, 18].

Complexes. A complex is a sequence of R-modules

$$C = \cdots \to C_2 \overset{d_2}{\to} C_1 \overset{d_1}{\to} C_0 \overset{d_0}{\to} C_{-1} \overset{d_{-1}}{\to} C_{-2} \to \cdots$$

together with homomorphisms such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The nth cycle (nth boundary) of C is defined as $\mathrm{Ker}d_n$ ($\mathrm{Im}d_{n+1}$) and is denoted by $Z_n(C)$ ($B_n(C)$). C is called exact or acyclic if $\mathrm{Ker}d_n = \mathrm{Im}d_{n+1}$ for each $n \in \mathbb{Z}$. $\mathcal{E}$ denotes the class of all exact complexes. We use $\mathrm{Ch}(R)$ to denote the category of complexes of R-modules.

Duality pairs. A duality pair over R is a pair $(\mathcal{L}, \mathcal{A})$, where $\mathcal{L}$ is a class of left R-modules and $\mathcal{A}$ is a class of right R-modules, satisfying the following conditions:

(1) $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{L}, \mathcal{A})$ is called perfect if $\mathcal{L}$ contains the module ${}_RR$, and is closed under coproducts and extensions. $\{\mathcal{L}, \mathcal{A}\}$ is a symmetric duality pair over R if $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{L})$ are duality pairs. A duality pair $(\mathcal{L}, \mathcal{A})$ is complete if $\{\mathcal{L}, \mathcal{A}\}$ is a symmetric duality pair and $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair over R.

Throughout this paper, $(\mathcal{L}, \mathcal{A})$ stands for a complete duality pair.

3. Gorenstein homological algebra relative to a duality pair and model structures

The goal of this section is to investigate Gorenstein homological algebra with respect to a given complete duality pair $(\mathcal{L}, \mathcal{A})$ and construct some model structures.

An R-module M is called Gorenstein $(\mathcal{L}, \mathcal{A})$-projective if there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence

$$\mathbb{P} : \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

with each $P_i \in \mathcal{P}$ such that $M \cong \mathrm{Ker}(P_{-1} \to P_{-2})$. $\mathcal{GP}$ denotes the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules (see [13]). It is clear that all the kernels, the images and the cokernels of $\mathbb{P}$ are in $\mathcal{GP}$. $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$. $\mathcal{P} \subseteq \mathcal{GP}$ via the exact sequence $0 \to P \xrightarrow{1} P \to 0$ for any $P \in \mathcal{P}$. It follows from [13] that $\mathcal{GP}$ is closed under direct sums, extensions, direct summands and kernels of epimorphisms. Since the perfect duality pair $(\mathcal{L}, \mathcal{A})$ implies $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{L}$ by [13, Proposition 2.3], a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module is Ding projective, and of course, it is Gorenstein projective (see [6,7]). We use $\mathcal{DP}$ and $\mathcal{GP}(R)$ to denote the class of all Ding projective modules and the class of all Gorenstein projective modules, respectively.

The following result gives some equivalent characterizations of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules.

Proposition 3.1. *For any R-module M, the following conditions are equivalent.*

(1) *M is a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module.*

(2) *There exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence*

$$0 \to M \to P_{-1} \to P_{-2} \to P_{-3} \to \cdots$$

with each $P_i \in \mathcal{P}$ and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$.

(3) *There exists an exact sequence $0 \to M \to P \to G \to 0$ with $P \in \mathcal{P}$ and $G \in \mathcal{GP}$.*

(4) *There exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_1,\ G_0 \in \mathcal{GP}$ and $\mathrm{Ext}^1_R(M, L) = 0$ for any $L \in \mathcal{L}$.*

(5) *There exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence*

$$0 \to M \to G_{-1} \to G_{-2} \to G_{-3} \to \cdots$$

with each $G_i \in \mathcal{GP}$ and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$.

(6) *There exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence*

$$\mathbb{P} : \cdots \to G_1 \to G_0 \to G_{-1} \to G_{-2} \to \cdots$$

with each $G_i \in \mathcal{GP}$ such that $M \cong \mathit{Ker}(G_{-1} \to G_{-2})$.

(7) *There exists some class of R-modules $\mathcal{U}$ with $\mathcal{P} \subseteq \mathcal{U} \subseteq \mathcal{GP}$ and there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence*

$$0 \to M \to U_{-1} \to U_{-2} \to U_{-3} \to \cdots$$

with each $U_i \in \mathcal{U}$ and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$.

(8) *There exists some class of R-modules $\mathcal{U}$ with $\mathcal{P} \subseteq \mathcal{U} \subseteq \mathcal{GP}$ and there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence*

$$\mathbb{P} : \cdots \to U_1 \to U_0 \to U_{-1} \to U_{-2} \to \cdots$$

with each $U_i \in \mathcal{U}$ such that $M \cong \mathit{Ker}(U_{-1} \to U_{-2})$.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Pick a projective resolution of M, $\cdots \to P_1 \to P_0 \to M \to 0$. By (2), the exact sequence

$$\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

is $\mathrm{Hom}_R(-,\mathcal{L})$-exact and each $P_i \in \mathcal{P}$. Thus M is a Gorenstein $(\mathcal{L},\mathcal{A})$-projective module.

(1) $\Rightarrow$ (3) Since M is a Gorenstein $(\mathcal{L},\mathcal{A})$-projective module, there exists a $\mathrm{Hom}_R(-,\mathcal{L})$-exact exact sequence

$$\mathbb{P}: \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

with each $P_i \in \mathcal{P}$ such that $M \cong \mathrm{Ker}(P_{-1} \to P_{-2})$. Let $P = P_{-1}$ and $G = \mathrm{Ker}(P_{-2} \to P_{-3})$. Then $0 \to M \to P \to G \to 0$ is an exact sequence with $P \in \mathcal{P}$ and $G \in \mathcal{GP}$, as desired.

(3) $\Rightarrow$ (2) Since $G \in \mathcal{GP}$, there exists a $\mathrm{Hom}_R(-,\mathcal{L})$-exact exact sequence

$$0 \to G \to P_{-1} \to P_{-2} \to P_{-3} \to \cdots$$

with each $P_i \in \mathcal{P}$ and $\mathrm{Ext}^i_R(G, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$. Then the exact sequence $0 \to M \to P \to G \to 0$ is $\mathrm{Hom}_R(-,\mathcal{L})$-exact and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$ by dimension shift. So the exact sequence

$$0 \to M \to P \to P_{-1} \to P_{-2} \to P_{-3} \to \cdots$$

is $\mathrm{Hom}_R(-,\mathcal{L})$-exact with $P \in \mathcal{P}$ and each $P_i \in \mathcal{P}$ and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$.

(1) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) Since $G_1 \in \mathcal{GP}$, there exists an exact sequence $0 \to G_1 \to P \to G \to 0$ with $P \in \mathcal{P}$ and $G \in \mathcal{GP}$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $G_0,\ G \in \mathcal{GP}$ and $\mathcal{GP}$ is closed under extensions, $X \in \mathcal{GP}$. The sequence $0 \to P \to X \to M \to 0$ is split since $\mathrm{Ext}^1_R(M, P) = 0$. Then M is a direct summand of X. So M is a Gorenstein $(\mathcal{L},\mathcal{A})$-projective module.

(2) $\Rightarrow$ (5) It immediately follows from $\mathcal{P} \subseteq \mathcal{GP}$.

(5) $\Rightarrow$ (2) By assumption, there exists a $\mathrm{Hom}_R(-,\mathcal{L})$-exact exact sequence $0 \to M \to G_{-1} \to K_{-1} \to 0$, where we put $K_{-1} = \mathrm{Ker}(G_{-2} \to G_{-3})$ for the exact sequence in (5). Since $G_{-1} \in \mathcal{GP}$, there exists an exact sequence

$0 \to G_{-1} \to P_{-1} \to G \to 0$ with $P_{-1} \in \mathcal{P}$ and $G \in \mathcal{GP}$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G_{-1} \arrow[r] \arrow[d] & K_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & P_{-1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

One easily checks that $0 \to M \to P_{-1} \to X \to 0$ is $\mathrm{Hom}_R(-, \mathcal{L})$-exact and $\mathrm{Ext}^i_R(X, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$. Pick an exact sequence $0 \to K_{-1} \to G_{-2} \to K_{-2} \to 0$ with $K_{-2} = \mathrm{Ker}(G_{-3} \to G_{-4})$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K_{-1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_{-2} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & G \arrow[r] & 0 \\
 & K_{-2} \arrow[r, equal] \arrow[d] & K_{-2} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since G_{-2}, $G \in \mathcal{GP}$, $Y \in \mathcal{GP}$. Because $0 \to K_{-1} \to G_{-2} \to K_{-2} \to 0$ is $\mathrm{Hom}_R(-, \mathcal{L})$-exact, $G \in \mathcal{GP}$ and Snake lemma, $0 \to X \to Y \to K_{-2} \to 0$ is $\mathrm{Hom}_R(-, \mathcal{L})$-exact. By repeating this process, we get a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence

$$0 \to M \to P_{-1} \to P_{-2} \to P_{-3} \to \cdots$$

with each $P_i \in \mathcal{P}$.

(1) $\Rightarrow$ (6) It immediately follows from the definition of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules and $\mathcal{P} \subseteq \mathcal{GP}$.

(6) ⇒ (5) By assumption, one easily gets a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence

$$0 \to M \to G_{-1} \to G_{-2} \to G_{-3} \to \cdots$$

with each $G_i \in \mathcal{GP}$. By Five lemma and dimension shift, one can obtain that $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$.

(2) ⇒ (7) One can set $\mathcal{U} = \mathcal{P}$.

(7) ⇒ (5) It immediately follows from $\mathcal{U} \subseteq \mathcal{GP}$.

(1) ⇒ (8) Set $\mathcal{U} = \mathcal{P}$. It immediately follows from the definition of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules.

(8) ⇒ (6) It immediately follows from $\mathcal{U} \subseteq \mathcal{GP}$. This completes the proof. □

Denote ${}^{\perp\infty}\mathcal{L} := \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(X, L) = 0, \ \forall\ L \in \mathcal{L} \text{ and } \forall\ i \geqslant 1\}$. It is easy to see that ${}^{\perp\infty}\mathcal{L}$ is the class of $\mathcal{L}$-projective modules.

Proposition 3.2. *Assume that $\mathcal{L}$ is closed under kernels of epimorphisms. Then $\mathcal{GP} = \mathcal{DP} \cap {}^{\perp\infty}\mathcal{L}$.*

Proof. It is easy to see that $\mathcal{GP} \subseteq \mathcal{DP} \cap {}^{\perp\infty}\mathcal{L}$. Next, let $M \in \mathcal{DP} \cap {}^{\perp\infty}\mathcal{L}$. Then there is an exact sequence $0 \to M \to P_0 \to M_0 \to 0$ with $P_0 \in \mathcal{P}$ and $M_0 \in \mathcal{DP}$, and $\mathrm{Ext}^i_R(M, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$. One can get $\mathrm{Ext}^i_R(M_0, L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 2$. We have an exact sequence $0 \to L_0 \to P \to L \to 0$ with $P \in \mathcal{P}$. By assumption, $L_0 \in \mathcal{L}$. We get an exact sequence $0 = \mathrm{Ext}^1_R(M_0, P) \to \mathrm{Ext}^1_R(M_0, L) \to \mathrm{Ext}^2_R(M_0, L_0) = 0$. Thus $\mathrm{Ext}^1_R(M_0, L) = 0$. Note that $M_0 \in \mathcal{DP} \cap {}^{\perp\infty}\mathcal{L}$. Continuing this process, we get a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence $0 \to M \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ with each $P_i \in \mathcal{P}$. By Proposition 3.1, M is a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module. Therefore, $\mathcal{GP} = \mathcal{DP} \cap {}^{\perp\infty}\mathcal{L}$. □

Remark 3.3. Suppose that $\mathcal{L}$ is closed under kernels of epimorphisms. With the similar method in the proof of Proposition 3.2, one can get that $\mathcal{GP} = \mathcal{GP}(R) \cap {}^{\perp\infty}\mathcal{L}$.

Proposition 3.4. *Let M be a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module. Then the following conditions are equivalent.*

(1) *M is projective.*

(2) *M is of finite projective dimension.*

(3) *M is flat.*

(4) *M is of finite flat dimension.*

(5) *M is in $\mathcal{L}$.*

(6) *M is of finite $\mathcal{L}$-resolution dimension.*

Proof. (1) ⇒ (2), (1) ⇒ (3) ⇒ (5), (3) ⇒ (4) and (5) ⇒ (6) are obvious.

(2) ⇒ (1) Assume $pd_R M = n < \infty$. Since M is a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module, there exists an exact sequence

$$0 \to G_n \to P_{n-1} \overset{d_{n-1}}{\to} P_{n-2} \overset{d_{n-2}}{\to} \cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0$$

with each $P_i \in \mathcal{P}$ and each $\operatorname{Ker} d_i \in \mathcal{GP}$. Then G_n is projective. Put $G_{n-1} = \operatorname{Ker} d_{n-2}$. Note that $0 \to G_n \to P_{n-1} \to G_{n-1} \to 0$ is split. Then G_{n-1} is projective. Repeating this process, we get that M is projective.

(4) $\Rightarrow$ (1) It can be immediately given by the similar way in (2) $\Rightarrow$ (1).

(6) $\Rightarrow$ (5) Let the $\mathcal{L}$-resolution dimension of M be n. Then there exists an exact sequence

$$0 \to L_n \to L_{n-1} \to L_{n-2} \to \cdots \to L_2 \to L_1 \to L_0 \to M \to 0$$

with each $L_i \in \mathcal{L}$. Since $L_n,\ L_{n-1} \in \mathcal{L}$ and $M \in \mathcal{GP}$, by dimension shift, we have $\operatorname{Ext}^i_R(M, K_{n-1}) = 0$ for all $i \geqslant 1$, where $K_{n-1} = \operatorname{Im}(L_{n-1} \to L_{n-2})$. By continuing this process, $\operatorname{Ext}^1_R(M, K_1) = 0$ where $K_1 = \operatorname{Im}(L_1 \to L_0)$. Then $0 \to K_1 \to L_0 \to M \to 0$ splits. Since $(\mathcal{L}, \mathcal{A})$ is a complete duality pair, $\mathcal{L}$ is closed under direct summands. Then $M \in \mathcal{L}$.

At last, we prove (5) $\Rightarrow$ (1). Since M is a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module, there exists an exact sequence $0 \to M \to P \to G \to 0$ with $P \in \mathcal{P}$ and $G \in \mathcal{GP}$ by Proposition 3.1. By assumption, $M \in \mathcal{L}$, $\operatorname{Ext}^1_R(G, M) = 0$, and hence $0 \to M \to P \to G \to 0$ splits. So M is projective. □

The proposition above shows that $\mathcal{GP} \cap \mathcal{P}^\wedge = \mathcal{GP} \cap \mathcal{F}^\wedge = \mathcal{GP} \cap \mathcal{L}^\wedge = \mathcal{GP} \cap \mathcal{L} = \mathcal{GP} \cap \mathcal{F} = \mathcal{P}$, which is useful in the sequel.

Remark 3.5. If R has finite weak global dimension, then M is Gorenstein $(\mathcal{L}, \mathcal{A})$-projective if and only if M is projective. For instance, a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module is projective over the von Neumann regular ring. In particular, if R has finite global dimension, then M is Gorenstein $(\mathcal{L}, \mathcal{A})$-projective if and only if M is projective.

Since $\mathcal{GP}$ is closed under extensions, it follows from [18, 4.1] that $(\mathcal{GP}, \tau)$ is an exact category, where τ denotes the class of all exact sequences of the form $0 \to L \to M \to N \to 0$ with all terms in $\mathcal{GP}$. If readers want to know more details about exact categories, please refer to [4, 10]. Next, we will devote to showing that $\mathcal{GP}^\wedge$ is closed under extensions. Further, $(\mathcal{GP}^\wedge, \tau)$ is an exact category.

Proposition 3.6. *Let n be the $\mathcal{GP}$-resolution dimension of M. Then for each exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \overset{d_0}{\to} M \to 0$ with each $P_i \in \mathcal{P}$, K_n is Gorenstein $(\mathcal{L}, \mathcal{A})$-projective.*

Proof. By assumption, there exists an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_1 \overset{g_1}{\to} G_0 \overset{g_0}{\to} M \to 0$ with each $G_i \in \mathcal{GP}$. For the exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \overset{d_0}{\to} M \to 0$, since each $P_i \in \mathcal{P}$, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r, "g_1"] & G_0 \arrow[r, "g_0"] & M \arrow[r] & 0
\end{tikzcd}$$

such that the mapping cone is exact. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & M \arrow[r, "1"] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_n \arrow[r] \arrow[d, equal] & G_n \oplus P_{n-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & G_1 \oplus P_0 \arrow[r, "\beta"] \arrow[d, equal] & G_0 \oplus M \arrow[r, "{(g_0,1)}"] \arrow[d, "{(1,0)}"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_n \arrow[r] & G_n \oplus P_{n-1} \arrow[r] & \cdots \arrow[r] & G_1 \oplus P_0 \arrow[r, "{(g_1,f_0)}"] & G_0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $\alpha = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $\beta = \begin{pmatrix} g_1 & f_0 \\ 0 & -d_0 \end{pmatrix}$. In fact, this is a short exact sequence of complexes. Thus the sequence $0 \to K_n \to G_n \oplus P_{n-1} \to \cdots \to G_2 \oplus P_1 \to G_1 \oplus P_0 \to G_0 \to 0$ is exact with $G_0 \in \mathcal{GP}$ and each $G_i \oplus P_{i-1} \in \mathcal{GP}$. Because $\mathcal{GP}$ is closed under kernels of epimorphisms, K_n is Gorenstein $(\mathcal{L}, \mathcal{A})$-projective. $\square$

Proposition 3.7. $\mathcal{GP}^{\wedge}$ *is closed under extensions.*

Proof. By Proposition 3.6, it is obtained by a version of the Horseshoe lemma. $\square$

We recall Hovey-Gillespie correspondence (refer to [10, Theorem 3.3]). Assume that $(\mathcal{B}, \tau)$ is an exact category with an exact model structure. Let $\mathcal{Q}$ be the class of cofibrant objects, $\mathcal{R}$ the class of fibrant objects and $\mathcal{W}$ the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathcal{B}$ and both $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in $\mathcal{B}$. If $(\mathcal{B}, \tau)$ is weakly idempotent complete, then the converse holds. That is, given a thick subcategory $\mathcal{W}$ and classes $\mathcal{Q}$ and $\mathcal{R}$ making $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ complete cotorsion pairs, then there is an exact model structure on $\mathcal{B}$, where $\mathcal{Q}$ is the class of cofibrant objects, $\mathcal{R}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects. Hence, we denote an exact model structure as a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, and call it a Hovey triple. If readers want to further learn model structures, please refer to [10–13, 17].

Example 3.8. $(\mathcal{GP}, \mathcal{P}^{\wedge})$ is a left AB-context in R-Mod. Furthermore, $(\mathcal{GP}, \mathcal{P})$ is a strong left Frobenius pair by [1, Proposition 6.10]. In addition, $\mathcal{GP}$ is a Frobenius category with $\mathcal{P}$ a class of relative projective-injective objects and $\underline{\mathcal{GP}} := \mathcal{GP}/\mathcal{P}$ is a triangulated category. There exists a Frobenius model structure $(\mathcal{GP}, \mathcal{P}, \mathcal{GP})$ on the exact category $\mathcal{GP}$, where $\mathcal{P}$ is the class of trivial objects, and $\mathcal{GP}$ is both the class of cofibrant objects and the class of fibrant objects. There exists an exact model structure $(\mathcal{GP}, \mathcal{P}^{\wedge}, \mathcal{GP}^{\wedge})$ on the exact category $\mathcal{GP}^{\wedge}$, where $\mathcal{GP}$ is the class of cofibrant objects, $\mathcal{P}^{\wedge}$ is the class of trivial objects and $\mathcal{GP}^{\wedge}$ is the class of fibrant objects.

Remark 3.9. The canonical example of a complete duality pair is the level duality pair $(\mathcal{L}, \mathcal{A})$ over any ring, given in [3], where $\mathcal{L}$ represents the class of level modules and $\mathcal{A}$ represents the class of absolutely clean modules. In this case, the class of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules coincides with the class of Gorenstein AC-projective modules, denoted by $\mathcal{G}_{ac}\mathcal{P}$ (refer to [3]). We obtain that $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{P}^{\wedge})$ is a left AB-context in R-Mod. $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{P})$ is a strong left Frobenius pair by [1, Corollary 6.11]. Furthermore, $\mathcal{G}_{ac}\mathcal{P}$ is a Frobenius category with $\mathcal{P}$ a class of relative projective-injective objects and hence $\underline{\mathcal{G}_{ac}\mathcal{P}} := \mathcal{G}_{ac}\mathcal{P}/\mathcal{P}$ is a triangulated category. Moreover, there exists a Frobenius model structure $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{P}, \mathcal{G}_{ac}\mathcal{P})$ on the exact category $\mathcal{G}_{ac}\mathcal{P}$ and there exists an exact model structure $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{P}^{\wedge}, \mathcal{G}_{ac}\mathcal{P}^{\wedge})$ on the exact category $\mathcal{G}_{ac}\mathcal{P}^{\wedge}$.

Actually, one can get that $(\mathcal{GP}, \mathcal{F})$ and $(\mathcal{GP}, \mathcal{L})$ are left Frobenius pairs when $\mathcal{F} \subseteq \mathcal{GP}$ and $\mathcal{L} \subseteq \mathcal{GP}$, respectively. In the following, we give some characterizations of rings via Frobenius pairs.

Recall that R is a left perfect ring if every flat left R-module is projective.

Proposition 3.10. *The following conditions are equivalent for any ring R.*

(1) $(\mathcal{GP}, \mathcal{F})$ *is a strong left Frobenius pair.*
(2) $(\mathcal{GP}, \mathcal{F})$ *is a left Frobenius pair.*
(3) R *is a left perfect ring.*

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Assume that $(\mathcal{GP}, \mathcal{F})$ is a left Frobenius pair. Then $\mathcal{F}$ is a relative cogenerator in $\mathcal{GP}$. For any $M \in \mathcal{F}$, there exists an exact sequence $0 \to M \to P \to G \to 0$ with $P \in \mathcal{P}$ and $G \in \mathcal{GP}$ by Proposition 3.1. The sequence $0 \to M \to P \to G \to 0$ splits since $\mathrm{Ext}^1_R(G, M) = 0$. Then M is projective. So R is a left perfect ring.

(3) $\Rightarrow$ (1) When R is a left perfect ring, $\mathcal{F}$ is exactly $\mathcal{P}$. It immediately follows from [1, Proposition 6.10]. $\square$

Similarly, we obtain the next result.

Proposition 3.11. *The following conditions are equivalent for any ring R.*

(1) $(\mathcal{GP}, \mathcal{L})$ *is a strong left Frobenius pair.*
(2) $(\mathcal{GP}, \mathcal{L})$ *is a left Frobenius pair.*
(3) $\mathcal{L} = \mathcal{P}$.

In this case, R is a left perfect ring.

It is known that R is a right coherent ring if and only if all level left R-modules are flat.

Example 3.12. Let $(\mathcal{L}, \mathcal{A})$ be the level duality pair over any ring R. Then the following conditions are equivalent.
(1) $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{L})$ is a strong left Frobenius pair.
(2) $(\mathcal{G}_{ac}\mathcal{P}, \mathcal{L})$ is a left Frobenius pair.
(3) $\mathcal{L} = \mathcal{P}$.
(4) R is a right coherent and left perfect ring.

An R-module M is called Gorenstein $(\mathcal{L}, \mathcal{A})$-flat if there exists an $\mathcal{A}\otimes_R-$-exact exact sequence

$$\mathbb{F}: \cdots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

with each $F_i \in \mathcal{F}$ such that $M \cong \mathrm{Ker}(F_{-1} \to F_{-2})$. $\mathcal{GF}$ denotes the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules (see [13]).

Many scholars have been interested in when Gorenstein projective modules are Gorenstein flat for two decades. However, when Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are Gorenstein $(\mathcal{L}, \mathcal{A})$-flat is trivial.

Remark 3.13. By [3, Theorem A.6], $\mathrm{Hom}_R(\mathbb{P}, \mathcal{L})$ is exact if and only if $\mathcal{A}\otimes_R\mathbb{P}$ is exact, where $\mathbb{P}$ is an exact sequence of projective modules. It follows from definitions of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective and Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules that Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are Gorenstein $(\mathcal{L}, \mathcal{A})$-flat over any ring. In particular, Gorenstein AC-projective modules are Gorenstein AC-flat.

In [19], the authors introduced and studied a kind of Gorenstein $(\mathcal{X}, \mathcal{Y})$-flat modules with respect to two classes of modules $\mathcal{X}$ and $\mathcal{Y}$. An R-module M is called Gorenstein $(\mathcal{X}, \mathcal{Y})$-flat if there exists a $\mathcal{Y}\otimes_R-$-exact exact sequence

$$\mathbb{X}: \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

with each $X_i \in \mathcal{X}$ such that $M \cong \mathrm{Ker}(X_{-1} \to X_{-2})$. Denote $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$ the class of all Gorenstein $(\mathcal{X}, \mathcal{Y})$-flat modules.

Remark 3.14. It is well known that $\mathcal{P} \subseteq \mathcal{GP} \subseteq \mathcal{GF}$ and $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{GF}$. If $\mathcal{X} = \mathcal{L}$ and $\mathcal{Y} = \mathcal{A}$, then $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{L} \subseteq \mathcal{GF}_{(\mathcal{L},\mathcal{A})}(R)$ and $\mathcal{GP} \subseteq \mathcal{GF} \subseteq \mathcal{GF}_{(\mathcal{L},\mathcal{A})}(R)$. Furthermore, if $\mathcal{L}$ is the class of flat left R-modules and $\mathcal{A}$ is the class of injective right R-modules over a right Noetherian ring, then $\mathcal{GF} = \mathcal{GF}_{(\mathcal{L},\mathcal{A})}(R)$.

Complete cotorsion pairs play an important role in constructing model structures. When $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair, $(\mathcal{L}, \mathcal{L}^{\perp})$ is a perfect cotorsion pair. This fact builds the bridge between duality pairs and cotorsion pairs. In [19, Proposition 2.18], the authors gave some equivalent characterizations of $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$ under some conditions, where $(\mathcal{X}, \mathcal{Y})$ is a complete duality pair. Let $(\mathcal{U}, \mathcal{V})$ be a complete and hereditary cotorsion pair. An R-module M is called Gorenstein $\mathcal{U}$-object if there exists a $\mathrm{Hom}_R(-, \mathcal{U} \cap \mathcal{V})$-exact exact sequence

$$\mathbb{U}: \cdots \to U_1 \to U_0 \to U_{-1} \to U_{-2} \to \cdots$$

with each $U_i \in \mathcal{U}$ such that $M \cong \mathrm{Ker}(U_{-1} \to U_{-2})$. Gorenstein $\mathcal{U}$-objects with respect to a complete and hereditary cotorsion pair $(\mathcal{U}, \mathcal{V})$ were introduced in [20]. Denote $\mathcal{G}(\mathcal{U})$ the class of Gorenstein $\mathcal{U}$-objects. We observe that $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$ is exactly $\mathcal{G}(\mathcal{X})$ under the conditions of [19, Proposition 2.18]. So one will obtain exact model structures associated to $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$.

Theorem 3.15. *Let $(\mathcal{X}, \mathcal{Y})$ be a complete duality pair, $\mathcal{X}$ closed under kernels of epimorphisms and $\mathrm{Tor}_i^R(Y, X) = 0$ for all $Y \in \mathcal{Y}$, $X \in \mathcal{X}$ and all $i \geqslant 1$. Then $(\mathcal{X}, \mathcal{X} \cap \mathcal{X}^{\perp})$ is a left Frobenius pair and $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R), (\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge})$ is a left AB-context in R-Mod. Furthermore, assume that $\mathcal{X}^{\perp}$ is closed under kernels of epimorphisms. Then $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}, \mathcal{X} \cap \mathcal{X}^{\perp})$ is a strong left Frobenius pair, $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$ is a Frobenius category with $\mathcal{X} \cap \mathcal{X}^{\perp}$ a class of relative projective-injective objects and $\underline{\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}} := \mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}/\mathcal{X} \cap \mathcal{X}^{\perp}$ is a triangulated category. In addition, there exists a Frobenius model structure $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}, \mathcal{X} \cap \mathcal{X}^{\perp}, \mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})$ on the exact category $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$, and there exists an exact model structure $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}, (\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge}, (\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge})$ on the exact category $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge}$, where $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$ is the class of cofibrant objects, $(\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge}$ is the class of trivial objects and $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge}$ is the class of fibrant objects.*

Proof. For one thing, since $(\mathcal{X}, \mathcal{Y})$ is a perfect duality pair, $(\mathcal{X}, \mathcal{X}^{\perp})$ is a perfect cotorsion pair by [16, Theorem 3.1]. Naturally, $(\mathcal{X}, \mathcal{X}^{\perp})$ is a complete cotorsion pair. For another, since $\mathcal{X}$ is closed under kernels of epimorphisms, $(\mathcal{X}, \mathcal{X}^{\perp})$ is a hereditary cotorsion pair by [12, Lemma 2.4]. $(\mathcal{X}, \mathcal{X}^{\perp})$ is a complete and hereditary cotorsion pair. Then $\mathrm{Ext}_R^i(X, Z) = 0$ for all $X \in \mathcal{X}$, $Z \in \mathcal{X}^{\perp}$ and all $i \geqslant 1$. So $\mathrm{id}_{\mathcal{X}} \mathcal{X} \cap \mathcal{X}^{\perp} = 0$, namely, $\mathcal{X} \cap \mathcal{X}^{\perp}$ is $\mathcal{X}$-injective. For any $X \in \mathcal{X}$, there exists an exact sequence $0 \to X \to Z \to X' \to 0$ with $Z \in \mathcal{X}^{\perp}$ and $X' \in \mathcal{X}$ by the completeness of $(\mathcal{X}, \mathcal{X}^{\perp})$. Because $\mathcal{X}$ is closed under extensions, $Z \in \mathcal{X} \cap \mathcal{X}^{\perp}$. Thus $\mathcal{X} \cap \mathcal{X}^{\perp}$ is a relative cogenerator in $\mathcal{X}$. Note that $\mathcal{X}$ is left thick and $\mathcal{X} \cap \mathcal{X}^{\perp}$ is closed under direct summands. So $(\mathcal{X}, \mathcal{X} \cap \mathcal{X}^{\perp})$ is a left Frobenius pair. By [19, Proposition 2.18], M is in $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$ if and only if there exists an exact sequence of left R-modules in $\mathcal{X}$

$$\mathbb{X} : \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

such that $M \cong \mathrm{Ker}(X_{-1} \to X_{-2})$ and $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{X}^{\perp})$ leaves the sequence exact. One can obtain that $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R)$ is exactly $\mathcal{G}(\mathcal{X})$. By [5, Proposition 4.1], $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R), (\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge})$ is a left AB-context in R-Mod. Now, assume that $\mathcal{X}^{\perp}$ is closed under kernels of epimorphisms. By [5, Theorem 4.2], $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}, \mathcal{X} \cap \mathcal{X}^{\perp})$ is a strong left Frobenius pair, $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$ is a Frobenius category with $\mathcal{X} \cap \mathcal{X}^{\perp}$ a class of relative projective-injective objects and $\underline{\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}} := \mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}/\mathcal{X} \cap \mathcal{X}^{\perp}$ is a triangulated category. Thus there exists a Frobenius model structure $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap$

$\mathcal{X}^{\perp}, \mathcal{X} \cap \mathcal{X}^{\perp}, \mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})$ on the exact category $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$, and there exists an exact model structure

$$(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}, (\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge}, (\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge})$$

on the exact category $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge}$, where $\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp}$ is the class of cofibrant objects, $(\mathcal{X} \cap \mathcal{X}^{\perp})^{\wedge}$ is the class of trivial objects and $(\mathcal{GF}_{(\mathcal{X},\mathcal{Y})}(R) \cap \mathcal{X}^{\perp})^{\wedge}$ is the class of fibrant objects. □

4. Strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules

In this section, we introduce and investigate strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules with respect to a given complete duality pair $(\mathcal{L}, \mathcal{A})$. Meanwhile, we attempt to construct some relevant model structures and recollements.

Definition 4.1. An R-module M is called strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective if there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence

$$\mathbb{P} : \cdots \to P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

with $P \in \mathcal{P}$ such that $M \cong \operatorname{Ker} f$.

We use $\mathcal{SGP}$ to denote the class of all strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules. One has $\mathcal{SGP} \subseteq \mathcal{GP}$. It is easy to obtain that $\mathcal{SGP}$ is closed under direct sums.

In the following, we give some homological characterizations of $\mathcal{SGP}$.

Proposition 4.2. *For any R-module M, the following conditions are equivalent.*

(1) *M is strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective.*

(2) *There exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence $0 \to M \to P \to M \to 0$ with $P \in \mathcal{P}$.*

(3) *There exists an exact sequence $0 \to M \to P \to M \to 0$ with $P \in \mathcal{P}$ and $\mathrm{Ext}^1_R(M, L) = 0$ for all $L \in \mathcal{L}$.*

(4) *There exists an exact sequence $0 \to M \to P \to M \to 0$ with $P \in \mathcal{P}$ and $\mathrm{Ext}^i_R(M, L) = 0$ for all $L \in \mathcal{L}$ and all $i \geqslant 1$.*

Proof. It is straightforward. □

Proposition 4.3. *Every projective module P is strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective.*

Proof. For a projective module P, there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence $0 \to P \xrightarrow{\alpha} P \oplus P \xrightarrow{(0,1)} P \to 0$ with $\alpha = \binom{1}{0}$. By Proposition 4.2, P is strongly Gorenstein $(\mathcal{L}, \mathcal{A})$-projective. □

It is clear that $\mathcal{P} \subseteq \mathcal{SGP} \subseteq \mathcal{GP} \subseteq \mathcal{GF} \subseteq \mathcal{GF}_{(\mathcal{L},\mathcal{A})}(R)$.

Proposition 4.4. *Let G be a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module. Then the following statements hold.*

(1) *$G^{\perp}$ is a thick subcategory of R-Mod.*

(2) *$({}^{\perp}(G^{\perp}),G^{\perp})$ is a projective cotorsion pair, cogenerated by a set.*

Proof. (1) Obviously, $G^{\perp}$ is closed under direct summands and extensions. Let $0 \to L \to M \to N \to 0$ be an exact sequence. At first, assume that $L \in G^{\perp}$ and $M \in G^{\perp}$. Then there is an exact sequence $0 = \mathrm{Ext}^1_R(G,M) \to \mathrm{Ext}^1_R(G,N) \to \mathrm{Ext}^2_R(G,L)$. Since G is strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective, there exists an exact sequence $0 \to G \to P \to G \to 0$ with $P \in \mathcal{P}$, which induces an exact sequence $0 = \mathrm{Ext}^1_R(G,L) \to \mathrm{Ext}^2_R(G,L) \to \mathrm{Ext}^2_R(P,L) = 0$. Then $\mathrm{Ext}^2_R(G,L) = 0$ and hence $\mathrm{Ext}^1_R(G,N) = 0$, that is $N \in G^{\perp}$. Next, assume that $M \in G^{\perp}$ and $N \in G^{\perp}$. Then there is an exact sequence $0 = \mathrm{Ext}^1_R(G,N) \to \mathrm{Ext}^2_R(G,L) \to \mathrm{Ext}^2_R(G,M)$=0. Hence $\mathrm{Ext}^2_R(G,L) = 0$. There is an exact sequence $0 = \mathrm{Ext}^1_R(P,L) \to \mathrm{Ext}^1_R(G,L) \to \mathrm{Ext}^2_R(G,L)$=0. Then $\mathrm{Ext}^1_R(G,L) = 0$. So $G^{\perp}$ is a thick subcategory of R-Mod.

(2) Put a set $\mathcal{S} = \{G\}$. By [15, Theorem 6.11], $({}^{\perp}(G^{\perp}),G^{\perp})$ is a complete cotorsion pair, cogenerated by the set $\mathcal{S}$. Since $\mathcal{P} \subseteq \mathcal{L}$, $\mathcal{P} \subseteq G^{\perp}$. It follows from [12, Proposition 3.7] that $({}^{\perp}(G^{\perp}),G^{\perp})$ is a projective cotorsion pair. □

Theorem 4.5. *Let G be a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module. Then there exists a hereditary abelian model structure $(\mathcal{GP},\mathcal{W},G^{\perp})$ on R-Mod, where $\mathcal{W}$ can be described in the following two ways*:

$$\begin{aligned}\mathcal{W} = \{W \in R\text{-Mod} \mid \exists \text{ an exact sequence } 0 \to W \to A \to B \to 0 \\ \text{with } B \in {}^{\perp}(G^{\perp}),\ A \in \mathcal{GP}^{\perp}\} \\ = \{W \in R\text{-Mod} \mid \exists \text{ an exact sequence } 0 \to A^{'} \to B^{'} \to W \to 0 \\ \text{with } B^{'} \in {}^{\perp}(G^{\perp}),\ A^{'} \in \mathcal{GP}^{\perp}\}.\end{aligned}$$

Proof. By [13, Theorem 4.9], we know that $(\mathcal{GP},\mathcal{GP}^{\perp})$ is a projective cotorsion pair. Because $(\mathcal{GP},\mathcal{GP}^{\perp})$ and $({}^{\perp}(G^{\perp}),G^{\perp})$ are two projective cotorsion pairs, $\mathcal{GP} \cap \mathcal{GP}^{\perp} = \mathcal{P} = {}^{\perp}(G^{\perp}) \cap G^{\perp}$. Since $G \in \mathcal{SGP} \subseteq \mathcal{GP}$, $\mathcal{GP}^{\perp} \subseteq G^{\perp}$ and ${}^{\perp}(G^{\perp}) \subseteq \mathcal{GP}$. It immediately follows from [11, Theorem 1.1]. □

Remark 4.6. When one chooses different strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G, the different model structure may be obtained.

The following result gives relations between $\mathcal{SGP}$ and $\mathcal{GP}$.

Proposition 4.7. *For any R-module M, the following conditions are equivalent.*

(1) *M is a Gorenstein $(\mathcal{L},\mathcal{A})$-projective module.*

(2) *M is a direct summand of some strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module.*

(3) *There exists a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G such that $M \in {}^{\perp}(G^{\perp})$.*

Proof. (1) $\Rightarrow$ (2) It is easy to prove this result.

(2) $\Rightarrow$ (3) Let M be a direct summand of a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G. Then $M \in {}^{\perp}(G^{\perp})$ since $G \in {}^{\perp}(G^{\perp})$.

(3) $\Rightarrow$ (1) By Proposition 4.4, we have an exact sequence $0 \to M \to P_0 \to L_0 \to 0$ with $P_0 \in G^{\perp}$ and $L_0 \in {}^{\perp}(G^{\perp})$. Thus $P_0 \in G^{\perp} \cap {}^{\perp}(G^{\perp}) = \mathcal{P}$. Continuing this process, we get an exact sequence $0 \to M \to P_0 \overset{f_0}{\to} P_{-1} \overset{f_{-1}}{\to} P_{-2} \overset{f_{-2}}{\to} \cdots$ with each $P_i \in \mathcal{P}$ and each $\operatorname{Ker} f_i \in {}^{\perp}(G^{\perp})$. Since $\mathcal{L} \subseteq G^{\perp}$, this exact sequence is $\operatorname{Hom}_R(-,\mathcal{L})$-exact. Note that $({}^{\perp}(G^{\perp}), G^{\perp})$ is a hereditary cotorsion pair, so $\operatorname{Ext}^i_R(M,L) = 0$ for any $L \in \mathcal{L}$ and any $i \geqslant 1$. By Proposition 3.1, M is a Gorenstein $(\mathcal{L},\mathcal{A})$-projective module. $\square$

In the following, we recall some notions and basic facts.

Let $\mathcal{S}$ be a class of R-modules. A complex X is in $dw\mathcal{S}$ if $X_j \in \mathcal{S}$ for any $j \in \mathbb{Z}$.

An exact complex X is in $ex\mathcal{S}$ if $X_j \in \mathcal{S}$ for any $j \in \mathbb{Z}$.

An exact complex X is in $\widetilde{\mathcal{S}}$ if $Z_j(X) \in \mathcal{S}$ for any $j \in \mathbb{Z}$.

Let $(\mathcal{S},\mathcal{T})$ be a cotorsion pair in R-Mod. A complex Y is a dg$\mathcal{S}$ complex if each $Y_j \in \mathcal{S}$ for any $j \in \mathbb{Z}$ and if each map $Y \to U$ is null homotopic for each complex $U \in \widetilde{\mathcal{T}}$. The definition of a dg$\mathcal{T}$ complex is dual. We use $dg\mathcal{S}$ and $dg\mathcal{T}$ to denote the class of all dg$\mathcal{S}$ complexes and the class of all dg$\mathcal{T}$ complexes respectively.

By Proposition 4.4, we know that for a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G, $({}^{\perp}(G^{\perp}), G^{\perp})$ is a projective cotorsion pair, cogenerated by a set $\{G\}$. It immediately follows from [12, Proposition 7.3] that there are six projective cotorsion pairs in $\operatorname{Ch}(R)$, cogenerated by sets, namely, $(dw^{\perp}(G^{\perp}), (dw^{\perp}(G^{\perp}))^{\perp})$, $(ex^{\perp}(G^{\perp}), (ex^{\perp}(G^{\perp}))^{\perp})$, $(\widetilde{{}^{\perp}(G^{\perp})}, dgG^{\perp})$, $({}^{\perp}(dwG^{\perp}), dwG^{\perp})$, $({}^{\perp}(exG^{\perp}), exG^{\perp})$, and $(dg^{\perp}(G^{\perp}), \widetilde{G^{\perp}})$. Of course, for $(\mathcal{P}, R\text{-Mod})$, there exist four distinct projective cotorsion pairs in $\operatorname{Ch}(R)$. They are $(dw\mathcal{P}, (dw\mathcal{P})^{\perp})$, $(ex\mathcal{P}, (ex\mathcal{P})^{\perp})$, $(\widetilde{\mathcal{P}}, \operatorname{Ch}(R))$, and $(dg\mathcal{P}, \mathcal{E})$. In the following, we will use these projective cotorsion pairs to construct some recollements.

Let $\mathcal{G}$ be a class of R-modules and $(\mathcal{S},\mathcal{T})$ a cotorsion pair in R-Mod. we use $D(R)$ to denote the derived category of R-modules, $K(\mathcal{G})$ to denote the homotopy category of $\mathcal{G}$, $K_{ex}(\mathcal{G})$ to denote the homotopy category consisting of exact complexes of $\mathcal{G}$, $K(dg\mathcal{S})$ to denote the homotopy category consisting of dg$\mathcal{S}$ complexes, $K_{ex}(dg\mathcal{S})$ to denote the homotopy category consisting of exact dg$\mathcal{S}$ complexes, $K({}^{\perp}(dw\mathcal{T}))$ to denote the homotopy category consisting of complexes in ${}^{\perp}(dw\mathcal{T})$, and $K({}^{\perp}(ex\mathcal{T}))$ to denote the homotopy category consisting of complexes in ${}^{\perp}(ex\mathcal{T})$.

Theorem 4.8. *There exist five recollements associated to a strongly Gorenstein $(\mathcal{L},\mathcal{A})$-projective module G, where three recollements are relative to the derived category $D(R)$ as follows*:

(1)

$$\begin{tikzcd} K_{ex}({}^\perp(G^\perp)) \arrow[r] & K({}^\perp(G^\perp)) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & D(R), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(2)

$$\begin{tikzcd} K_{ex}(dg^\perp(G^\perp)) \arrow[r] & K(dg^\perp(G^\perp)) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & D(R), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(3)

$$\begin{tikzcd} K({}^\perp(dwG^\perp)) \arrow[r] & K({}^\perp(exG^\perp)) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & D(R), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(4)

$$\begin{tikzcd} K(\mathcal{P}) \arrow[r] & K({}^\perp(G^\perp)) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & K({}^\perp(dwG^\perp)), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(5)

$$\begin{tikzcd} K_{ex}(\mathcal{P}) \arrow[r] & K({}^\perp(G^\perp)) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & K({}^\perp(exG^\perp)). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Proof. (1) We have projective cotorsion pairs $(dw^\perp(G^\perp), (dw^\perp(G^\perp))^\perp)$, $(ex^\perp(G^\perp), (ex^\perp(G^\perp))^\perp)$, and $(dg\mathcal{P}, \mathcal{E})$. They satisfy that $dg\mathcal{P} \subseteq dw^\perp(G^\perp)$ since $\mathcal{P} \subseteq {}^\perp(G^\perp)$ and $dw^\perp(G^\perp) \cap \mathcal{E} = ex^\perp(G^\perp)$. The first recollement follows from [12, Theorem 4.7] and $K(dg\mathcal{P}) \cong D(R)$.

(2) There are projective cotorsion pairs $(dg^\perp(G^\perp), \widetilde{G^\perp})$, $(\widetilde{{}^\perp(G^\perp)}, dgG^\perp)$, and $(dg\mathcal{P}, \mathcal{E})$. We know that $dg\mathcal{P} \subseteq dg^\perp(G^\perp)$, $dg^\perp(G^\perp) \cap \mathcal{E} = \widetilde{{}^\perp(G^\perp)}$, and $K(dg\mathcal{P}) \cong D(R)$. By [12, Theorem 4.7], the second recollement is obtained.

(3) Consider three projective cotorsion pairs $({}^\perp(exG^\perp), exG^\perp)$, $({}^\perp(dwG^\perp), dwG^\perp)$, and $(dg\mathcal{P}, \mathcal{E})$. Since $exG^\perp \subseteq \mathcal{E}$, we get $dg\mathcal{P} \subseteq {}^\perp(exG^\perp)$. By [8, Theorem 7.4.3], we have a Hovey triple $({}^\perp(exG^\perp), \mathcal{E}, dwG^\perp)$. Thus $\mathcal{E} \cap {}^\perp(exG^\perp) = {}^\perp(dwG^\perp)$. By [12, Theorem 4.7], we get the third recollement.

(4) $(dw^\perp(G^\perp), (dw^\perp(G^\perp))^\perp)$, $(dw\mathcal{P}, (dw\mathcal{P})^\perp)$, and $({}^\perp(dwG^\perp), dwG^\perp)$ are projective cotorsion pairs. By [12, Proposition 7.3], ${}^\perp(dwG^\perp) \subseteq dw^\perp(G^\perp)$. However, $dw^\perp(G^\perp) \cap dwG^\perp = dw({}^\perp(G^\perp) \cap G^\perp) = dw\mathcal{P}$. Also, it immediately follows from [12, Theorem 4.7].

(5) Pick three projective cotorsion pairs $(dw^\perp(G^\perp)$, $(dw^\perp(G^\perp))^\perp)$, $(ex\mathcal{P}, (ex\mathcal{P})^\perp)$, and $({}^\perp(exG^\perp), exG^\perp)$. We get that $exG^\perp \cap dw^\perp(G^\perp) = (\mathcal{E} \cap dwG^\perp) \cap dw^\perp(G^\perp) = \mathcal{E} \cap dw({}^\perp(G^\perp) \cap G^\perp) = ex\mathcal{P}$, and by [12, Proposition 7.3], ${}^\perp(exG^\perp) \subseteq dw^\perp(G^\perp)$. By [12, Theorem 4.7], we obtain the last recollement. This completes the proof. ☐

Remark 4.9. In this section, when $(\mathcal{L}, \mathcal{A})$ is the level duality pair, one can obtain some specific results about $\mathcal{G}_{ac}\mathcal{P}$.

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