Bull. Korean Math. Soc. **60** (2023), No. 2, pp. 425–441
https://doi.org/10.4134/BKMS.b220170
pISSN: 1015-8634 / eISSN: 2234-3016

ON MEROMORPHIC SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS OF FIRST ORDER IN SEVERAL COMPLEX VARIABLES

Qibin Cheng, Yezhou Li, and Zhixue Liu

Abstract. This paper is concerned with the value distribution for meromorphic solutions f of a class of nonlinear partial differential-difference equation of first order with small coefficients. We show that such solutions f are uniquely determined by the poles of f and the zeros of $f-c, f-d$ (counting multiplicities) for two distinct small functions c, d.

1. Introduction and the main theorem

For a point $z_0 \in \mathbb{C}^m$, f is holomorphic at z_0 if it can be written as $f(z) = \sum_{i=0}^{\infty} Q_i(z-z_0)$ on a neighborhood $U \subset \mathbb{C}^m$ of z_0, where the term $Q_i(z-z_0)$ is either identically zero or a homogeneous polynomial of degree i. We say f is a nonzero "meromorphic" function on $\mathbb{C}^m$ if for all $z_0 \in \mathbb{C}^m$, one can choose non-zero holomorphic functions f_1 and f_2 on a neighborhood U of z_0 such that $f = \frac{f_1}{f_2}$ on U and $\dim\{z \in \mathbb{C}^m \mid f_1(z) = f_2(z) = 0\} \le m-2$.

We assume that the readers are familiar with standard notations of Nevanlinna value distribution theory (see, e.g., [7,12,21,22,29]), such as the proximity function $m(r,f)$, the (integrated) counting function $N(r,f)$, and the Nevanlinna characteristic function $T(r,f)$. The order and hyper-order of f are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \ \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

A meromorphic function $\alpha(z)$ is called a small function with respect to f if $T(r,\alpha) = o(T(r,f))$ holds for all r possibly outside of a set E with finite logarithmic measure, i.e., $lm(E) = \int_E \frac{dt}{t} < \infty$. Denote by $S(f)$ the family of all small functions with respect to f, and write $\widehat{S}(f) = S(f) \cup \{\infty\}$. For

Received March 2, 2022; Revised September 30, 2022; Accepted November 10, 2022.
2020 *Mathematics Subject Classification.* Primary 39A14, 32H30, 32A20.
Key words and phrases. Several complex variables, meromorphic solutions, partial differential-difference equation, value distribution.
This work was financially supported by the National Natural Science Foundation of China (Grant Nos.12171050, 12101068, 12071047).

$\alpha \in \widehat{S}(f) \cap \widehat{S}(g)$, we say that two meromorphic functions f and g share α CM (IM), provided that $f-\alpha$ and $g-\alpha$ have the same zeros counting multiplicities (ignoring multiplicities).

In 1926, Nevanlinna [20] proved the celebrated five-value theorem that for two non-constant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same preimages (ignoring multiplicities) for five distinct values in $\mathbb{P}^1(\mathbb{C})$, then $f = g$. Afterwards, Nevanlinna five-value theorem was also considered into the case of small functions [18,31]. Various uniqueness theories of meromorphic or entire functions in different research directions had been established by many scholars (see, e.g., [6, 12, 18, 19, 29]). For instance, in 1983, Mues and Steinmetz [19] and Gundersen [6] proved that a nonconstant meromorphic function f and its first derivative f' are identical if they share two finite values CM.

The study of uniqueness problems of meromorphic functions plays a significant role in value distribution theory of complex analysis. The investigation of uniqueness problems has also been extended to the case when f is a meromorphic solution of differential equations (see, e.g., [3, 8, 10, 11, 14, 15]). A typical example is that Brosch [3] pointed out that a meromorphic solution to the complex differential equation of Malmquist-Yosida type $(w')^n = \sum_{k=0}^{2n} a_k w^k$ is uniquely determined by three distinct values counting mutiplicities, where the coefficients a_k $(k = 1, 2, \ldots, 2n)$ are small with respect to the solution.

In recent years, many scholars have shown great interest in the uniqueness problems in the case of higher dimension (see, e.g., [9, 13, 16, 24, 25]). In 2011, Hu and Li [9] focused their attentions on a special class of nonlinear partial differential equation of first order in m (≥ 1) independent complex variables, that is,

$$\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}}(\partial u)^{\mathbf{i}} = \sum_{j=0}^{n} b_j u^j, \tag{1}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in \mathbb{Z}^m$ denotes an index of m dimension with $|\mathbf{i}| = i_1 + i_2 + \cdots + i_m$, $a_{\mathbf{i}} = a_{\mathbf{i}}(z)$ and $b_j = b_j(z)$ $(j = 0, 1, \ldots, n)$ are meromorphic functions in $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$,

$$(\partial u)^{\mathbf{i}} = (u_{z_1})^{i_1}(u_{z_2})^{i_2} \cdots (u_{z_m})^{i_m},\ u_{z_k} = \frac{\partial u}{\partial z_k}\ (k = 1, 2, \ldots, m).$$

Hu and Li [9] showed that a meromorphic solution f to partial differential equation (1) could be uniquely determined by the poles of f and the zeros of $f - c_l$, where c_l $(l = 1, 2)$ are two distinct finite complex numbers.

In the 1970s and 1980s, Bank and Kaufman [1], Shimomura [23], Yanagihara [27, 28] and other scholars had acquired some results on the existence of meromorphic solutions to several classes of difference equations. Compared with a few decades ago, complex difference equations, the discrete counterparts of differential equations, has gained much more attention today. Xu and Cao [24] studied entire and meromorphic solutions of a Fermat-type partial differential-difference equation

$$\left(\frac{\partial u(z_1, z_2)}{\partial z_1}\right)^p + u^q(z_1 + \zeta_1, z_2 + \zeta_2) = 1 \tag{2}$$

in $\mathbb{C}^2$, and Xu and Wang [25] investigated the existence of transcendental entire solutions of finite order for a kind of Fermat-type partial differential-difference equation

$$\left(\frac{\partial u(z_1, z_2)}{\partial z_1} + \frac{\partial u(z_1, z_2)}{\partial z_2}\right)^p + u^q(z_1 + \zeta_1, z_2 + \zeta_2) = 1 \tag{3}$$

in $\mathbb{C}^2$.

Based on the above results, we will make further study on meromorphic solutions of a class of nonlinear partial differential-difference equation of first order in $\mathbb{C}^m$. The equation is of the form

$$\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [u_{z_k}(z + \xi_k)]^{i_k} = \sum_{j=0}^{n} b_j u^j(z + \eta_j), \tag{4}$$

where $\mathbf{i} = (i_1, i_2, \dots, i_m) \in \mathbb{Z}^m$ denotes an index of m dimension with $|\mathbf{i}| = i_1 + i_2 + \cdots + i_m$, $a_{\mathbf{i}} = a_{\mathbf{i}}(z)$ and $b_j = b_j(z)$ $(j = 0, 1 \dots, n)$ are meromorphic functions in $z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$, $u_{z_k}(z + \xi_k) = \frac{\partial u}{\partial z_k}(z + \xi_k)$ $(k = 1, 2, \dots, m)$, and ξ_k, η_j $(k = 1, 2, \dots, m; j = 0, 1, \dots, n)$ are m-dimensional vectors in $\mathbb{C}^m$.

As we can see, (1), (2) and (3) can be regarded as some specific cases of (4). The left and the right of (4) are shifts of the corresponding sides of (1), respectively. This paper is concerned with the value distribution for meromorphic solutions f of (4) with small functions coefficients, which shows that f could be uniquely determined by its poles and the zeros of $f - c$, $f - d$ with c, $d \in S(f)$ (Notice that c, d are not limited to finite complex constants). We prove the following result:

Theorem 1.1. *Suppose that f is a nonconstant meromorphic solution of* (4) *in $\mathbb{C}^m$ with $b_n \not\equiv 0$ and $a_{\mathbf{i}}, b_j \in S(f)$ $(0 < |\mathbf{i}| < n,\ 0 \le j \le n)$ such that*

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{r} = 0, \tag{5}$$

holds for all $r \notin E$, where E is a set with zero upper density measure, i.e.,

$$\overline{dens}E = \limsup_{r\to\infty} \frac{1}{r} \int_{E\cap[1,r]} dt = 0.$$

Let $c(z), d(z)$ be distinct small functions with respect to f satisfying $H(z, c) = \sum_{j=0}^{n} b_j(z) c^j(z + \eta_j) \not\equiv 0$ and $H(z, d) = \sum_{j=0}^{n} b_j(z) d^j(z + \eta_j) \not\equiv 0$. If f and a meromorphic function g in $\mathbb{C}^m$ share $c(z), d(z)$ and ∞ CM, then $f = g$.

Noting that the set of meromorphic functions satisfying (5) consists of all meromorphic functions with some being of hyper-order $\rho_2(f) < 1$ and the others being of hyper-order $\rho_2(f) = 1$. In fact, let $f(z)$ be a meromorphic function such that $T(r,f) = \exp\{r \cdot (\log r)^{-\lambda}\}$ $(\lambda > 0)$, we know easily that $\limsup\limits_{r\to\infty} \frac{\log T(r,f)}{r} = 0$ and $\rho_2(f) = 1$.

The following counterexamples indicate the strictness of the conditions in Theorem 1.1 in some aspects.

Example 1.2. The condition $|\mathbf{i}| < n$ in (4) cannot be improved. Let $m = 3, n = 1$, we consider the following partial differential-difference equation:

$$\begin{aligned}&\frac{\partial u}{\partial z_1}(z_1+\pi i, z_2+\pi i, z_3) + \frac{\partial u}{\partial z_2}(z_1+2\pi i, z_2+\pi i, z_3+\pi i)\\ &= 2u\left(z_1+\frac{\pi i}{2}, z_2+\frac{3\pi i}{2}, z_3\right),\end{aligned}$$

where i is the imaginary unit. Then $f(z_1,z_2,z_3) = e^{z_1+z_2+z_3}$ is an entire solution of the above equation. We can easily check that $f(z_1,z_2,z_3)$ and $g(z_1,z_2,z_3) = e^{-(z_1+z_2+z_3)}$ share $1, -1, \infty$ CM with $H(z,1) \not\equiv 0$, $H(z,-1) \not\equiv 0$. However, $f \not\equiv g$.

Example 1.3. The conditions $H(z,c) \not\equiv 0$ and $H(z,d) \not\equiv 0$ in Theorem 1.1 cannot be dropped. For instance, $f(z_1,z_2,z_3) = \tan(\alpha z_1 + \beta z_2 + \gamma z_3)$ is a meromorphic solution of the partial differential-difference equation

$$\begin{aligned}&\frac{\partial u}{\partial z_1}\left(z_1+\frac{\pi}{4\alpha}, z_2+\frac{\pi}{4\beta}, z_3+\frac{\pi}{2\gamma}\right) + \frac{\partial u}{\partial z_2}\left(z_1+\frac{\pi}{6\alpha}, z_2+\frac{\pi}{2\beta}, z_3+\frac{\pi}{3\gamma}\right)\\ &+\frac{\partial u}{\partial z_3}\left(z_1+\frac{\pi}{8\alpha}, z_2+\frac{3\pi}{4\beta}, z_3+\frac{\pi}{8\gamma}\right)\\ &= (\alpha+\beta+\gamma)\left[1+u^2\left(z_1+\frac{2\pi}{3\alpha}, z_2+\frac{\pi}{6\beta}, z_3+\frac{\pi}{6\beta}\right)\right].\end{aligned}$$

Let i be the imaginary unit, we know that $i, -i$ are two exceptional values of f, which implies $f(z_1,z_2,z_3)$ and $g(z_1,z_2,z_3) = -\tan(\alpha z_1 + \beta z_2 + \gamma z_3)$ share i, $-i$, ∞ CM. However, $f \not\equiv g$. The reason is that $H(z,i) \equiv 0$, $H(z,-i) \equiv 0$.

Example 1.4. The number of shared small functions cannot be reduced. Considering the following equation:

$$\begin{aligned}&\frac{\partial u}{\partial z_1}\left(z_1+\frac{\pi i}{2}, z_2-\frac{\pi i}{2}, z_3\right) + \frac{\partial u}{\partial z_2}(z_1, z_2, z_3+2\pi i)\\ =&\ -u^2(z_1+\pi i, z_2-\pi i, z_3+4\pi i),\end{aligned}$$

where i is the imaginary unit. We have that $f(z_1,z_2,z_3) = \frac{2}{z_1+z_2+e^{z_3}}$ is a meromorphic solution to the equation. Further, we can verify that $f(z_1,z_2,z_3)$ and $g(z_1,z_2,z_3) = z_1+z_2+e^{z_3}$ share $\sqrt{2}$ and $-\sqrt{2}$ CM with $H(z,\sqrt{2}) \not\equiv 0$ and $H(z,-\sqrt{2}) \not\equiv 0$. Obviously, $f \not\equiv g$.

2. Preliminary lemmas

Here, we give some auxiliary lemmas which are of great importance to the proof of our main result.

Lemma 2.1 (The First Main Theorem [17]). *Let f be a meromorphic function on $\mathbb{C}^m$, and $a \in \widehat{S}(f)$. Then, we have*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + o(T(r,f))$$

for all $r \notin E$ with $lm(E) < \infty$.

Lemma 2.2 (The Second Main Theorem [4, 5, 26]). *Let f be a meromorphic function on $\mathbb{C}^m$, and $a_1, a_2, \dots, a_q \in \widehat{S}(f)$ are distinct $(q \geq 3)$. Then, we have*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + o(T(r,f))$$

for all $r \notin E$ with $lm(E) < \infty$.

Lemma 2.3 (The Logarithmic Derivative Lemma [12, 30]). *Suppose that f is a nonconstant meromorphic function on $\mathbb{C}^m$. Then*

$$m\left(r, \frac{f_{z_k}}{f}\right) = o(T(r,f))$$

for all $r \notin E$ with $lm(E) < \infty$ and for any $k \in \{1, 2, \dots, m\}$, where $f_{z_k} = \frac{\partial f}{\partial z_k}$.

Lemma 2.4 (The Logarithmic Difference Lemma [4, 32]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^m$, and let $c \in \mathbb{C}^m \setminus \{\mathbf{0}\}$. If*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0,$$

then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o(T(r,f))$$

for all $r \notin E$ with $\overline{dens}E = 0$.

Lemma 2.5 ([4]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^m$. If*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0,$$

then

$$T(r, f(z+c)) = T(r,f) + o(T(r,f)),$$
$$N(r, f(z+c)) = N(r,f) + o(T(r,f))$$

for any $c \in \mathbb{C}^m \setminus \{\mathbf{0}\}$ and all $r \notin E$ with $\overline{dens}E = 0$.

Remark 2.6. Both Lemma 2.4 and Lemma 2.5 imply $m(r, f(z+c)) = m(r, f) + o(T(r, f))$ for any $c \in \mathbb{C}^m \setminus \{\mathbf{0}\}$ and all $r \notin E$ with $\overline{dens}E = 0$, provided that f is a nonconstant meromorphic function on $\mathbb{C}^m$ satisfying $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$.

Lemma 2.7 ([2,12]). *Let f_j $(j = 1, 2, \ldots, n)$ $(n \geq 2)$ be meromorphic functions and g_j $(j = 1, 2, \ldots, n)$ be entire functions in $\mathbb{C}^m$ satisfying*

(i) $\sum_{j=1}^{n} f_j e^{g_j} = 0$;
(ii) $g_j - g_k$ *are not constants for* $1 \leq j < k \leq n$;
(iii) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ *for all* $r \notin E$ *with* $lm(E) < \infty$.

Then, $f_j = 0$ $(j = 1, 2, \ldots, n)$.

A slight modification to the proof of [9, Lemma 2.2] yields the following lemma, which lays a foundation for the proof of Theorem 1.1.

Lemma 2.8. *Let f be a nonconstant meromorphic solution of* (4) *with $b_n \not\equiv 0$ on $\mathbb{C}^m$ such that*

$$\sum_{0<|\mathbf{i}|<n} T(r, a_{\mathbf{i}}) + \sum_{j=0}^{n} T(r, b_j) = o(T(r, f)) \tag{6}$$

and

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{r} = 0$$

hold for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Then we have

$$m(r, f) = o(T(r, f)) \tag{7}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Furthermore, if $d \in S(f)$ satisfies

$$H(z, d) = \sum_{j=0}^{n} b_j(z) d^j(z + \eta_j) \not\equiv 0,$$

then

$$m\left(r, \frac{1}{f - d}\right) = o(T(r, f)) \tag{8}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$.

Proof. From the assumption that f is a nonconstant meromorphic solution of (4), we get

$$\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} [f_{z_1}(z + \xi_1)]^{i_1} [f_{z_2}(z + \xi_2)]^{i_2} \cdots [f_{z_m}(z + \xi_m)]^{i_m} = \sum_{j=0}^{n} b_j f^j(z + \eta_j).$$

Thus,

$$f(z+\eta_n) = \frac{1}{b_n}\left(\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \frac{[f_{z_1}(z+\xi_1)]^{i_1}\cdots[f_{z_m}(z+\xi_m)]^{i_m}}{f^{n-1}(z+\eta_n)} - \sum_{j=0}^{n-1} b_j \frac{f^j(z+\eta_j)}{f^{n-1}(z+\eta_n)}\right). \tag{9}$$

As we know, $m(r, f(z+\eta_n)) = 0$ when $|f(z+\eta_n)| < 1$. So it suffices to consider the term $m(r, f(z+\eta_n))$ in the case of $|f(z+\eta_n)| \geq 1$. For $0 \leq k \leq n-1$, $\frac{1}{|f(z+\eta_n)|^k} \leq 1$. Hence, by (9) we have the following estimation:

$$|f(z+\eta_n)| \leq \left|\frac{1}{b_n}\right| \left(\sum_{0<|\mathbf{i}|<n} |a_{\mathbf{i}}| \left|\frac{f_{z_1}(z+\xi_1)}{f(z+\eta_n)}\right|^{i_1} \cdots \left|\frac{f_{z_m}(z+\xi_m)}{f(z+\eta_n)}\right|^{i_m} + \sum_{j=0}^{n-1} |b_j| \left|\frac{f(z+\eta_j)}{f(z+\eta_n)}\right|^j\right)$$

when $|f(z+\eta_n)| \geq 1$, which implies that

$$\begin{aligned} m(r,f) &= m(r, f(z+\eta_n)) + o(T(r,f)) \\ &\leq \sum_{0<|\mathbf{i}|<n} \left\{ m(r, a_{\mathbf{i}}) + i_1 m\left(r, \frac{f_{z_1}(z+\xi_1)}{f(z+\eta_n)}\right) \right. \\ &\quad \left. + \cdots + i_m m\left(r, \frac{f_{z_m}(z+\xi_m)}{f(z+\eta_n)}\right) \right\} + \sum_{j=0}^{n-1} m(r, b_j) \\ &\quad + \sum_{j=0}^{n-1} j m\left(r, \frac{f(z+\eta_j)}{f(z+\eta_n)}\right) + m\left(r, \frac{1}{b_n}\right) + o(T(r,f)) \end{aligned}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$. By Lemmas 2.3, 2.4 and Remark 2.6, we further get from the assumption (6) that $m(r,f) = o(T(r,f))$, i.e., (7) holds.

Next, we prove (8). Let $g = f - d$, where $d \in S(f)$. Since $m\left(r, \frac{1}{g}\right) = 0$ for $|g(z)| > 1$, we only need to consider the case that $|g(z)| \leq 1$. The solution $f = g + d$ of (4) ensures that

$$\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [g_{z_k}(z+\xi_k) + d_{z_k}(z+\xi_k)]^{i_k} = \sum_{j=0}^{n} b_j [g(z+\eta_j) + d(z+\eta_j)]^j. \tag{10}$$

For the term $\prod_{k=1}^{m} [g_{z_k}(z+\xi_k) + d_{z_k}(z+\xi_k)]^{i_k}$, we rewrite it as follows:

$$\prod_{k=1}^{m} [g_{z_k}(z+\xi_k) + d_{z_k}(z+\xi_k)]^{i_k} = [g_{z_1}(z+\xi_1)]^{i_1} \cdots [g_{z_m}(z+\xi_m)]^{i_m} + Q_{\mathbf{i}}$$

$$= \prod_{k=1}^{m} [g_{z_k}(z+\xi_k)]^{i_k} + Q_{\mathbf{i}},$$

where $Q_{\mathbf{i}}$ is a polynomial in $g_{z_1}(z+\xi_1), g_{z_2}(z+\xi_2), \ldots, g_{z_m}(z+\xi_m)$ with degree less than $|\mathbf{i}| = i_1 + i_2 + \cdots + i_m$ and with coefficients being small functions with respect to f (thus are also small with respect to g). Hence, the left-hand side of (10) can be reformulated as follows:

$$\begin{aligned} & \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [g_{z_k}(z+\xi_k) + d_{z_k}(z+\xi_k)]^{i_k} \\ = & \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [g_{z_k}(z+\xi_k)]^{i_k} + \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} Q_{\mathbf{i}} \\ = & \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [g_{z_k}(z+\xi_k)]^{i_k} + Q_*, \end{aligned} \tag{11}$$

where $Q_* = \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} Q_{\mathbf{i}}$ is a polynomial in $g_{z_1}(z+\xi_1), g_{z_2}(z+\xi_2), \ldots, g_{z_m}(z+\xi_m)$ with degree less than $n-1$ and with small coefficients.

For the right-hand side of (10), we have

$$\begin{aligned} & \sum_{j=0}^{n} b_j(z)[g(z+\eta_j) + d(z+\eta_j)]^j \\ = & \sum_{j=0}^{n} b_j(z) \sum_{s=0}^{j} \binom{j}{s} g^s(z+\eta_j) d^{j-s}(z+\eta_j) \\ = & \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} g^s(z+\eta_j) d^{j-s}(z+\eta_j) + H(z,d), \end{aligned} \tag{12}$$

where $H(z,d) = \sum_{j=0}^{n} b_j(z) d^j(z+\eta_j)$. It follows from (10)-(12) that

$$\begin{aligned} & \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} [g_{z_k}(z+\xi_k)]^{i_k} + Q_* \\ = & \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} g^s(z+\eta_j) d^{j-s}(z+\eta_j) + H(z,d). \end{aligned}$$

Divide both sides of the above equation by $g(z)H(z,d)$, we obtain

$$\begin{aligned} \frac{1}{g(z)} = & \frac{1}{H(z,d)} \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \frac{[g_{z_1}(z+\xi_1)]^{i_1} [g_{z_2}(z+\xi_2)]^{i_2} \cdots [g_{z_m}(z+\xi_m)]^{i_m}}{g(z)} \\ & + \frac{1}{H(z,d)} \frac{Q_*}{g(z)} - \frac{1}{H(z,d)} \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} \frac{g^s(z+\eta_j)}{g(z)} d^{j-s}(z+\eta_j). \end{aligned} \tag{13}$$

When $|g(z)| \leq 1$, we get the following inequality

$$\frac{1}{|g(z)|} \leq \left|\frac{1}{H(z,d)}\right| \sum_{0<|\mathbf{i}|<n} |a_{\mathbf{i}}| \left|\frac{g_{z_1}(z+\xi_1)}{g(z)}\right|^{i_1} \left|\frac{g_{z_2}(z+\xi_2)}{g(z)}\right|^{i_2} \cdots \left|\frac{g_{z_m}(z+\xi_m)}{g(z)}\right|^{i_m}$$
$$+ \left|\frac{1}{H(z,d)}\right| \left|\frac{Q_*}{g}\right| + \left|\frac{1}{H(z,d)}\right| \sum_{j=1}^{n} |b_j(z)| \sum_{s=1}^{j} \binom{j}{s} \left|\frac{g(z+\eta_j)}{g(z)}\right|^{s} |d(z+\eta_j)|^{j-s},$$

which implies

$$m\left(r, \frac{1}{g}\right) \leq O\left(\sum_{0<|\mathbf{i}|<n} m\left(r, a_{\mathbf{i}}\right) + \sum_{k=1}^{m} m\left(r, \frac{g_{z_k}(z+\xi_k)}{g(z)}\right) + m\left(r, \frac{1}{H(z,d)}\right)\right)$$
$$+ O\left(\sum_{j=1}^{n} \left\{m\left(r, \frac{g\left(z+\eta_j\right)}{g(z)}\right) + m\left(r, b_j\right) + m\left(r, d(z+\eta_j)\right)\right\}\right). \tag{14}$$

By Lemmas 2.3, 2.4, we know

$$m\left(r, \frac{g_{z_k}(z+\xi_k)}{g(z)}\right) \leq m\left(r, \frac{g_{z_k}(z+\xi_k)}{g(z+\xi_k)}\right) + m\left(r, \frac{g(z+\xi_k)}{g(z)}\right) = o(T(r,g)),$$

and

$$m\left(r, \frac{g(z+\eta_j)}{g(z)}\right) = o(T(r,g))$$

for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Together with (6), it can be concluded from (14) that

$$m\left(r, \frac{1}{f-d}\right) = m\left(r, \frac{1}{g}\right) = o(T(r,f))$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$. □

3. Proof of Theorem 1.1

Since f and g share c, d, ∞ CM, we get

$$\frac{f-c}{g-c} = e^{\alpha}, \ \frac{f-d}{g-d} = e^{\beta}, \tag{15}$$

where α and β are two entire functions on $\mathbb{C}^m$. By Lemma 2.2,

$$\begin{aligned} T(r,g) &\leq N(r,g) + N\left(r, \frac{1}{g-c}\right) + N\left(r, \frac{1}{g-d}\right) + o(T(r,g)) \\ &= N(r,f) + N\left(r, \frac{1}{f-c}\right) + N\left(r, \frac{1}{f-d}\right) + o(T(r,g)) \\ &= 3T(r,f) + o(T(r,g)) \end{aligned}$$

holds for all $r \notin E$ with $lm(E) < \infty$. Furthermore, we deduce that

$$T(r,g) \leq (3 + o(1))\, T(r,f).$$

In view of (15), we have

$$T(r,e^{\alpha}) \le T(r,f)+T(r,g)+o(T(r,f)) \le (4+o(1))T(r,f), \tag{16}$$

$$T(r,e^{\beta}) \le T(r,f)+T(r,g)+o(T(r,f)) \le (4+o(1))T(r,f) \tag{17}$$

hold for all $r \notin E$ with $lm(E) < \infty$.

By (15), if $z_0 \in \mathbb{C}^m$ is a zero of $f-c$, z_0 must be a zero of $g-c$. It is easy to check that z_0 is also a zero of $\frac{g-d}{f-d}-1$ or $c-d$. So by Lemma 2.1 and Lemma 2.8, we obtain

$$\begin{aligned}
T(r,f) &= N\left(r,\frac{1}{f-c}\right)+m\left(r,\frac{1}{f-c}\right)+o(T(r,f))\\
&\le N\left(r,\frac{1}{\frac{g-d}{f-d}-1}\right)+N\left(r,\frac{1}{c-d}\right)+o(T(r,f))\\
&\le T\left(r,\frac{g-d}{f-d}\right)+o(T(r,f)) = m\left(r,\frac{g-d}{f-d}\right)+o(T(r,f))\\
&\le m(r,g)+o(T(r,f))
\end{aligned} \tag{18}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$.

By simple calculation and analysis, it can be concluded from (15) that $f = g$ if and only if $e^{\alpha} = 1$ or $e^{\beta} = 1$ or $e^{\alpha} = e^{\beta}$. We now suppose to the contrary that $f \ne g$, which means

$$e^{\alpha} \ne 1,\ e^{\beta} \ne 1,\ e^{\alpha} \ne e^{\beta},\ e^{\beta-\alpha} \ne 1.$$

From (15), we get the following expression for f and g:

$$f = c+(d-c)\frac{e^{\beta}-1}{e^{\beta-\alpha}-1},\ g = d+(d-c)\frac{1-e^{-\alpha}}{e^{\beta-\alpha}-1}.$$

For $k \in \{1,2,\ldots,m\}$ and $j \in \{0,1,\ldots,n\}$, we have

$$\begin{aligned}
f_{z_k} = c_{z_k} &+ \frac{(d_{z_k}-c_{z_k})(e^{\beta}-1)}{(e^{\beta-\alpha}-1)}\\
&+ \frac{(d-c)(\alpha_{z_k}e^{2\beta-\alpha}-\beta_{z_k}e^{\beta}+\beta_{z_k}e^{\beta-\alpha}-\alpha_{z_k}e^{\beta-\alpha})}{(e^{\beta-\alpha}-1)^2}
\end{aligned}$$

and

$$f(z+\eta_j) = c(z+\eta_j)+(d(z+\eta_j)-c(z+\eta_j))\frac{(e^{\beta(z+\eta_j)}-1)}{e^{\beta(z+\eta_j)-\alpha(z+\eta_j)}-1}.$$

For the sake of simplicity, we will use short notations

$$\overline{f}^{1k}(z) = f(z+\xi_k),\ \overline{f_k}^{1k}(z) = f_{z_k}(z+\xi_k),\ \overline{f}^{2j}(z) = f(z+\eta_j)$$

for meromorphic function f and $k \in \{1, 2, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$. Since f is a solution to equation (4), we get the equality

$$\begin{aligned}
&\sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} \Bigg\{ \frac{\overline{d}^{1k} - \overline{c}^{1k}}{\left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right)^2} \Big[\overline{\alpha_k}^{1k} e^{2\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - \overline{\beta_k}^{1k} e^{\overline{\beta}^{1k}} \\
&+ \left(\overline{\beta_k}^{1k} - \overline{\alpha_k}^{1k}\right) e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} \Big] + \overline{c_k}^{1k} + \left(e^{\overline{\beta}^{1k}} - 1\right) \frac{\overline{d_k}^{1k} - \overline{c_k}^{1k}}{e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1} \Bigg\}^{i_k} \\
=& \sum_{j=0}^{n} b_j \left[\overline{c}^{2j} + \left(\overline{d}^{2j} - \overline{c}^{2j}\right) \frac{e^{\overline{\beta}^{2j}} - 1}{e^{\overline{\beta}^{2j}-\overline{\alpha}^{2j}} - 1} \right]^j .
\end{aligned}$$

Multiplying $\prod_{k=1}^{m} \left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right)^{2n} \prod_{j=0}^{n} \left(e^{\overline{\beta}^{2j}-\overline{\alpha}^{2j}} - 1\right)^j$ to both sides of the above equality, we obtain

$$\begin{aligned}
&\prod_{j=0}^{n} \left(e^{\overline{\beta}^{2j}-\overline{\alpha}^{2j}} - 1\right)^j \sum_{0<|\mathbf{i}|<n} a_{\mathbf{i}} \prod_{k=1}^{m} \left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right)^{2n-2i_k} \Bigg\{ \Big[\overline{\alpha_k}^{1k} e^{2\overline{\beta}^{1k}-\overline{\alpha}^{1k}} \\
&- \overline{\beta_k}^{1k} e^{\overline{\beta}^{1k}} + \left(\overline{\beta_k}^{1k} - \overline{\alpha_k}^{1k}\right) e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} \Big] \left(\overline{d}^{1k} - \overline{c}^{1k}\right) + \overline{c_k}^{1k} \left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right)^2 \\
&+ \left(e^{\overline{\beta}^{1k}} - 1\right)\left(\overline{d_k}^{1k} - \overline{c_k}^{1k}\right)\left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right) \Bigg\}^{i_k} \\
=& \prod_{k=1}^{m} \left(e^{\overline{\beta}^{1k}-\overline{\alpha}^{1k}} - 1\right)^{2n} \sum_{j=0}^{n} b_j \Big[\overline{c}^{2j} \left(e^{\overline{\beta}^{2j}-\overline{\alpha}^{2j}} - 1\right) \\
&+ \left(\overline{d}^{2j} - \overline{c}^{2j}\right)\left(e^{\overline{\beta}^{2j}} - 1\right) \Big]^j \prod_{s \neq j} \left(e^{\overline{\beta}^{2s}-\overline{\alpha}^{2s}} - 1\right)^s . \qquad (19)
\end{aligned}$$

For any vector $\tau \in \mathbb{C}^m \setminus \{\mathbf{0}\}$, by (16), (17) and Lemma 2.5 we know

$$T\left(r, e^{\alpha(z+\tau)}\right) = T\left(r, e^{\alpha}\right) + o(T\left(r, e^{\alpha}\right)) = T\left(r, e^{\alpha}\right) + o(T\left(r, f\right))$$

and

$$T\left(r, e^{\beta(z+\tau)}\right) = T(r, e^{\beta}) + o(T(r, e^{\beta})) = T(r, e^{\beta}) + o\left(T\left(r, f\right)\right)$$

hold for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$, which indicate that $e^{\alpha(z)}$ (resp. $e^{\beta(z)}$) grows as fast as its shift $e^{\alpha(z+\tau)}$ (resp. $e^{\beta(z+\tau)}$) does. Be aware that

$$e^{\alpha(z+\tau)} = e^{\alpha(z)} e^{\alpha(z+\tau)-\alpha(z)},$$

$e^{\alpha(z+\tau)}$ can be represented as the product of $e^{\alpha(z)}$ and a small function with respect to f on account of (16), (17) and Lemma 2.4. Similarly, $e^{\beta(z+\tau)}$ can be represented as the product of $e^{\beta(z)}$ and a small function with respect to f.

Based on the discussion above, (19) can be rewritten as the form

$$\sum_{p=0}^{K}\sum_{q=0}^{K}\varphi_{p,q}e^{p\beta-q\alpha}=\sum_{p=0}^{K}\sum_{q=0}^{K}\psi_{p,q}e^{p\beta-q\alpha}, \tag{20}$$

where $K=2nm+\frac{n(n+1)}{2}$, $\varphi_{p,q}$ are either 0 or polynomials in $\overline{\alpha}_k^{1k}$ and $\overline{\beta}_k^{1k}$ with coefficients being products of a nonzero integer, $a_{\mathbf{i}}$, $\overline{c}_k^{1k}$, $\overline{d}_k^{1k}-\overline{c}_k^{1k}$, $\overline{d}^{1k}-\overline{c}^{1k}$, $e^{\overline{\beta}^{1k}-\beta}$, $e^{\alpha-\overline{\alpha}^{1k}}$ and $e^{\overline{\beta}^{2j}-\beta+\alpha-\overline{\alpha}^{2j}}$, where $k\in\{1,2,\ldots,m\}$, $j\in\{0,1,\ldots,n\}$. And $\psi_{p,q}$ are either 0 or polynomials in b_j with coefficients being products of a nonzero integer, $\overline{c}^{2j}$, $\overline{d}^{2j}-\overline{c}^{2j}$, $e^{\overline{\beta}^{2j}-\beta}$, $e^{\alpha-\overline{\alpha}^{2j}}$ and $e^{\overline{\beta}^{1k}-\beta+\alpha-\overline{\alpha}^{1k}}$, where $j\in\{0,1,\ldots,n\}$, $k\in\{1,2,\ldots,m\}$. In conclusion, $\varphi_{p,q}$ and $\psi_{p,q}$ are small with respect to meromorphic function f for any $p\in\{0,1,\ldots,2nm+\frac{n(n+1)}{2}\}$ and $q\in\{0,1,\ldots,2nm+\frac{n(n+1)}{2}\}$. From the observation of $e^{n\beta-0\alpha}$ on each side of (20), we deduce $\varphi_{n,0}=0$ and

$$\psi_{n,0}=(-1)^{2nm+\frac{n(n-1)}{2}}b_n\left(\overline{d}^{2n}-\overline{c}^{2n}\right)^n e^{n(\overline{\beta}^{2n}-\beta)}\neq 0.$$

Furthermore, we rewrite (20) as

$$\sum_{p=0}^{K}\sum_{q=0}^{K}\phi_{p,q}e^{p\beta-q\alpha}=0, \tag{21}$$

where $K=2nm+\frac{n(n+1)}{2}$, $\phi_{p,q}=\varphi_{p,q}-\psi_{p,q}$ and $\phi_{n,0}=\varphi_{n,0}-\psi_{n,0}\neq 0$. By Lemma 2.3, it is concluded from (16) and (17) that

$$m(r,\alpha_{z_k})=m\left(r,\frac{(e^{\alpha})_{z_k}}{e^{\alpha}}\right)=o(T(r,e^{\alpha}))=o(T(r,f)),$$
$$m(r,\beta_{z_k})=m\left(r,\frac{(e^{\beta})_{z_k}}{e^{\beta}}\right)=o(T(r,e^{\beta}))=o(T(r,f))$$

hold for all $r\notin E=E_1\cup E_2$ with $lm(E_1)<\infty$ and $\overline{dens}E_2=0$ and for any $k\in\{1,2,\ldots,m\}$. Since $a_{\mathbf{i}}$, c, d, c_{z_k}, d_{z_k}, $e^{\overline{\beta}^{1k}-\beta}$, $e^{\alpha-\overline{\alpha}^{1k}}$, $e^{\overline{\beta}^{2j}-\beta}$, $e^{\alpha-\overline{\alpha}^{2j}}$ and b_j $(1\leq k\leq m,\ 0\leq j\leq n)$ are all small functions with respect to f, it follows that

$$T(r,\varphi_{p,q})=o(T(r,f)),\ T(r,\phi_{p,q})=o(T(r,f)) \tag{22}$$

hold for all $r\notin E=E_1\cup E_2$ with $lm(E_1)<\infty$ and $\overline{dens}E_2=0$.

Next, we are going to make an estimation of the proximity function of $e^{u\alpha+v\beta}$ for any pair of integers u,v with $(u,v)\neq(0,0)$. Our discussion will be divided into four cases.

Case 1: $u\geq 0$ and $v\geq 0$. By Lemma 2.8, it is easy to verify that

$$m\left(r,e^{u\alpha+v\beta}\right)=m\left(r,\left(\frac{f-c}{g-c}\right)^u\left(\frac{f-d}{g-d}\right)^v\right)$$

$$\begin{aligned}
&\leq m\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) + o(T(r,f))\\
&\leq m\left(r, \left(\frac{f-c}{g-c}\right)^u \left(\frac{f-d}{g-d}\right)^v\right) + m\left(r, \frac{1}{(f-c)^u(f-d)^v}\right)\\
&\quad + o(T(r,f))\\
&= m\left(r, e^{u\alpha+v\beta}\right) + o(T(r,f)),
\end{aligned}$$

which implies

$$m\left(r, e^{u\alpha+v\beta}\right) = m\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) + o(T(r,f)) \tag{23}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Notice that

$$\begin{aligned}
m\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) &= T\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) - N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right)\\
&= (u+v)T(r,g) - N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right)\\
&\quad + o(T(r,f))
\end{aligned} \tag{24}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Owing to the assumption that f and g share c, d CM, by Lemma 2.8 we further gets

$$\begin{aligned}
N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) &= uN\left(r, \frac{1}{g-c}\right) + vN\left(r, \frac{1}{g-d}\right)\\
&= uN\left(r, \frac{1}{f-c}\right) + vN\left(r, \frac{1}{f-d}\right)\\
&= (u+v)T(r,f) + o(T(r,f))
\end{aligned} \tag{25}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Combine (23)-(25) with the assumption that meromorphic functions f and g share ∞ CM, we know

$$\begin{aligned}
m\left(r, e^{u\alpha+v\beta}\right) &= (u+v)(T(r,g) - T(r,f)) + o(T(r,f))\\
&= (u+v)(m(r,g) - m(r,f)) + o(T(r,f))\\
&= (u+v)m(r,g) + o(T(r,f)) \geq m(r,g) + o(T(r,f)),
\end{aligned}$$

that is, when $(u,v) \neq (0,0)$, $m\left(r, e^{u\alpha+v\beta}\right) \geq m(r,g) + o(T(r,f))$ holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$.

Case 2: $u \geq 0$, $v \leq 0$ and $(u,v) \neq (0,0)$. Making use of a similar reasoning as in Case 1, we have

$$\begin{aligned}
m\left(r, e^{u\alpha+v\beta}\right) &= m\left(r, \left(\frac{f-c}{g-c}\right)^u \left(\frac{f-d}{g-d}\right)^v\right)\\
&= m\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) + o(T(r,f))
\end{aligned}$$

$$= T\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) - N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) + o(T(r,f))$$
$$= \max\{u,-v\}T(r,g) - N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) + o(T(r,f)).$$

For the second term in the right-hand side of the equality above, we obtain the following estimation

$$\begin{aligned} & N\left(r, \frac{1}{(g-c)^u(g-d)^v}\right) \\ \le\ & uN\left(r, \frac{1}{g-c}\right) + \max\{0,-u-v\}N(r,g-d) \\ =\ & uN\left(r, \frac{1}{f-c}\right) + \max\{0,-u-v\}N(r,f) + o(T(r,f)) \\ =\ & (uT(r,f) + \max\{0,-u-v\}T(r,f)) + o(T(r,f)) \\ =\ & \max\{u,-v\}T(r,f) + o(T(r,f)). \end{aligned}$$

Hence,

$$\begin{aligned} m\left(r, e^{u\alpha+v\beta}\right) &= \max\{u,-v\}T(r,g) - \max\{u,-v\}T(r,f) + o(T(r,f)) \\ &= \max\{u,-v\}m(r,g) + o(T(r,f)), \end{aligned}$$

which yields that when $u \ge 0$, $v \le 0$ and $(u,v) \ne (0,0)$, $m\left(r, e^{u\alpha+v\beta}\right) \ge m(r,g) + o(T(r,f))$ holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$.

Case 3: $u \le 0$, $v \ge 0$ and $(u,v) \ne (0,0)$. In virtue of Lemma 2.1, we have

$$\begin{aligned} m\left(r, e^{u\alpha+v\beta}\right) &= T\left(r, e^{u\alpha+v\beta}\right) - N\left(r, e^{u\alpha+v\beta}\right) = T\left(r, e^{-u\alpha-v\beta}\right) + O(1) \\ &= m\left(r, e^{-u\alpha-v\beta}\right) + o(T(r,f)), \end{aligned}$$

that is,

$$m\left(r, e^{u\alpha+v\beta}\right) = m\left(r, e^{-u\alpha-v\beta}\right) + o(T(r,f)) \tag{26}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$. Thus, by a similar discussion as in Case 2, we deduce from (26) that $m\left(r, e^{u\alpha+v\beta}\right) \ge m(r,g) + o(T(r,f))$ holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$ when $u \le 0$, $v \ge 0$ and $(u,v) \ne (0,0)$.

Case 4: $u \le 0$, $v \le 0$ and $(u,v) \ne (0,0)$. In this case, $-u \ge 0$ and $-v \ge 0$. As discussed in Case 1 for $-u, -v$, we see from (26) that $m\left(r, e^{u\alpha+v\beta}\right) \ge m(r,g) + o(T(r,f))$ holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens}E_2 = 0$ when $u \le 0$, $v \le 0$ and $(u,v) \ne (0,0)$.

Therefore, for any pair of integers u, v with $(u,v) \ne (0,0)$,

$$m\left(r, e^{u\alpha+v\beta}\right) \ge m(r,g) + o(T(r,f))$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$. Combine this with (18), for $(u,v) \neq (0,0)$, we have

$$\begin{aligned} T(r,f) &\leq m(r,g) + o(T(r,f)) \leq m\left(r, e^{u\alpha+v\beta}\right) + o(T(r,f)) \\ &= T\left(r, e^{u\alpha+v\beta}\right) + o(T(r,f)) \end{aligned} \tag{27}$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$. In view of (22) and (27), we get

$$T(r, \phi_{p,q}) = o(T(r, e^{u\alpha+v\beta})), \ (u,v) \neq (0,0)$$

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\overline{dens} E_2 = 0$. Then by Lemma 2.7, we deduce that all coefficients of the exponential function in (21) vanish identically, that is, $\phi_{p,q} = 0$ for all $p \in \{0, 1, \ldots, 2nm + \frac{n(n+1)}{2}\}$ and $q \in \{0, 1, \ldots, 2nm + \frac{n(n+1)}{2}\}$. It yields a contradiction to $\phi_{n,0} \neq 0$. Consequently, $f = g$ holds on $\mathbb{C}^m$.

Acknowledgment. The authors would like to express a sincere gratitude to the editor and the anonymous reviewer for valuable suggestions and careful comments in helping to improve this paper.

References

[1] S. B. Bank and R. P. Kaufman, *On the gamma function and the Nevanlinna characteristic*, Analysis **6** (1986), no. 2-3, 115–133. https://doi.org/10.1524/anly.1986.6.23.115

[2] C. Berenstein, D.-C. Chang, and B. Q. Li, *A note on Wronskians and linear dependence of entire functions in $\mathbb{C}^n$*, Complex Variables Theory Appl. **24** (1994), no. 1-2, 131–144. https://doi.org/10.1080/17476939408814706

[3] G. Brosch, *Eindeutigkeitssätze für meromorphe Funktionen*, Dissertation, Technical University of Aachen, 1989.

[4] T. Cao and L. Xu, *Logarithmic difference lemma in several complex variables and partial difference equations*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 2, 767–794. https://doi.org/10.1007/s10231-019-00899-w

[5] Z. Chen and Q. Yan, *Uniqueness problem of meromorphic functions sharing small functions*, Proc. Amer. Math. Soc. **134** (2006), no. 10, 2895–2904. https://doi.org/10.1090/S0002-9939-06-08475-9

[6] G. G. Gundersen, *Meromorphic functions that share two finite values with their derivative*, Pacific J. Math. **105** (1983), no. 2, 299–309. http://projecteuclid.org/euclid.pjm/1102723331

[7] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[8] P.-C. Hu and B. Q. Li, *Unicity of meromorphic solutions of partial differential equations*, J. Math. Sci. (N.Y.) **173** (2011), no. 2, 201–206; translated from Sovrem. Mat. Prilozh. (2010), No. 67, Uravneniya s Chastnymi Proizvodnymi 78–83. https://doi.org/10.1007/s10958-011-0242-9

[9] P.-C. Hu and B. Q. Li, *On meromorphic solutions of nonlinear partial differential equations of first order*, J. Math. Anal. Appl. **377** (2011), no. 2, 881–888. https://doi.org/10.1016/j.jmaa.2010.12.004

[10] P.-C. Hu and B. Q. Li, *On meromorphic solutions of linear partial differential equations of second order*, J. Math. Anal. Appl. **393** (2012), no. 1, 200–211. https://doi.org/10.1016/j.jmaa.2012.03.012

[11] P.-C. Hu and B. Q. Li, *A note on meromorphic solutions of linear partial differential equations of second order*, Complex Anal. Oper. Theory **8** (2014), no. 6, 1173–1182. https://doi.org/10.1007/s11785-013-0314-6

[12] P.-C. Hu, P. Li, and C.-C. Yang, *Unicity of meromorphic mappings*, Advances in Complex Analysis and its Applications, 1, Kluwer Academic Publishers, Dordrecht, 2003. https://doi.org/10.1007/978-1-4757-3775-2

[13] P.-C. Hu and Q.-Y. Wang, *On unicity of meromorphic solutions of differential-difference equations*, J. Korean Math. Soc. **55** (2018), no. 4, 785–795. https://doi.org/10.4134/JKMS.j170387

[14] P.-C. Hu and C.-C. Yang, *Global solutions of homogeneous linear partial differential equations of the second order*, Michigan Math. J. **58** (2009), no. 3, 807–831. https://doi.org/10.1307/mmj/1260475702

[15] P.-C. Hu and C.-C. Yang, *A linear homogeneous partial differential equation with entire solutions represented by Bessel polynomials*, J. Math. Anal. Appl. **368** (2010), no. 1, 263–280. https://doi.org/10.1016/j.jmaa.2010.03.048

[16] R. Korhonen, *A difference Picard theorem for meromorphic functions of several variables*, Comput. Methods Funct. Theory **12** (2012), no. 1, 343–361. https://doi.org/10.1007/BF03321831

[17] B. Q. Li, *Uniqueness of entire functions sharing four small functions*, Amer. J. Math. **119** (1997), no. 4, 841–858.

[18] Y. Li and J. Qiao, *The uniqueness of meromorphic functions concerning small functions*, Sci. China Ser. A **43** (2000), no. 6, 581–590. https://doi.org/10.1007/BF02908769

[19] E. Mues and N. Steinmetz, *Meromorphe Funktionen, die mit ihrer Ableitung zwei Werte teilen*, Results Math. **6** (1983), no. 1, 48–55. https://doi.org/10.1007/BF03323323

[20] R. Nevanlinna, *Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen*, Acta Math. **48** (1926), no. 3-4, 367–391. https://doi.org/10.1007/BF02565342

[21] J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Grundlehren der mathematischen Wissenschaften, 350, Springer, Tokyo, 2014. https://doi.org/10.1007/978-4-431-54571-2

[22] M. Ru, *Nevanlinna Theory and Its Relation to Diophantine Approximation*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. https://doi.org/10.1142/9789812810519

[23] S. Shimomura, *Entire solutions of a polynomial difference equation*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 2, 253–266.

[24] L. Xu and T. Cao, *Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math. **15** (2018), no. 6, Paper No. 227, 14 pp. https://doi.org/10.1007/s00009-018-1274-x

[25] H. Y. Xu and H. Wang, *Notes on the existence of entire solutions for several partial differential-difference equations*, Bull. Iranian Math. Soc. **47** (2021), no. 5, 1477–1489. https://doi.org/10.1007/s41980-020-00453-y

[26] K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math. **192** (2004), no. 2, 225–294. https://doi.org/10.1007/BF02392741

[27] N. Yanagihara, *Meromorphic solutions of some difference equations of higher order*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), no. 1, 21–24. http://projecteuclid.org/euclid.pja/1195516181

[28] N. Yanagihara, *Meromorphic solutions of some difference equations of higher order. II*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), no. 7, 284–286. http://projecteuclid.org/euclid.pja/1195515917

[29] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557, Kluwer Academic Publishers Group, Dordrecht, 2003.

[30] Z. Ye, *On Nevanlinna's second main theorem in projective space*, Invent. Math. **122** (1995), no. 3, 475–507. https://doi.org/10.1007/BF01231453

[31] Q. D. Zhang, *A uniqueness theorem for meromorphic functions with respect to slowly growing functions*, Acta Math. Sinica **36** (1993), no. 6, 826–833.

[32] J. Zheng and R. Korhonen, *Studies of differences from the point of view of Nevanlinna theory*, Trans. Amer. Math. Soc. **373** (2020), no. 6, 4285–4318. `https://doi.org/10.1090/tran/8069`

Qibin Cheng
School of Science
Beijing University of Posts and Telecommunications
Beijing 100876, P. R. China
Email address: `18707672503@163.com`

Yezhou Li
School of Science
Beijing University of Posts and Telecommunications
Beijing 100876, P. R. China
Email address: `yezhouli2019@outlook.com`

Zhixue Liu
School of Science
Beijing University of Posts and Telecommunications
Beijing 100876, P. R. China
Email address: `zxliumath@bupt.edu.cn`