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PRIMARY DECOMPOSITION OF SUBMODULES OF A FREE MODULE OF FINITE RANK OVER A BÉZOUT DOMAIN

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Abstract. Let R be a commutative ring with identity. In this paper, we characterize the prime submodules of a free R-module F of finite rank with at most n generators, when R is a GCD domain. Also, we show that if R is a Bézout domain, then every prime submodule with n generators is the row space of a prime matrix. Finally, we study the existence of primary decomposition of a submodule of F over a Bézout domain and characterize the minimal primary decomposition of this submodule.

1. Introduction

Throughout this article, all rings are assumed to be commutative with identity and F denotes a free R-module of finite rank n ($n \geq 2$). Let M be an R-module. A proper submodule N of M is called a *prime submodule* if, for $r \in R$, $m \in M$ and $rm \in N$, we have $m \in N$ or $r \in (N : M)$, where $(N : M) := \{r \in R \,|\, rM \subseteq N\}$. Note that, if N is a prime submodule of an R-module M, then $(N : M)$ is a prime ideal of R. We use the notation $R^{(n)}$ for $\underbrace{R \oplus \cdots \oplus R}_{n\text{-times}}$.

Let R be a commutative domain and K be the quotient field of R. The integral domain R is a *valuation domain* if for every $0 \neq x \in K$, either $x \in R$ or $x^{-1} \in R$. Equivalently, the set of all ideals of R is totally ordered by inclusion. An integral domain R is a *Prüfer domain* if each non-zero finitely generated ideal of R is invertible. It can be shown that an integral domain R is a Prüfer domain if and only if R_P is a valuation domain for every maximal ideal P of R, see [2]. An integral domain R is a GCD domain if any two elements of R have a greatest common divisor. A *Bézout domain* is an integral domain in which the sum of two principal ideals is again principal. Note that a Bézout domain is a GCD domain, see [6, 7]. Any PID is a Bézout domain but a Bézout domain need not be a PID or a UFD. For example, let R be the

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ring of entire complex valued functions. Then R is a Bézout domain and so is a GCD domain. Since the irreducible elements of R are linear polynomials and there are functions with infinitely many roots, R is not a UFD, see [8, Fact 2.3]. Also, note that a GCD domain is not necessarily a UFD. For example, suppose that R is the ring of algebraic integers. Then, R is a Bézout domain and hence is a GCD domain. For every non-zero and non-unit element $a \in R$, we have $a = \sqrt{a}\sqrt{a}$. But $\sqrt{a}$ is a non-unit in R and hence R is not a UFD. Moreover, $\dim(R) = \dim(\mathbb{Z}) = 1$, see [6, Theorem 102].

Prime submodules of a module over a commutative ring have been studied in [1,9,10,13,14] and prime submodules of a finitely generated free module over a PID have been studied in [3,5]. The authors in [3,5] have described prime submodules of a free module $F = R^{(n)}$ $(n \geq 2)$ with at most n generators over a UFD. They have characterized the prime submodules of $F = R^{(n)}$ $(n \geq 2)$, over a PID. In [11,12], we have extended some of these results to Dedekind, Prüfer and valuation domains.

In this paper, we extend some results obtained in [3–5] to a Bézout domain and a GCD domain. Also, we study the existence of primary decomposition of a submodule of F, where R is a Bézout domain and characterize its minimal primary decomposition.

2. Prime submodules of $F = R^{(n)}$ with at most n generators over a Bézout domain R

In this section, we characterize the prime submodules of $F = R^{(n)}$ with at most n generators, when R is a Bézout domain. Also, we show that when R is a Bézout domain, every prime submodule with n generators is the row space of a prime matrix.

The following notations and results obtained from [11], will be frequently used in this article. Let $F := R^{(n)}$ and $X := (x_{i1}, \ldots, x_{in}) \in F$ for some $x_{ij} \in R$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq m \leq n)$. We put

$$B_{m\times n} := [X_1 \cdots X_m] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \ddots & & \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \in M_{m\times n}(R).$$

Thus the jth row of the matrix $[X_1 \cdots X_m]$ consists the components of X_j. We use $N := \langle B \rangle$ to denote the non-zero submodule of F generated by the rows of B. Also $B(j_1, \ldots, j_k) \in M_{m\times k}(R)$ to denote the submatrix of B consisting of the columns $j_1, \ldots, j_k \in \{1, \ldots, n\}$. Setting $\psi := \{X_i = (x_{i1}, \ldots, x_{in}) \in F \mid i \in \Omega\}$, where $\Omega(\subseteq \mathbb{N})$ is an index set, we have:

Lemma 2.1. *Let R be a GCD domain, $F = R^{(n)}$ and $B = [X_1 \cdots X_m] \in M_{m\times n}(R)$ $(m < n)$ with* $\operatorname{rank} B = m$. *Let d be a GCD of determinants of all $m \times m$ submatrices of B. Let $X \in A = \{X' = (x_1, \ldots, x_n) \in F \mid \det \beta(i_1, \ldots,$

$i_{m+1}) = 0$ for every $i_1, \ldots, i_{m+1} \in \{1, \ldots, n\}\}$, where $\beta = [X'X_1 \ldots X_m]$. Then $X = \frac{1}{d}(r_1, \ldots, r_m)B$ for some $r_i \in R$ $(1 \leq i \leq m)$.

Proof. Suppose that $\{B_i(j_{i1}, \ldots, j_{im})\}_{i=1}^k$ is the family of all $m \times m$ submatrices of B with $\det B_i(j_{i1}, \ldots, j_{im}) \neq 0$. Since $\operatorname{rank} B = m$, $k \geq 1$. Let $k = 1$. Then $d = \det B_1$ and by [3, Lemma 2.2] other columns of B are zero. If $X = (x_1, \ldots, x_n) \in A$, then by [11, Lemma 1.5] $(\det B_1)(x_1, \ldots, x_n) = (r_1, \ldots, r_m)B$ for some $r_i \in R$ $(1 \leq i \leq m)$. So $\det B_1(x_{j_{11}}, \ldots, x_{j_{1m}}) = (r_1, \ldots, r_m)B_1$ and hence $\det B_1(x_{j_{11}}, \ldots, x_{j_{1m}})B_1' = (r_1, \ldots, r_m)\det B_1$, where B_1' is the adjoint matrix of B_1. It follows that $(x_{j_{11}}, \ldots, x_{j_{1m}})B_1' = (r_1, \ldots, r_m)$ and so $(x_{j_{11}}, \ldots, x_{j_{1m}})\det B_1 = (r_1, \ldots, r_m)B_1$.

Thus $(x_{j_{11}}, \ldots, x_{j_{1m}}) = \frac{1}{d}(r_1, \ldots, r_m)B_1$. Since $x_i = 0$ for $1 \leq i \leq m$, $i \neq j_{1k}$ $(1 \leq k \leq m)$, we have $X = \frac{1}{d}(r_1, \ldots, r_m)B$. Now assume that $k > 1$ and let $d_1 = \det B_1$ and $d_{i+1} = \gcd(d_i, \det B_{i+1})$ $(1 \leq i \leq k-1)$. Then $d_k = d$. Suppose that $X = (x_1, \ldots, x_n) \in A$. Then by [11, Lemma 2.5] $(\det B_1)(x_1, \ldots, x_n) = (r_1, \ldots, r_m)B$ for some $r_i \in R$ $(1 \leq i \leq m)$. By induction on t $(1 \leq t \leq k-1)$, we show that there exist $r_{t1}, \ldots, r_{tm} \in R$ such that $(x_{j_{i1}}, \ldots, x_{j_{im}}) = \frac{1}{d_{t+1}}(r_{t1}, \ldots, r_{tm})B_i$ for all $1 \leq i \leq t+1$. Since $(\det B_1)(x_1, \ldots, x_n) = (r_1, \ldots, r_m)B$ for some $r_i \in R$ $(1 \leq i \leq m)$, we have $(\det B_1)(x_{j_{i1}}, \ldots, x_{j_{im}}) = (r_1, \ldots, r_m)B_i$ for all i $(1 \leq i \leq k)$. It follows that $(\det B_1)(x_{j_{21}}, \ldots, x_{j_{2m}})B_2' = (r_1, \ldots, r_m)B_2$. Since $(x_{j_{11}}, \ldots, x_{j_{1m}})B_1' = (r_1, \ldots, r_m)$, so $(\det B_1)(x_{j_{21}}, \ldots, x_{j_{2m}})B_2' = (x_{j_{11}}, \ldots, x_{j_{1m}})B_1' \det B_2$. We know that $d_2 = \gcd(\det B_1, \det B_2)$ and $(\frac{\det B_1}{d_2}, \frac{\det B_2}{d_2}) = 1$.

Then $(x_{j_{11}}, \ldots, x_{j_{1m}})B_1' = (r_{11}, \ldots, r_{1m})\frac{\det B_1}{d_2}$ for some $r_{1i} \in R$ $(1 \leq i \leq m)$. Hence $(x_{j_{11}}, \ldots, x_{j_{1m}}) = \frac{1}{d_2}(r_{11}, \ldots, r_{1m})B_1$. With substituting $(x_{j_{11}}, \ldots, x_{j_{1m}})$ in $(\det B_1)(x_{j_{21}}, \ldots, x_{j_{2m}})B_2' = (x_{j_{11}}, \ldots, x_{j_{1m}})B_1' \det B_2$, we have $(x_{j_{21}}, \ldots, x_{j_{2m}})B_2' = (r_1, \ldots, r_m)\frac{\det B_2}{d_2}$ and hence $(x_{j_{21}}, \ldots, x_{j_{2m}}) = \frac{1}{d_2}(r_1, \ldots, r_m)B_2$.

Therefore the assertion is true for $t = 1$. Now suppose that the assertion is true for some t $(1 \leq t < k-1)$. With similar to the way of $t = 1$, we have $(\det B_1)(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}})B_{t+2}' = (x_{j_{11}}, \ldots, x_{j_{1m}})B_1' \det B_{t+2}$. Since $\det B_1 \neq 0$ and by induction hypothesis, we have $d_{t+1}(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}})B_{t+2}'$ $= (r_{t1}, \ldots, r_{tm})\det B_{t+2}$. Since $\gcd(\frac{\det B_{t+2}}{d_{t+2}}, \frac{d_{t+1}}{d_{t+2}}) = 1$, we have

$$(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}})B_{t+2}' = (r_{(t+1)1}, \ldots, r_{(t+1)m})\frac{\det B_{t+2}}{d_{t+2}}.$$

It follows that $(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}}) = \frac{1}{d_{t+2}}(r_{(t+1)1}, \ldots, r_{(t+1)m})B_{t+2}$ for some $r_{(t+1)i} \in R$ $(1 \leq i \leq m)$.

With substituting $(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}})$ in $d_{t+1}(x_{j_{(t+2)1}}, \ldots, x_{j_{(t+2)m}})B_{t+2}' = (r_{t1}, \ldots, r_{tm})\det B_{t+2}$, we have $(r_{t1}, \ldots, r_{tm}) = \frac{d_{t+1}}{d_{t+2}}(r_{(t+1)1}, \ldots, r_{(t+1)m})$.

So by the induction hypothesis, $(x_{j_{i1}}, \ldots, x_{j_{im}}) = \frac{1}{d_{t+1}}(r_{t1}, \ldots, r_{tm})B_i = \frac{1}{d_{t+2}}(r_{(t+1)1}, \ldots, r_{(t+1)m})B_i$ for all $1 \leq i \leq t+2$. Hence by induction, the assertion is true for all t $(1 \leq t \leq k-1)$. Let $t = k-1$. Then $(x_{j_{i1}}, \ldots, x_{j_{im}}) =$

$\frac{1}{d_k}(r_{(k-1)1},\dots,r_{(k-1)m})B_i$ for all i $(1\le i\le k)$. Therefore by [3, Lemma 2.2], $X=\frac{1}{d_k}(r_{(k-1)1},\dots,r_{(k-1)m})B$ for some $r_{(k-1)i}\in R$ $(1\le i\le m)$. □

Theorem 2.2. *Suppose that R is a GCD domain and $F=R^{(n)}$ $(n\ge 2)$. Let $B=[X_1\cdots X_m]\in M_{m\times n}(R)$ for some $X_i\in F$ $(1\le i\le m, m<n)$ and* $\operatorname{rank}B=m$. *Then $N:=\langle B\rangle$ is a prime submodule of F if and only if a GCD of the determinants of all $m\times m$ submatrices of B is unit.*

Proof. Let the GCD of the determinants of all $m\times m$ submatrices of B be unit. Put $M:=\{X=(x_1,\dots,x_n)\in F\mid \det\beta(i_1,\dots,i_{m+1})=0$ for every $i_1,\dots,i_{m+1}\in\{1,\dots,n\}\}$, where $\beta=[XX_1\cdots X_m]$. Since $X_i\in M$ $(1\le i\le m)$, we have $N\subseteq M$. Now suppose that $X\in M$. Then by Lemma 2.1, $X\in N=\langle B\rangle$. Thus $N=M$ and by [11, Corollary 1.9], N is a prime submodule of F. Now let $N=\langle B\rangle$ be a prime submodule of F and d be a GCD of all $m\times m$ submatrices of B. Let d be a non-unit. Since d is a GCD of all $m\times m$ submatrices of B and by [3, Lemma 2.2], there exists a submatrix $B(j_1,\dots,j_m)$ of B with $j_1<j_2<\cdots<j_m$ of $\{1,\dots,n\}$ such that $\det B(j_1,\dots,j_m)=dr$ and $\gcd(r,d)=u$ for some $r\in R$ and unit $u\in R$. Now let $B(j_1,\dots,j_m)=(t_{ij})$ and let $B'(j_1,\dots,j_m)=(t'_{ij})$ be the adjoint matrix of $B(j_1,\dots,j_m)$. Fix i $(1\le i\le m)$ and consider $(x_1,\dots,x_n)=(t'_{i1},\dots,t'_{im})B\in N$. Since $B'(j_1,\dots,j_m)B(j_1,\dots,j_m)=\det B(j_1,\dots,j_m)I_m$, we have $x_{j_i}=\det B(j_1,\dots,j_m)$ and $x_{j_k}=0$ $(1\le k\le m, k\ne i)$. Also $x_j=I\det C_j$, where $C_j=B(j_1,\dots,j_{i-1},j,j_{i+1},\dots,j_m)$ for all $j\in\{1,\dots,n\}\setminus\{j_1,\dots,j_m\}$. Since $\det C_j\in\langle\det B(j_1,\dots,j_m)\rangle$, we have $j\in\{1,\dots,n\}\setminus\{j_1,\dots,j_m\}$. Then $d\mid x_j$ and hence $(\frac{x_1}{d},\dots,\frac{x_n}{d})\in F$. Note that $d(\frac{x_1}{d},\dots,\frac{x_n}{d})\in N$. Since N is prime, $dF\subseteq N$ or $(\frac{x_1}{d},\dots,\frac{x_n}{d})\in N$.

If $dF\subseteq N$, then $(0,\dots,0,d,0,\dots,0)\in N$ with d as the j_0th component for some $j_0\in\{1,\dots,n\}\setminus\{j_1,\dots,j_m\}$. Hence there are $r_j\in R$ $(1\le j\le m)$ such that $(0,\dots,0,d,0,\dots,0)=(r_1,\dots,r_m)B(j_1,\dots,j_m)$. It follows that $r_j\det B(j_1,\dots,j_m)=0$ $(1\le j\le m)$ and hence $r_j=0$ $(1\le j\le m)$. Thus $\det B(j_1,\dots,j_m)=0$, which is a contradiction. So $(\frac{x_1}{d},\dots,\frac{x_n}{d})\in N$. Hence there are $s_j\in R$ $(1\le j\le m)$ such that $(\frac{x_1}{d},\dots,\frac{x_n}{d})=(s_1,\dots,s_m)B(j_1,\dots,j_m)$.

It follows that $(t'_{i1},\dots,t'_{im})B(j_1,\dots,j_m)=(x_1,\dots,x_n)=d(s_1,\dots,s_m)$ $B(j_1,\dots,j_m)$. So $(t'_{i1},\dots,t'_{im})B(j_1,\dots,j_m)=d(s_1,\dots,s_m)B(j_1,\dots,j_m)$.

Thus $t'_{ij}\det B(j_1,\dots,j_m)=ds_j\det B(j_1,\dots,j_m)$ and $t'_{ij}=ds_j$ $(1\le j\le m)$. Therefore $\det B'(j_1,\dots,j_m)=d^ms$ for some $0\ne s\in R$. But $\det B'(j_1,\dots,j_m)=(\det B(j_1,\dots,j_m))^{m-1}=(dr)^{m-1}$. Hence $d^{m-1}r^{m-1}=d^ms$ and $\gcd(d,r)=u$. It follows that $d\mid r^{m-1}$, which is a contradiction. □

Theorem 2.3. *Let R be a GCD domain and $F=R^{(n)}$ $(n\ge 2)$. Let $B\in M_{n\times n}(R)$ and* $\operatorname{rank}B=n$. *Suppose that* $\det B$ *has a decomposition into distinct prime elements of R. Then $N=\langle B\rangle$ is prime in F if and only if there exist an irreducible element $p\in R$, a unit $u\in R$ and a positive integer $\alpha\le n$ such that* $\det B=up^{\alpha}$ *and a GCD of entries of B' is $p^{\alpha-1}$.*

Proof. Since $\det B$ has a prime decomposition and by [12, Lemma 1.1] and [11, Lemma 1.1], the proof is similar to [3, Theorem 2.5(ii)]. □

Proposition 2.4. *Let R be a Bézout domain and $0 \neq P = \langle p \rangle$ be a prime ideal of R. Then P is a maximal ideal of R.*

Proof. Let $x \in R - P$ and $d := \gcd(p, x)$. Then $p = ad$ and $x = bd$ for some $a, b \in R$. Since $p = ad \in P$, $d \in P$ or $a \in P$. If $d \in P$, then $x \in P$, which is a contradiction. If $a \in P$, then we have $a = ps$ for some $s \in R$. So $p = ad = psd$ and hence $1 = sd$. Therefore d is a unit. So there exist $m, n \in R$ such that $1 = pm + xn$ and hence $1 + P = xn + P$. Thus R/P is a field and P is maximal. □

Theorem 2.5. *Suppose R is a Bézout domain and $F = R^{(n)}$ $(n \geq 2)$. Let $B \in M_{n\times n}(R)$ with $\operatorname{rank} B = n$. Then $N = \langle B \rangle$ is prime in F if and only if there exist a maximal ideal $P = \langle p \rangle$ of R and a positive integer $\alpha \leq n$ such that $\langle \det B \rangle = P^\alpha$ and the ideal J' of R generated by entries of B' is $P^{\alpha-1}$, where B' is the adjoint matrix of B.*

Proof. Since R is a Bézout domain, R is a Prüfer domain [6, Theorem 62]. Hence by [12, Theorem 3.2], [12, Proposition 2.2] and Proposition 2.4, the proof is complete. □

The notion of prime matrix is introduced in [4]. Now we show that for a Bézout domain R every finitely generated prime submodule of $R^{(n)}$ $(n \geq 2)$, with n generators is the row space of a prime matrix.

Definition. Suppose R is a domain. Let $m = \langle p \rangle$ be a principal maximal ideal of R for some $p \in R$. Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of $\{1, \ldots, n\}$. A matrix $B = (b_{ij}) \in M_{n\times n}(R)$ is said to be a p-prime matrix if it satisfies the following conditions:
(i) B is upper triangular.
(ii) For all i, $1 \leq i \leq n$, $a_{ii} = p$ if $i \in J$ and $a_{ii} = 1$ if $i \notin J$.
(iii) For all $i, j \in \{1, \ldots, n\}$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.
Sometimes we call J the set of integers associated with B and denote it by J_B. By (i) and (ii), it is clear that $\det B = p^\alpha$.

We recall that for $A \in M_{n\times s}(R)$ and $Y \in M_{n\times 1}(R)$, the augmented matrix $[A : Y]$ is a matrix whose first n columns are the columns of the matrix A and its last column is Y.

Lemma 2.6. *Suppose R is a Bézout domain and $m = \langle p \rangle$ is a principal maximal ideal of R. Let s and n be positive integers such that $s < n$. Also, suppose that $A \in M_{n\times s}(R)$, $Y \in M_{n\times 1}(R)$ and $X = [x_1 \cdots x_s]^T \in M_{s\times 1}(R)$. Let $C \in M_{n\times(s+1)}(R)$ be the augmented matrix $[A : Y]$. If p does not divide the determinant of at least one $s \times s$ submatrix of A, then the system of equations $AX \equiv Y (\mathrm{mod}\ p)$ has a solution if and only if p divides the determinants of all $(s+1) \times (s+1)$ submatrices of C.*

Proof. Suppose $AX \equiv Y (\text{mod } p)$ has a solution and C_0 is an $(s+1)\times(s+1)$ submatrix of C. The proof is similar to [12, Lemma 2.6].

Now let $p \nmid \det A^T(i_1,\ldots,i_s)$. By [11, Lemma 1.5(ii)], $(\det A^T(i_1,\ldots,i_s))Y^T \in \langle p\rangle F + \langle A^T\rangle$. Since $\det A^T(i_1,\ldots,i_s) \notin m$, $1 = s(\det A^T(i_1,\ldots,i_s)) + rp$ for some $r, s \in R$. Then $Y^T = s(\det A^T(i_1,\ldots,i_s))Y^T + rpY^T \in \langle p\rangle F + \langle A^T\rangle$ and so the system of equations $AX \equiv Y (\text{mod } p)$ has a solution. □

Theorem 2.7. *Suppose R is a Bézout domain and $m = \langle p\rangle$ is a principal maximal ideal of R. Let s, n and α be positive integers such that $s \le n$ $(1 \le \alpha \le n)$ and $A \in M_{s\times n}(R)$. Then $\langle A\rangle \subseteq \langle B\rangle$ for some p-prime matrix $B \in M_{n\times n}(R)$ with $\det B = p^\alpha$ if and only if p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A.*

Proof. See [12, Theorem 2.7]. □

Proposition 2.8. *Suppose R is a Bézout domain, n is a positive integer and $m = \langle p\rangle$ is a principal maximal ideal of R. Let $A \in M_{n\times n}(R)$ and $1 \le \alpha \le n$ be the greatest integer such that $p^\alpha \mid \det A$ and $p^{\alpha-1}$ divides all entries of A', where A' is the adjoint matrix of A. Then p divides the determinants of all $(n-\alpha+1)\times(n-\alpha+1)$ submatrices of A.*

Proof. Let $C = (c_{ij})$ be an $(n-\alpha+1)\times(n-\alpha+1)$ submatrix of A. Since R is a Bézout domain, R_m is a valuation domain and $p \mid \det C_m$ [12, Proposition 2.8], where C_m is a matrix with entries $(C_m)_{ij} = \frac{c_{ij}}{1} \in R_m$. So $\frac{\det C}{1} = \frac{p}{1}\cdot\frac{r}{s}$ and $s(\det C) = pr$ for some $r \in R$, $s \in R\backslash m$. Since $s \notin m$, $1 = xs + py$ for some $x, y \in R$. Then $\det C = xs(\det C) + py(\det C)$. It follows that $p \mid \det(C)$. □

Theorem 2.9. *Suppose R is a Bézout domain and $F = R^{(n)}$ $(n \ge 2)$. Let $A \in M_{n\times n}(R)$. Then $N = \langle A\rangle$ is a prime submodule of F if and only if $(N : F)$ is a principal maximal ideal of R and N is the row space of a prime matrix.*

Proof. Let N be a prime submodule of F. By Theorem 2.5, $m = (N : F) = \langle p\rangle$ is a principal maximal ideal of R, $\langle \det A\rangle = \langle p^\alpha\rangle$ and the ideal J' of R generated by entries of A' is $\langle p^{\alpha-1}\rangle$, where A' is the adjoint matrix of A. So by Proposition 2.8 and Theorem 2.7, $N \subseteq \langle B\rangle$ for some prime matrix B with $\det B = p^\alpha$. Thus $A = CB$ for some $C \in M_{n\times n}(R)$ and therefore $up^\alpha = \det A = (\det C)(\det B) = (\det C)p^\alpha$. Thus $\det(C) = u$ and so C is invertible. Hence $C^{-1}B = A$. It follows that $\langle B\rangle \subseteq N = \langle A\rangle$. Therefore $N = \langle B\rangle$. Conversely by Theorem 2.5, the row space of every prime matrix is a prime submodule. □

3. Primary Decomposition of submodules of a free module of finite rank over a Bézout domain

In this section, we describe primary decomposition of submodules of a free module of finite rank over a Bézout domain.

Lemma 3.1. *Suppose R is a Bézout domain and $m = \langle p \rangle$ is a principal maximal ideal of R. Let n, s and α be positive integers such that $s < n$. Also, suppose that $A \in M_{n\times s}(R)$, $Y \in M_{n\times 1}(R)$ and $X = [x_1 \cdots x_s]^T \in M_{s\times 1}(R)$. Let $C \in M_{n\times(s+1)}(R)$ be the augmented matrix $[A : Y]$. If p does not divide the determinant of at least one $s \times s$ submatrix of A, then the system of equations $AX \equiv Y \pmod{p^\alpha}$ has a solution if and only if p^α divides the determinants of all $(s+1)\times(s+1)$ submatrices of C.*

Proof. The proof is similar to Lemma 2.6. □

Proposition 3.2. *Let R be a domain, n be a positive integer and $F = R^n$ $(n \geq 2)$. Suppose that $B \in M_{n\times n}(R)$ and $0 \neq \det B$ has a decomposition into distinct prime elements of R. Then $\langle B \rangle$ is a primary submodule of F if and only if $\det B = up^\alpha$ for some unit $u \in R$, a prime element $p \in R$ and a positive integer α.*

Proof. Suppose that $\det B = up^\alpha$. Let $r(x_1, \dots, x_n) \in \langle B \rangle$ for some $r \in R$ and $x_i \in R$ $(1 \leq i \leq n)$. By [11, Lemma 2.1], $up^\alpha \mid r \sum_{i=1}^n x_i b'_{ij}$ for every j $(1 \leq j \leq n)$. If $p \nmid r$, then we have $up^\alpha \mid \sum_{i=1}^n x_i b'_{ij}$ for every j $(1 \leq j \leq n)$ and again by [11, Lemma 2.1], $(x_1, \dots, x_n) \in \langle B \rangle$.

If $p \mid r$, then $up^\alpha \mid r^\alpha b'_{ij}$ for every i and j $(1 \leq i, j \leq n)$. Again by [11, Lemma 2.1], $(0, \dots, 0, r^\alpha, 0, \dots, 0) \in \langle B \rangle$ with r^α as the ith component $(1 \leq i \leq n)$. Then $r^\alpha F \subseteq \langle B \rangle$. Thus $\langle B \rangle$ is a primary submodule of F. Conversely, let $\langle B \rangle$ be primary. Assume that $\det B = rs$ for some non-unit relatively prime elements $r, s \in R$. By [11, Lemma 2.1], $r(0, \dots, 0, s, 0, \dots, 0) \in \langle B \rangle$ with s as the ith component $(1 \leq i \leq n)$. Since $\langle B \rangle$ is primary, $sF \subseteq \langle B \rangle$ or $r^m F \subseteq \langle B \rangle$ for some positive integer m. If $sF \subseteq \langle B \rangle$, then $(\det B) \mid sb'_{ij}$ for every i and j $(1 \leq i, j \leq n)$. Thus $(\det B)^n \mid s^n \det(B') = s^n (\det B)^{n-1}$. It follows that $(\det B) \mid s^n$, i.e., $rs \mid s^n$ and thus $r \mid s^{n-1}$, which contradicts the fact that r and s are relatively prime. Similarly, the case $r^m F \subseteq \langle B \rangle$ implies that $(\det B) \mid r^{mn}$. So $s \mid r^{mn-1}$, which is a contradiction. We conclude that $\det B = up^\alpha$ for some unit $u \in R$, a prime element $p \in R$ and a positive integer α. □

Let $m \leq n$ be positive integers and $B \in M_{m\times n}(R)$. Suppose that t $(1 \leq t \leq m)$, $1 \leq i_1 < \cdots < i_t \leq m$ and $1 \leq j_1 < \cdots < j_t \leq n$ are some integers. Then $B\left[\begin{smallmatrix} i_1 & \cdots & i_t \\ j_1 & \cdots & j_t \end{smallmatrix}\right]$ denotes the determinant of the $t \times t$ submatrix of B consisting of rows $i_1, \dots, i_t$ and columns $j_1, \dots, j_t$.

Theorem 3.3. *Suppose R is a Bézout domain. Let $m \leq n$ be positive integers and $B \in M_{m\times n}(R)$. Also, suppose that $p \in R$ is a prime element and let α be the greatest integer such that $p^\alpha \mid B\left[\begin{smallmatrix} 1 & \cdots & m \\ 1 & \cdots & m \end{smallmatrix}\right]$. Then there exists an upper triangular matrix $A \in M_{n\times n}(R)$ with $\det A = p^\alpha$ such that $\langle B \rangle \subseteq \langle A \rangle$.*

Proof. By Lemma 3.1, the proof is similar to [4, Theorem 2.4]. □

Theorem 3.4. *Let R be a Bézout domain and $F = R^{(n)}$. Let $B \in M_{n\times n}(R)$ such that* $\det B$ *is non-unit and non-zero. Suppose that* $\det B = p_1^{\beta_1} \cdots p_t^{\beta_t}$ *is a decomposition of* $\det B$ *into distinct prime elements of R and $\beta_i \in \mathbb{N}$ $(1 \leq i, j \leq t)$. Let A_k with* $\det A_k = p_k^{\beta_k}$ *$(1 \leq k \leq t)$ be the triangular matrix in Theorem 3.3. Then $\bigcap_{k=1}^{t} \langle A_k \rangle$ is a minimal primary decomposition of $\langle B \rangle$.*

Proof. Since $\langle B \rangle \subseteq \langle A_k \rangle$ for all k $(1 \leq k \leq t)$, $\langle B \rangle \subseteq \bigcap_{k=1}^{t} \langle A_k \rangle$. Take an element $(x_1, \ldots, x_n) \in \bigcap_{k=1}^{t} \langle A_k \rangle$. Then for all k $(1 \leq k \leq t)$, we have $(x_1, \ldots, x_n) \in \langle A_k \rangle$. So $(x_1, \ldots, x_n)A'_k = p_k^{\beta_k}(r_1, \ldots, r_n)$ for some $r_i \in R$ $(1 \leq i \leq n)$. Since $B = C_k A_k$ for some $C_k \in M_{n\times n}(R)$, we have $p_k^{\beta_k}(r_1, \ldots, r_n)C'_k = (x_1, \ldots, x_n)A'_k C'_k = (x_1, \ldots, x_n)B'$. So $p_k^{\beta_k} \mid \sum_{i=1}^{n} x_i b'_{ij}$ for every j and k $(1 \leq j \leq n,\ 1 \leq k \leq t)$. Thus $\det B = p_1^{\beta_1} \cdots p_t^{\beta_t} \mid \sum_{i=1}^{n} x_i b'_{ij}$ for every j $(1 \leq j \leq n)$ and hence $(x_1, \ldots, x_n) \in \langle B \rangle$. It follows that $\bigcap_{k=1}^{t} \langle A_k \rangle \subseteq \langle B \rangle$. So $\langle B \rangle = \bigcap_{k=1}^{t} \langle A_k \rangle$. Note that by Proposition 3.2, $\langle A_k \rangle$ is $\langle p_k \rangle$-primary. Suppose that $\bigcap_{i\neq k=1}^{t} \langle A_k \rangle \subseteq \langle A_i \rangle$ for some i $(1 \leq i \leq t)$. Then $\sqrt{(\bigcap_{i\neq k=1}^{t} \langle A_k \rangle : F)} \subseteq \sqrt{(\langle A_i \rangle : F)}$ and so $\bigcap_{i\neq k=1}^{t} \sqrt{(\langle A_k \rangle : F)} \subseteq \sqrt{(\langle A_i \rangle : F)}$. Since every $\langle A_k \rangle$ is a $\langle p_k \rangle$-primary submodule, hence $\bigcap_{i\neq k=1}^{t} \langle p_k \rangle \subseteq \langle p_i \rangle$. It follows that $\langle p_j \rangle \subseteq \langle p_i \rangle$, which is a contradiction. $\square$

Example 3.5. Let R be the ring of entire complex valued functions. By [8, Fact 2.3], R is a Bézout domain and so is a GCD domain. Since the irreducible elements of R are linear polynomials and there are functions with infinitely many roots, R is not a UFD.

Let $\begin{pmatrix} x+2 & x+2 & 0 \\ x+2 & x+3 & x \\ x^2(x+2) & x & x^2 \end{pmatrix} \in M_{3\times 3}(R)$. We shall find a minimal primary decomposition of $\langle B \rangle$. Since $\det B = x^3(x+2)^2$, by Theorem 3.3, there exist upper triangular matrices

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & x^3 \end{pmatrix}, \qquad \begin{pmatrix} x+2 & a'_{12} & a'_{13} \\ 0 & 1 & a'_{23} \\ 0 & 0 & x+2 \end{pmatrix}$$

such that

$$\begin{cases} (x+2)a_{13} \equiv 0 \pmod{x^3} \\ (x+2)a_{13} + a_{23} \equiv x \pmod{x^3} \\ x^2(x+2)a_{13} + (-x^3 - 2x^2 + x)a_{23} \equiv x^3 \pmod{x^3} \end{cases}$$

and

$$\begin{cases} a'_{13} + (x+1)a'_{23} \equiv 0 \pmod{x+2} \\ a'_{13} + (x+2)a'_{23} \equiv x \pmod{x+2} \\ x^2 a'_{13} + (x - x^2)a'_{23} \equiv x^2 \pmod{x+2}. \end{cases}$$

A solution for the above systems is

$$a_{13} = x^3, \quad a_{23} = -x^4 - x^3 + x, \quad a'_{13} = 3x + 4 \text{ and } a'_{23} = -2.$$

Hence

$$A_1 = \begin{pmatrix} 1 & 0 & x^3 \\ 0 & 1 & -x^4 - x^3 + x \\ 0 & 0 & x^3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} x+2 & 0 & 3x+4 \\ 0 & 1 & -2 \\ 0 & 0 & x+2 \end{pmatrix}.$$

By Theorem 3.4, $\langle B\rangle = \langle A_1\rangle \bigcap \langle A_2\rangle$ is a minimal primary decomposition.

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