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DOUBLE LINES IN THE QUINTIC DEL PEZZO FOURFOLD

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Abstract. Let Y be the quintic del Pezzo 4-fold defined by the linear section of $\mathrm{Gr}(2,5)$ by $\mathbb{P}^7$. In this paper, we describe the locus of double lines in the Hilbert scheme of coincs in Y. As a corollary, we obtain the desigularized model of the moduli space of stable maps of degree 2 in Y. We also compute the intersection Poincaré polynomial of the stable map space.

1. Introduction

1.1. Motivation

In previous series of papers [2, 4, 5], the authors completely solved the comparison problem of different moduli spaces (i.e., the stable map space, Hilbert scheme of curves and the stable sheaf space) of rational curves in a homogeneous variety X when the degree of curves is ≤ 3. As a result, we obtain the moduli theoretic birational model (in the sense of log minimal model program) and compute the cohomology group of the moduli spaces. In this case, the convexity of X provides the mild singularity of the moduli space of stable maps and thus one can set it as the initial point of comparison. But many of Fano varieties are not convex. As such a toy example is the minimal compactification of $\mathbb{C}^3$: the quintic del Pezzo 3-fold W_5 and Mukai variety W_{22}. In the case of W_5, our starting point of the comparison is the Hilbert scheme (which is isomorphic to the moduli space of stable sheaves). In [1], we obtain the desingularized model of the moduli space of stable maps in W_5. In this paper, as well-known example of the minimal compactification of $\mathbb{C}^4$, we study the rational curves in a quintic del Pezzo 4-fold Y which is unique up to isomorphism. We deal with the first non-trivial case, that is, the degree two rational curves in Y. We obtain the desingularized model of stable maps space and thus its intersection cohomology group. Similar to the 3-fold case ([1]), the crucial part is to classify types of the normal bundle of a line in Y. In general, the geometry of lines

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in a Fano variety has played an important role for determining the geometric properties of a Fano variety ([9, 12, 13]).

1.2. Results

Unless otherwise stated we define the quintic del Pezzo 4-fold Y by the linear section of $\mathrm{Gr}(2,5)$ by $\{p_{12}-p_{03}=p_{13}-p_{24}=0\}$, where $\{p_{ij}\}$ are the Plücker coordinates of $\mathbb{P}^9$ under the Plücker embedding $\mathrm{Gr}(2,5)\subset\mathbb{P}^9$. It is known that the normal bundle $N_{L/Y}$ of a line L in Y is one of the following types ([12, Lemma 1.6])

$$N_{L/Y}\cong\mathcal{O}_L(1)\oplus\mathcal{O}_L^{\oplus 2}\text{ or }\mathcal{O}_L(-1)\oplus\mathcal{O}_L(1)^{\oplus 2}.$$

Let us call the line of the first case (resp. the second case) by free line (resp. non-free line). Let $\mathbf{H}_d(Y)$ be the Hilbert scheme of curves C with Hilbert polynomial $\chi(\mathcal{O}_C(m))=dm+1$ in Y. Let us define the *double line* L^2 as the non-split extension sheaf F ($\cong\mathcal{O}_{L^2}$)

$$0\to\mathcal{O}_L(-1)\to F\to\mathcal{O}_L\to 0,$$

where L is a line. A double line L^2 in Y supported on L is classified by $\mathrm{Ext}^1(\mathcal{O}_L,\mathcal{O}_L(-1))\cong\mathrm{H}^0(N_{L/Y}(-1))$. Hence the double line L^2 for a (resp. non-free) free line L in Y is unique (resp. parameterized by $\mathbb{P}^1$). The main result of this paper is the following.

Theorem 1.1. *Let $\mathbf{D}(Y)$ be the locus of double lines in $\mathbf{H}_2(Y)$. Then $\mathbf{D}(Y)$ is a 4-dimensional smooth subvariety of $\mathbf{H}_2(Y)$.*

By combining the result of [3] and Theorem 1.1, it turns out that non-free lines in Y consist of lines meeting with the *dual conic* $C_v^\vee$ at a point (Corollary 3.2). Furthermore we obtain the designularized model (i.e., a subvariety of *complete conics*) of the moduli space of stable maps of degree 2 in Y, which enable to compute the intersection cohomology group of the moduli space. For detail description, see Corollary 3.4.

Notation 1.2.

- Let us denote by $\mathrm{Gr}(k,n)$ the Grassimannian variety parameterizing k-dimensional subspaces in a fixed vector space V with $\dim V=n$.
- We sometimes do not distinguish the moduli point $[x]\in\mathcal{M}$ and the object x parameterized by $[x]$ when no confusion can arise.
- Let us shortly denote the projectivized linear subspace $\mathbb{P}(e_i,\dots,e_j)=\mathbb{P}(\mathrm{span}\{e_i,\dots,e_j\})$ in $\mathbb{P}(V_5)$, where $\{e_0,e_2,\dots,e_4\}$ is the standard basis of the vector space V_5 of dimension 5.

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2. Preliminaries

In this section we collect some facts about the quintic del Pezzo fourfold which are mostly taken from [3] and [11].

One can define a Schburt variety relating with lines and planes in $\mathrm{Gr}(2,5)$ as follow. For fixed a flag $p \in \mathbb{P}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^3 \subset \mathbb{P}^4$, let

- $\sigma_{3,2} = \{\ell \,|\, p \in \ell \subset \mathbb{P}^2\}$,
- $\sigma_{3,1} = \{\ell \,|\, p \in \ell \subset \mathbb{P}^3\}$,
- $\sigma_{2,2} = \{\ell \,|\, \ell \subset \mathbb{P}^2\}$.

Clearly, $\sigma_{3,2}$ is a line in $\mathrm{Gr}(2,5)$ and thus it is parameterized by the *flag* variety $\mathrm{Gr}(1,3,5)$. Also, we note that the planes in $\mathrm{Gr}(2,5)$ with $\sigma_{3,1}$ (resp. $\sigma_{2,2}$)-type is parameterized by $\mathrm{Gr}(1,4,5)$ (resp. $\mathrm{Gr}(3,5)$). The projection maps $v_1 : \mathrm{Gr}(1,3,5) \to \mathrm{Gr}(1,5)$ and $v_2 : \mathrm{Gr}(1,4,5) \to \mathrm{Gr}(1,5)$ are called by the *vertex map.* In [11], the Hilbert scheme of lines and planes in Y are explicitly described. For a projective variety X with fixed embedding in $\mathbb{P}^N$, let $\mathbf{H}_1(X)$ (resp. $\mathbf{F}_2(X)$) be the Hilbert scheme of lines (resp. planes) in X.

Proposition 2.1 ([11, Proposition 2.7])**.** *Let $i : \mathbf{H}_1(Y) \subset \mathbf{H}_1(\mathrm{Gr}(2,5)) = \mathrm{Gr}(1,3,5)$ be the inclusion map and $v_1 : \mathrm{Gr}(1,3,5) \to \mathrm{Gr}(1,5)$ be the vertex map. Then the composition map $v_1 \circ i : \mathbf{H}_1(Y) \to \mathrm{Gr}(1,5)$ is a smooth blow-up along the smooth conic $C_v \subset \mathrm{Gr}(1,5)$.*

Let us call C_v by the *vertex conic* in Proposition 2.1.

Proposition 2.2 ([11, Proposition 2.2])**.** *The Hilbert scheme of planes in Y is isomorphic to*

$$\mathbf{F}_2(Y) \cong C_v \sqcup \{[S]\}.$$

Here each point $t \in C_v(\cong \mathbb{P}^1)$ parameterizes the $\sigma_{3,1}$-type planes P_t such that the vertex of the plane P_t is the point $\{t\}$ in C_v. Also the point $[S]$ parameterizes the $\sigma_{2,2}$-type plane S in Y determined by the linear spanning $\langle C_v\rangle \subset \mathrm{Gr}(1,5) \cong \mathbb{P}^4$ of C_v.

Let $\{e_0, e_1, e_2, e_3, e_4\}$ be the standard coordinate vectors of the space $V_5(\cong \mathbb{C}^5)$, which provides the original projective space $\mathbb{P}(V_5)(= \mathbb{P}^4)$. Let $\{p_{ij}\}_{0\le i<j\le 4}$ be the Plücker coordinates of $\mathbb{P}^9$. Let $\mathbb{P}^7 = H_1 \cap H_2$ be the linear subspace of $\mathbb{P}^9$ defined by $p_{12} - p_{03} = p_{13} - p_{24} = 0$. The vertex conic C_v is given by ([3, Lemma 6.3])

$$C_v = \{[a_0 : a_1 : a_2 : a_3 : a_4] \mid a_0a_4 + a_1^2 = a_2 = a_3 = 0\} \subset \mathbb{P}(V_5).$$

Remark 2.3. From the proof of [3, Lemma 6.3], we know that $\sigma_{3,1}$-type planes P_t in Y are $P_t = \mathbb{P}(V_1 \wedge V_4)$, where $V_1 = \mathrm{span}\{e_0 + te_1 - t^2e_4\}$ and $V_4 = \mathrm{span}\{e_0, e_1, e_2 + te_3, e_4\}$. Also the unique plane S in Y is given by $S = \mathbb{P}(\wedge^2 V_3)$ such that $V_3 = \mathrm{span}\{e_0, e_1, e_4\}$.

The positional relations of planes in Y are as follows.

Proposition 2.4 ([11, Proposition 2.2])**.** *Let P_t be a $\sigma_{3,1}$-type plane and S be the unique $\sigma_{2,2}$-type plane in Y. Then*

(1) *the intersection part* $P_t \cap S$ *is a tangent line of the dual conic*[1] $C_v^\vee$ *in* Y.
(2) *the intersection part* $P_t \cap P_{t'}$ *is a point in* S *for any* $t \neq t' \in C_v$.

The lines in Y have a stratification relating with the plane's type in Y.

Proposition 2.5 ([11, Corollary 3.7])**.** *Let* L *be a line in* Y *and* $R = \bigcup_{t\in C_v} P_t$ *be the union of planes in* Y. *Then there are five types of lines in* Y *such that the automorphism group* $\mathrm{Aut}(Y)$ *of* Y *transitively acts on each stratum.*

(a) $L \not\subset R \cup S$.
(b) $L \subset R$, $L \cap S = \{\text{pt.}\}$ *and* $L \cap C_v^\vee = \emptyset$.
(c) $L \subset R$, $L \cap S = L \cap C_v^\vee = \{\text{pt.}\}$.
(d) $L \subset S$ *and* L *is a tangent line of* $C_v^\vee$.
(e) $L \subset S$ *and* $L \cap C_v^\vee = \{p_1, p_2\}$ *for* $p_1 \neq p_2$.

In Section 6 of [3], the authors reproduce the results of Propositions 2.1, 2.2, and 2.4 by specifying the linear subspace $\mathbb{P}^7 \subset \mathbb{P}^9$.

Example 2.6. Let P_{t_0} be the plane determined by the vertex $\mathbb{P}(e_0)$ and the three dimensional space $\mathbb{P}(e_0, e_1, e_2, e_4)$. The intersection point is $P_{t_0} \cap C_v^\vee = \mathbb{P}(e_0 \wedge e_1)$ which is the tangent line of C_v at $\mathbb{P}(e_0)$. Furthermore, the example of lines in Proposition 2.5 is given in Table 1.

Table 1. Example of lines in Y

Type	Vertex	Plane
(a)	$\mathbb{P}(e_2)$	$\mathbb{P}(e_0, e_2, e_3)$
(b)	$\mathbb{P}(e_0)$	$\mathbb{P}(e_0, e_2, e_4)$
(c)	$\mathbb{P}(e_0)$	$\mathbb{P}(e_0, e_1, e_2)$
(d)	$\mathbb{P}(e_0)$	$\mathbb{P}(e_0, e_1, e_4)$
(e)	$\mathbb{P}(e_1)$	$\mathbb{P}(e_0, e_1, e_4)$

Let $\mathbf{H}_2(Y)$ be the Hilbert scheme of conics in Y. For a general conic C in Y, it determines linear spanning in two meanings: the linear space $\mathbb{P}^2$ containing C in $\mathbb{P}(\wedge^2 V_5) = \mathbb{P}^9$ and the linear space $\mathbb{P}^3$ containing two skew lines in $\mathbb{P}(V_5) = \mathbb{P}^5$. Motivated this observation, we have a birational model of $\mathbf{H}_2(Y)$ as follow. Let $\mathcal{U}$ be the universal subbundle on $\mathrm{Gr}(4,5)$ and

$$\mathcal{K} := \ker\{\wedge^2 \mathcal{U} \subset \wedge^2 \mathcal{O}^{\oplus 5} \to \mathcal{O}^{\oplus 2}\}$$

be the kernel of the composition map, where the arrow is given by $\{p_{12} - p_{03}, p_{13} - p_{24}\}$. Let $\mathbf{S}(Y) := \mathrm{Gr}(3, \mathcal{K})$ be the relative Grassmannian over $\mathrm{Gr}(4,5)$.

[1]That is, the curve is generated by the tangent lines of C_v.

Proposition 2.7 ([3, Proposition 6.7 and Remark 6.8])**.** *Under above definition and notations,* $\mathbf{H}_2(Y)$ *is obtained from* $\mathbf{S}(Y)$ *by a blow-down followed by a blow-up*

$$\begin{array}{ccc} & \widetilde{\mathbf{S}}(Y) & \\ \swarrow & & \searrow \\ \mathbf{S}(Y) & \overset{\Psi}{\dashrightarrow} & \mathbf{H}_2(Y), \end{array}$$

where

(1) *the blow-up center in* $\mathbf{S}(Y)$ (*resp.* $\mathbf{H}_2(Y)$) *is a disjoint union* $\mathbb{P}^1 \sqcup \mathbb{P}^1$ (*resp.* $\mathbb{P}^5$) *of* $\mathbb{P}^1$*'s and*
(2) *the space* $\widetilde{\mathbf{S}}(Y)$ *is a relative conics space over* $\mathrm{Gr}(4,5)$ *such the fiber over* $\mathrm{Gr}(4,5)$ *is the Hilbert scheme* $\mathbf{H}_2(\mathrm{Gr}(2,4)\cap H_1 \cap H_2)$ *of conics in the quadric surface* $\mathrm{Gr}(2,4)\cap H_1\cap H_2$.

In special $\mathbf{H}_2(Y)$ is an irreducible and smooth variety of dimension 7.

Remark 2.8. The relative Grassimannian $\mathbf{S}(Y) = \mathrm{Gr}(3, \mathcal{K})$ in Proposition 2.7 can be regarded as the incident variety

$$\mathbf{S}(Y) = \{(U_3, V_4) \mid U_3 \subset \mathcal{K}_{[V_4]}\} \subset \mathrm{Gr}(3, \wedge^2 V_5) \times \mathrm{Gr}(4, V_5), \tag{2.1}$$

where $\mathcal{K}_{[V_4]} = \ker\{\wedge^2 V_4 \subset \wedge^2 V_5 \overset{(p_{12}-p_{03})\oplus(p_{13}-p_{24})}{\to} \mathbb{C}\oplus\mathbb{C}\}$. Also the birational correspondence $\Psi : \mathbf{S}(Y) \dashrightarrow \mathbf{H}_2(Y)$ is $\Psi([(U_3, V_4)]) = \mathbb{P}(U_3) \cap \mathrm{Gr}(2, V_4)$. Note that the map Ψ is not defined at the two distinct points $[P_t]$ and $[S]$ over a linear subspace $\mathbb{P}^1(\cong C_v)$ in $\mathrm{Gr}(4,5)$ ([3, Remark 6.8]).

3. Results

In this section we prove Theorem 1.1. As corollaries, we have a description of the locus of non-free lines in Y (Corollary 3.2). Also we obtain the desinguarized model of stable maps space in Y and thus its intersection cohomology (Corollary 3.4).

3.1. Proof of Theorem 1.1

Firstly, we describe the closure of the birational inverse Ψ^{-1} of the double line in $\mathbf{H}_2(Y)$ in Proposition 2.7. Then we find explicitly the strict transform of the closure along the blow-up/down maps in Proposition 2.7.

Let $\bar{\mathbf{D}}(Y)$ be the locus of the pairs (U_3, V_4) in $\mathbf{S}(Y)$ such that the restriction $q_G|_{\mathbb{P}(U_3)}$ to $\mathbb{P}(U_3)$ of the quadric form q_G associated to $\mathrm{Gr}(2, V_4)$ is rank ≤ 1. Let

$$p = p_2 \circ i : \bar{\mathbf{D}}(Y) \to \mathrm{Gr}(4,5)$$

be the composition of the second projection map $p_2 : \mathbf{S}(Y) \to \mathrm{Gr}(4,5)$ in equation (2.1) and the inclusion map $i : \bar{\mathbf{D}}(Y) \subset \mathbf{S}(Y)$.

Lemma 3.1. *Under above definition and notations, the image* $p(\bar{\mathbf{D}}(Y)) := Q_3$ *is an irreducible quadric* 3*-fold in* $\mathrm{Gr}(4,5)$ *with the homogenous coordinates* x_0, x_1, x_2, x_3, x_4 *such that* Q_3 *is defined by* $x_1^2 + 4x_0x_2 = 0$.

Proof. For the chart $x_3 \neq 0$, let

$$[V_4] := \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}$$

be an affine chart of $\mathrm{Gr}(4,5)$ with $a = x_0/x_3$, $b = x_1/x_3$, $c = x_4/x_3$, $d = x_2/x_3$ and $V_4 = \mathrm{span}\{e_0 + ae_3, e_1 + be_3, e_2 + ce_3, e_4 + de_3\}$. Then for an affine chart of $\mathrm{Gr}(2, V_4)$

$$[V_2] = \begin{pmatrix} 1 & 0 & t_1 & t_3 \\ 0 & 1 & t_2 & t_4 \end{pmatrix},$$

the affine chart of $\mathrm{Gr}(2, V_4)$ in $\mathrm{Gr}(2,5)$ is

$$[V_2][V_4] = \begin{pmatrix} 1 & 0 & t_1 & a + ct_1 + dt_3 & t_3 \\ 0 & 1 & t_2 & b + ct_2 + dt_4 & t_4 \end{pmatrix}.$$

After eliminating the variables $\{t_1, t_2, t_3, t_4\}$ by the computer program ([8]), we have a defining equation of $\mathrm{Gr}(2, V_4) \cap H_1 \cap H_2$

$$\begin{aligned} &\langle bcp_{01}^2 + c^2 p_{01}p_{02} + cdp_{01}p_{04} - ap_{01}^2 + bp_{01}p_{04} + cp_{02}p_{04} \\ &\quad + dp_{04}^2 + dp_{01}p_{14} - p_{02}p_{14},\ h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6\rangle, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &h_1 = p_{03} - p_{12},\ h_2 = p_{12} - bp_{01} - cp_{02} - dp_{04},\ h_3 = p_{13} - p_{24}, \\ &h_4 = p_{23} + ap_{02} + bp_{12} - dp_{24},\ h_5 = p_{24} + ap_{01} - cp_{12} - dp_{14}, \\ &h_6 = p_{34} - ap_{04} - bp_{14} - cp_{24} \end{aligned}$$

in $\mathbb{P}^9 \times \mathbb{C}_{(a,b,c,d)}$.

For the chart $x_4 \neq 0$, let $a = x_0/x_4$, $b = x_1/x_4$, $u = x_3/x_4$, $d = x_2/x_4$. By doing the same calculation as before, we obtain the local equation of $\mathrm{Gr}(2, V_4) \cap H_1 \cap H_2$ as follows:

$$\begin{aligned} &\langle ap_{04}^2 + ap_{01}p_{14} + bp_{04}p_{14} - dp_{14}^2 - p_{01}p_{34} - aup_{04}p_{14} \\ &\quad - bup_{14}^2 - up_{04}p_{34} + u^2 p_{14}p_{34},\ k_1,\ k_2,\ k_3,\ k_4,\ k_5,\ k_6\rangle, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &k_1 = p_{02} + bp_{01} + dp_{04} - up_{12},\ k_2 = p_{03} - p_{12},\ k_3 = p_{12} - ap_{01} + dp_{14} - up_{24}, \\ &k_4 = p_{13} - p_{24},\ k_5 = p_{23} + ap_{12} + bp_{24} - dp_{34},\ k_6 = p_{24} + ap_{04} + bp_{14} - up_{34} \end{aligned}$$

in $\mathbb{P}^9 \times \mathbb{C}_{(a,b,u,d)}$.

Since the restricted form q_G associated to $\mathrm{Gr}(2, V_4)$ is of $\mathrm{rank}(q_G|_{\mathbb{P}(U_3)}) \leq 1$, the quadratics of the defining equations (3.1) and (3.2) is of rank ≤ 3. Therefore the defining equation of the image $p(\bar{\mathbf{D}}(Y))$ is given by

$$\langle b^2 + 4ad \rangle$$

in both cases. □

Obviously, the singular locus $\mathrm{Sing}(Q_3)(\cong \mathbb{P}^1)$ is defined by $I_{\mathrm{Sing}(Q_3)} = \langle x_0, x_1, x_2 \rangle$.

Proof of Theorem 1.1. **Step 1**. For each $[V_4] \in Q_3 \setminus \mathrm{Sing}(Q_3)$, the quadric surface $\mathrm{Gr}(2, V_4) \cap H_1 \cap H_2$ is of rank 3. Hence the fiber $p^{-1}([V_4])$ is isomorphic to $\mathbb{P}^1$ which parameterizes tangent planes (i.e., lines) of the quadric cone $\mathrm{Gr}(2, V_4) \cap H_1 \cap H_2$. If $[V_4] \in \mathrm{Sing}(Q_3)$, the singular quadric surface $\mathrm{Gr}(2, V_4) \cap H_1 \cap H_2$ is the union of the plane P_t and S. In fact, for the affine chart $x_3 \neq 0$ (similarly, $x_4 \neq 0$), it is defined by the union of $\sigma_{3,1}$-type planes:

$$\langle c^2 p_{01} + cp_{04} - p_{14}, p_{23}, cp_{24} - p_{34}, cp_{02} - p_{12}, -cp_{12} + p_{24}, p_{13} - p_{24}, p_{03} - p_{12} \rangle$$

and the $\sigma_{2,2}$-plane S:

$$\langle p_{02}, p_{23}, cp_{24} - p_{34}, cp_{02} - p_{12}, -cp_{12} + p_{24}, p_{13} - p_{24}, p_{03} - p_{12} \rangle$$

which matches with Remark 2.3 (by letting $c = t$). Hence the fiber $p^{-1}([V_4])$ is isomorphic to $\mathbb{P}^1$ which parameterizes planes containing the intersection line $P_t \cap S$. After all, $\bar{\mathbf{D}}(Y)$ is a $\mathbb{P}^1$-fiberation over Q_3.

Step 2. Note that the birational map Ψ in Proposition 2.7 is not defined for the two points: $\{[P_t], [S]\}$ over $\mathrm{Sing}(Q_3)(\cong \mathbb{P}^1)$. Hence the blow-up center of $\eta : \widetilde{\mathbf{S}}(Y) \to \mathbf{S}(Y)$ is contained in $\bar{\mathbf{D}}(Y)$ and thus the strict transform of $\bar{\mathbf{D}}(Y)$ by the blow-up map η is nothing but the blow-up $\widetilde{\mathbf{D}}(Y)$ of $\bar{\mathbf{D}}(Y)$ along the center $\mathbb{P}^1 \sqcup \mathbb{P}^1$. Since the blow-center of $\bar{\mathbf{D}}(Y)$ is of $\mathbb{Z}_2$-quotient singularity, one can easily check that $\widetilde{\mathbf{D}}(Y)$ is smooth and the exceptional divisor $\mathbf{E}$ in $\widetilde{\mathbf{D}}(Y)$ is a $\mathbb{P}(1,2,2)(\cong \mathbb{P}^2)$-bundle over $\mathbb{P}^1 \sqcup \mathbb{P}^1$. Each fiber $\mathbb{P}^2$ parameterizes the double line in the plane because any flat family in $\bar{\mathbf{D}}(Y)$ is obviously supported on lines by its construction.

Step 3. The restriction to each fiber $\mathbb{P}^1$ of the normal bundle $\mathcal{N}_{\mathbf{E}/\widetilde{\mathbf{D}}(Y)}$ of the exceptional divisor $\mathbf{E}$ is $\mathcal{N}_{\mathbf{E}/\widetilde{\mathbf{D}}(Y)}|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, the image $\mathbf{D}(Y)$ of the restriction to $\widetilde{\mathbf{D}}(Y)$ of the blow-down map $\widetilde{\mathbf{S}}(Y) \to \mathbf{H}_2(Y)$ is smooth by the Fujiki-Nakano criterion ([6]). So we finish the proof. □

3.2. Non-free lines in Y and the intersection cohomology of stable maps

Corollary 3.2. *Let Z be the locus of non-free lines in the Hilbert scheme $\mathbf{H}_1(Y)$ of lines in Y. Then Z is isomorphic to a $\mathbb{P}^1$-fiberation over the vertex conic $C_v(\cong \mathbb{P}^1)$.*

Proof. Geometrically, non-free lines are lines in Y meeting with the dual conic $C_v^\vee$ at a point uniquely. Since the automorphism of Y transitively acts on each stratum of Proposition 2.5, it is enough to check each case in Table 1. For the case (d) and (e), L is a non-free line if and only if L is a tangent line of $C_v^\vee$ by [3, Proposition 6.6]. Thus the lines of the case (d) are only non-free. For the case (c), the line L is defined by $p_{03} = p_{04} = p_{12} = p_{13} = p_{14} = p_{23} = p_{24} = p_{34} = 0$. Thus for the affine chart $x_3 \neq 0$, it lies on irreducible quadric cones defined by $dp_{04}^2+(dp_{01}-p_{02})p_{14} = h_1 = p_{12}-dp_{04} = h_3 = p_{23}-dp_{24} = p_{24}-dp_{14} = p_{34} = 0$ for $d \neq 0$. Hence there exists one parameter family of double lines supported on L. That is, L is non-free. For other cases (a) and (b), we know that each line is free by a similar computation. □

Let C be a projective connected reduced curve. A map $f : C \to Y$ is considered *stable* if C has at worst nodal singularities and $|\mathrm{Aut}(f)| < \infty$. Let $\mathcal{M}(Y, d)$ be the moduli space of isomorphism classes of stable maps $f : C \to Y$ with genus $g(C) = 0$ and $\deg(f^*\mathcal{O}_Y(1)) = d$. The moduli space $\mathcal{M}(Y, d)$ might be singular and reducible depending on the geometric property (for example, convexity) of Y.

Remark 3.3. Let $f : C \to L \subset Y$ be a stable map of the degree $\deg(f) = 2$ such that L is non-free. From the tangent bundle sequence of $L \subset Y$, $\mathrm{H}^1(f^*T_Y) \cong \mathrm{H}^1(f^*N_{L/Y}) \cong \mathrm{H}^1(f_*\mathcal{O}_C \otimes N_{L/Y}) \cong \mathbb{C}$. That is, Y is not convex and thus $\mathcal{M}(Y, 2)$ is not a smooth stack ([7]).

Let X be a quasi-projective variety. For the (resp. intersection) Hodge-Deligne polynomial $\mathrm{E}_c(X)(u, v)$ (resp. $\mathrm{IE}_c(X)(u, v)$) for compactly supported (resp. intersection) cohomology of X, let

$$\mathrm{P}(X) = \mathrm{E}_c(X)(-t, -t) \text{ (resp. } \mathrm{IP}(X) = \mathrm{IE}_c(X)(-t, -t))$$

be the *virtual* (resp. intersection) Poincaré polynomial of X. A map $\pi : X \to Y$ is *small* if for a locally closed stratification of $Y = \bigsqcup_i Y_i$ such that the restriction map $\pi|_{\pi^{-1}(Y_i)} : \pi^{-1}(Y_i) \to Y_i$ is *etale* locally trivial, the inequality

$$\dim \pi^{-1}(y) < \frac{1}{2}\mathrm{codim}_Y(Y_i)$$

holds for each closed point $y \in Y_i$ except a dense open stratum of Y. Let $\pi : X \to Y$ be a small map such that X has at most finite group quotient singularities. Then $\mathrm{P}(X) = \mathrm{IP}(Y)$ ([10, Definition 6.6.1 and Theorem 6.6.3]).

Corollary 3.4. *The intersection cohomology of the moduli space $\mathcal{M}(Y, 2)$ is given by*

$$\mathrm{IP}(\mathcal{M}(Y, 2)) = 1 + 4t^2 + 10t^4 + 15t^6 + 15t^8 + 10t^{10} + 4t^{12} + t^{14}.$$

Proof. By the same method of the proof of Theorem 1.2 in [1], one can show that the blow-up $\widetilde{\mathbf{H}}_2(Y)$ of $\mathbf{H}_2(Y)$ along $\mathbf{D}(Y)$ is smooth one and thus we have a birational morphism

$$\pi : \widetilde{\mathbf{H}}_2(Y) \to \mathcal{M}(Y, 2)$$

such that the exceptional divisor (i.e., $\mathbb{P}^2$-bundle over $\mathbf{D}(Y)$) contracts to a $\mathbb{P}^2$-bundle over $\mathbf{H}_1(Y)$. From Corollary 3.2, the map π is a small map and thus $\mathrm{P}(\widetilde{\mathbf{H}}_2(Y)) = \mathrm{IP}(\mathcal{M}(Y,2))$. By Proposition 2.7, Corollary 3.2, and Proposition 2.1, we have

$$\begin{aligned} P(\mathbf{H}_2(Y)) &= P(\mathbf{S}(Y)) + 2P(\mathbb{P}^1)\cdot(P(\mathbb{P}^5)-1) - P(\mathbb{P}^5)\cdot(P(\mathbb{P}^1)-1),\\ P(\mathbf{D}(Y)) &= P(\mathbf{H}_1(Y)) - P(Z) + P(Z)\cdot P(\mathbb{P}^1),\\ P(\mathbf{H}_1(Y)) &= P(\mathbb{P}^4) + P(C_v)\cdot(P(\mathbb{P}^2)-1). \end{aligned}$$

Also, by the equality

$$\mathrm{P}(\widetilde{\mathbf{H}}_2(Y)) = \mathrm{P}(\mathbf{H}_2(Y)) + (\mathrm{P}(\mathbb{P}^2)-1)\cdot\mathrm{P}(\mathbf{D}(Y)),$$

we obtain the result. □

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