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A NOTE ON COMPARISON PRINCIPLE FOR ELLIPTIC OBSTACLE PROBLEMS WITH L^1-DATA

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Abstract. In this note, we study a comparison principle for elliptic obstacle problems of p-Laplacian type with L^1-data. As a consequence, we improve some known regularity results for obstacle problems with zero Dirichlet boundary conditions.

1. Introduction

We consider obstacle problems related to inhomogeneous elliptic equations of the form

$$-\mathrm{div}\,(A(x,Du)) = f \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain and $f \in L^1(\Omega)$. The vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the second variable, with $\partial_z A(\cdot)$ being Carathéodory regular, and to satisfy the following growth and monotonicity assumptions

$$|A(x,z)| + |z||\partial_z A(x,z)| \leq L|z|^{p-1} \tag{2}$$

and

$$0 < (A(x,z_1) - A(x,z_2)) \cdot (z_1 - z_2) \tag{3}$$

for every $z, z_1, z_2 \in \mathbb{R}^n$ with $z \neq 0$, $z_1 \neq z_2$ and a.e. $x \in \Omega$, where $L > 0$ and $p > 1$ are fixed constants.

Before introducing the precise formulation and a notion of solutions to obstacle problems with L^1-data, let us first consider the classical assumptions on data and constraint. For an obstacle function $\psi \in W^{1,p}(\Omega)$, a Dirichlet boundary data $g \in W^{1,p}(\Omega)$ with $(\psi - g)^+ \in W_0^{1,p}(\Omega)$ and a function $f \in W^{-1,p'}(\Omega)$, the obstacle problem for (1) is formulated by the variational inequality

$$\int_\Omega A(x,Du) \cdot D(\phi - u)\,dx \geq \int_\Omega f(\phi - u)\,dx \qquad \forall\ \phi \in \mathcal{A}^g_\psi(\Omega), \tag{4}$$

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where the convex admissible set $\mathcal{A}_{\psi}^{g}(\Omega)$ is defined by

$$\mathcal{A}_{\psi}^{g}(\Omega)=\left\{\phi\in g+W_0^{1,p}(\Omega):\phi\geq\psi\text{ a.e. in }\Omega\right\}.$$

When $g\equiv 0$, we simply write $\mathcal{A}_{\psi}(\Omega)=\mathcal{A}_{\psi}^{g}(\Omega)$. The existence and uniqueness of such a weak solution u to the variational inequality (4) follow from the classical result [16].

However, the integral in the right-hand side of (4) is not well-defined when f is merely an L^1 function or a Borel measure. For the case of equation (1), Boccardo and Gallouët first proved the existence of a class of distributional solutions to elliptic and parabolic equations with measure data in the pioneering work [3], and the notion of solutions was extended to various settings, see for instance [13, Section 3.2] and references therein. Such solutions to measure data problems were obtained by approximating the right-hand side f, getting uniform a priori estimates for the gradient of solutions to the corresponding regularized problems, and then taking the limit. Later on, the approximation argument was extended to elliptic obstacle problems with measure data, see Definition 1 below. The uniqueness of solutions to general measure data problems is still an open problem, while it is known for $f\in L^1$. We refer to [21] for the existence of solutions to obstacle problems with measure data and [2] for the uniqueness results for equations with L^1-data.

Note that such approximation procedures for the class of solutions usually involve some truncation arguments. We introduce the truncation operators

$$T_t(y)=\min\{t,\max\{-t,y\}\},\qquad y\in\mathbb{R}^n$$

for any $t>0$. Then, for a given boundary data $g\in W^{1,p}(\Omega)$, we set

$$\mathcal{T}_g^{1,p}(\Omega)=\left\{u\text{ is measurable in }\Omega\,:\,T_t(u-g)\in W_0^{1,p}(\Omega)\text{ for every }t>0\right\}.$$

For any $u\in\mathcal{T}_g^{1,p}(\Omega)$, there exists a unique measurable function $Z_u:\Omega\to\mathbb{R}^n$ such that

$$D[T_t(u)]=\chi_{\{|u|<t\}}Z_u\qquad\text{a.e. in }\Omega$$

for every $t>0$. If $u\in\mathcal{T}_g^{1,p}(\Omega)\cap W^{1,1}(\Omega)$, then Z_u coincides with the weak derivative Du of u. In what follows, we denote Z_u by Du for the simplicity of notation.

Our results will be obtained for *a limit of approximating solutions* defined as follows:

Definition 1. Assume that $\psi,g\in W^{1,p}(\Omega)$ with $(\psi-g)^+\in W_0^{1,p}(\Omega)$ and $f\in L^1(\Omega)$. We say that a function $u\in\mathcal{T}_g^{1,p}(\Omega)$ with $u\geq\psi$ a.e. in Ω is a *limit of approximating solutions* to the obstacle problem $OP(\psi;f)$ if there is a sequence of functions

$$\{f_k\}_{k\in\mathbb{N}}\subset L^\infty(\Omega)\quad\text{with }f_k\to f\text{ in }L^1(\Omega)\tag{5}$$

and a sequence of solutions $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{A}^g_\psi(\Omega)$ to

$$\int_\Omega A(x, Du_k)\cdot D(\phi - u_k)\,dx \geq \int_\Omega f_k(\phi - u_k)\,dx \qquad \forall\, \phi \in \mathcal{A}^g_\psi(\Omega)$$

with the following convergence

$$\begin{cases} u_k \to u & \text{a.e. in } \Omega, \\ \displaystyle\int_\Omega |u_k - u|^r\,dx \to 0 & \text{for every } 0 < r < \frac{n(p-1)}{n-p}, \\ \displaystyle\int_\Omega |Du_k - Du|^q\,dx \to 0 & \text{for every } 0 < q < \frac{n(p-1)}{n-1}. \end{cases} \tag{6}$$

Note that if $p > 2 - \frac{1}{n}$, then $\frac{n(p-1)}{n-1} > 1$ and Du_k converges to Du in $L^q(\Omega)$ for every $\max\{1, p-1\} \leq q < \frac{n(p-1)}{n-1}$ in Definition 1. Hence, in this case a *limit of approximating solutions* u belongs to the Sobolev space $W^{1,1}(\Omega)$.

We refer to [21, Lemma 3.4] for the proof of the existence of a *limit of approximating solutions* under assumptions (2) and (8). It is worth mentioning that in [21], $\{f_k\}_{k\in\mathbb{N}}$ is taken to be a sequence in $W^{-1,p'}(\Omega)\cap L^1(\Omega)$ which is not contained in $L^\infty(\Omega)$ in general. However, if one takes $\{f_k\}_{k\in\mathbb{N}}$ as the sequence of mollifications of f, then the sequence is a subset of $L^\infty(\Omega)$ which satisfies the assumptions in Definition 1. Hence, it is not restrictive to take $\{f_k\}_{k\in\mathbb{N}}$ in $L^\infty(\Omega)$ in Definition 1. Moreover, such a construction gives the strong L^1-convergence (5) for L^1-data, while only weak* convergence can be assured for measure data. This will play a crucial role in the proof of uniqueness results, see Lemma 3.2 below.

In this paper, we provide a comparison principle for obstacle problems with L^1-data. As a consequence of the comparison principle, we show that the solution to a given obstacle problem with zero Dirichlet boundary data is indeed affected by only the positive part of the obstacle, instead of the whole obstacle.

2. Preliminaries

In what follows, we denote a generic constant depending only on n, p, ν, L by $c \geq 1$ which may vary from line to line. For any $q > 1$, $q' = q/(q-1)$ is the Hölder conjugate exponent of q.

We denote by $B_r(x_0)$ the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. If there is no confusion, we simply denote $B_r = B_r(x_0)$. The Lebesgue measure of a measurable set $S \subset \mathbb{R}^n$ is denoted by $|S|$. For an integrable map $f : S \to \mathbb{R}^k$, with $k \geq 1$ and $0 < |S| < \infty$, we write

$$(f)_S := \fint_S f\,dx := \frac{1}{|S|}\int_S f\,dx$$

to mean the integral average of f over S.

We now introduce additional assumptions on the vector field $A(\cdot)$ for regularity results. We say that $A(\cdot)$ is strongly elliptic if

$$\nu|z|^{p-2}|\xi|^2 \leq \partial_z A(x,z)\xi \cdot \xi \tag{7}$$

holds for some $\nu > 0$ and for every $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. It is readily seen that (7) implies the following monotonicity condition

$$\frac{1}{c}|V(z_1) - V(z_2)|^2 \leq (A(x,z_1) - A(x,z_2)) \cdot (z_1 - z_2) \tag{8}$$

for any $z_1, z_2 \in \mathbb{R}^n$, where the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$V(z) = |z|^{\frac{p-2}{2}} z, \quad z \in \mathbb{R}^n.$$

Note that $V(\cdot)$ is a locally bi-Lipschitz bijection on $\mathbb{R}^n$ satisfying $V(0) = 0$ and

$$\frac{1}{c}(|z_1| + |z_2|)^{\frac{p-2}{2}}|z_1 - z_2| \leq |V(z_1) - V(z_2)| \leq c(|z_1| + |z_2|)^{\frac{p-2}{2}}|z_1 - z_2|.$$

Hence, (7) implies (3). Moreover, (2) and (7) further give

$$\frac{1}{c}|V(z_1) - V(z_2)|^2 \leq (A(x,z_1) - A(x,z_2)) \cdot (z_1 - z_2) \leq c|V(z_1) - V(z_2)|^2.$$

3. Comparison principles and their applications

The comparison principle for weak solutions to obstacle problems is well-known, which we state as follows:

Lemma 3.1. *Assume that $g, \psi_1, \psi_2 \in W^{1,p}(\Omega)$ satisfy $(\psi_1 - g)^+, (\psi_2 - g)^+ \in W_0^{1,p}(\Omega)$ and $f_1, f_2 \in L^\infty(\Omega)$. Under assumptions* (2) *and* (3)*, let $u_1 \in \mathcal{A}^g_{\psi_1}(\Omega)$ and $u_2 \in \mathcal{A}^g_{\psi_2}(\Omega)$ be the unique weak solutions to* (4) *with $(\psi, f) = (\psi_1, f_1)$ and $(\psi, f) = (\psi_2, f_2)$, respectively. Then*

$$\psi_1 \leq \psi_2,\ f_1 \leq f_2 \text{ implies } u_1 \leq u_2 \quad \text{a.e. in } \Omega.$$

We refer to [20, Theorem 3.2] for the proof of Lemma 3.1, where such a comparison principle is obtained for inhomogeneous double obstacle problems with general growth. Its proof works for Lemma 3.1 in a similar way, see [20, Remark 3.7]. We note that such a comparison principle is obtained in the context of the Lewy-Stampacchia inequalities in an abstract form. We further refer to [22] for similar results in the setting of nonlocal problems.

In order to extend Lemma 3.1 to any *limits of approximating solutions*, we need the following uniqueness result.

Lemma 3.2. *Assume that $g, \psi \in W^{1,p}(\Omega)$ satisfy $(\psi - g)^+ \in W_0^{1,p}(\Omega)$ and $f \in L^1(\Omega)$. Under assumptions* (2) *and* (8)*, there exists a unique limit of approximating solutions $u \in \mathcal{T}^{1,p}_g(\Omega)$ to $OP(\psi; f)$.*

Proof. As mentioned above, the existence of u is proved in [21, Lemma 3.4]. To show the uniqueness, let u and $\bar{u}$ be two *limits of approximating solutions* to $OP(\psi; f)$. Then there are sequences of functions $\{f_k\}_{k\in\mathbb{N}}, \{\bar{f}_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ satisfying $f_k \to f$ and $\bar{f}_k \to f$ in $L^1(\Omega)$, and corresponding sequences of weak solutions $\{u_k\}_{k\in\mathbb{N}}, \{\bar{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_\psi^g(\Omega)$ to (4) with the data $\{f_k\}_{k\in\mathbb{N}}, \{\bar{f}_k\}_{k\in\mathbb{N}}$, respectively.

We then observe that $u_k + T_t(\bar{u}_k - u_k), \bar{u}_k + T_t(u_k - \bar{u}_k) \in \mathcal{A}_\psi^g(\Omega)$ for each $t > 0$. Testing $u_k + T_t(\bar{u}_k - u_k)$ to (4) with (u_k, f_k) and $\bar{u}_k + T_t(u_k - \bar{u}_k)$ to (4) with $(\bar{u}_k, \bar{f}_k)$ and subtracting them, we have

$$\begin{aligned} &\int_\Omega \chi_{\{|u_k-\bar{u}_k|\le t\}}(A(x,Du_k) - A(x,D\bar{u}_k))\cdot(Du_k - D\bar{u}_k)\,dx \\ &\le \int_\Omega (f_k - \bar{f}_k)T_t(u_k - \bar{u}_k)\,dx \end{aligned} \tag{9}$$

for $k \in \mathbb{N}$. The last convergence in (6) implies $Du_k \to Du$ a.e. in Ω, so we apply Fatou's lemma to (9) to discover

$$\int_\Omega \chi_{\{|u-\bar{u}|\le t\}}(A(x,Du) - A(x,D\bar{u}))\cdot(Du - D\bar{u})\,dx = 0,$$

where we have also used (3). Then $Du = D\bar{u}$ a.e. in the set $\{|u - \bar{u}| \le t\}$ for every $t > 0$. Taking into account the fact that $u, \bar{u} \in \mathcal{T}_g^{1,p}(\Omega)$, we obtain $T_t(u - \bar{u}) = 0$ for each $t > 0$, from which the desired uniqueness follows. □

Note that if a *limit of approximating solutions* u to $OP(\psi; f)$ under (2) and (8) belongs to the energy space $W^{1,p}(\Omega)$, then Lemma 3.2 implies that u is the unique weak solution to (4).

Theorem 3.3. *Assume that $g, \psi_1, \psi_2 \in W^{1,p}(\Omega)$ satisfy $(\psi_1 - g)^+, (\psi_2 - g)^+ \in W_0^{1,p}(\Omega)$ and $f_1, f_2 \in L^1(\Omega)$. Under assumptions (2) and (8), let $u_1 \in \mathcal{T}_g^{1,p}(\Omega)$ and $u_2 \in \mathcal{T}_g^{1,p}(\Omega)$ be the limits of approximating solutions to $OP(\psi_1; f_1)$ and $OP(\psi_2; f_2)$, respectively. Then*

$$\psi_1 \le \psi_2,\ f_1 \le f_2 \text{ implies } u_1 \le u_2 \quad \text{a.e. in } \Omega.$$

Proof. Assume that $\psi_1 \le \psi_2$ and $f_1 \le f_2$. We now extend f_1 and f_2 by zero outside Ω and then take $f_{1,k} = \eta_{1/k} * f_1$ and $f_{2,k} = \eta_{1/k} * f_2$ for each $k \in \mathbb{N}$, where $\eta_{1/k}$ is the standard mollifier. Let $u_{1,k}$ and $u_{2,k}$ be the weak solutions to (4) with $(\psi, f) = (\psi_1, f_{1,k})$ and $(\psi, f) = (\psi_2, f_{2,k})$, respectively. Then, since $f_{1,k} \le f_{2,k}$, Lemma 3.1 implies that $u_{1,k} \le u_{2,k}$ for every k. From Lemma 3.2 and Definition 1, we conclude that $u_1 \le u_2$ a.e. in Ω. □

We now consider problems with zero Dirichlet boundary condition and nonnegative data. It is readily seen that if $g \equiv 0$ and $0 \le f \in W^{-1,p'}(\Omega)$, then the unique weak solution u to (4) with the obstacle function $\psi \in W^{1,p}(\Omega)$ is a weak supersolution to (1). Then the maximum principle implies $u \ge 0$ a.e. in Ω, and hence it is the weak solution to (4) with the obstacle function $\psi^+ \in W^{1,p}(\Omega)$. This fact, together with the approximating procedure and the uniqueness result described in Lemma 3.2, yields the following corollary.

Corollary 3.4. *Assume that $g \equiv 0$, $\psi \in W^{1,p}(\Omega)$ satisfy $\psi^+ \in W_0^{1,p}(\Omega)$ and $0 \le f \in L^1(\Omega)$. Under assumptions* (2) *and* (8)*, the limit of approximating solutions u to $OP(\psi; f)$ is indeed the limit of approximating solutions to $OP(\psi^+; f)$.*

We note that the *limit of approximating solutions* to an obstacle problem is equal to the obstacle in a set called the contact set, so the regularity of the solution is at best limited to that of the obstacle. Moreover, Corollary 3.4 implies that, in the case of zero Dirichlet boundary condition and nonnegative L^1-data, the contact set is contained in the set $\{\psi \ge 0\}$.

In the following sections, we apply Corollary 3.4 to two regularity results for elliptic obstacle problems. One is a gradient potential estimate and the other is a Calderón-Zygmund type estimate. In what follows, we assume the Dirichlet boundary data $g \equiv 0$, the right-hand side $f \ge 0$ and the vector field $A(\cdot)$ satisfies (2) and (7).

3.1. An application to gradient potential estimates

In this section, we assume that $A(\cdot)$ does not depend on the variable x. In the recent paper [12], pointwise gradient estimates for *limits of approximating solutions* to obstacle problems with measure data are obtained. Note that such pointwise estimates are actually consequences of the oscillation estimates in [12, Theorem 1.2 and Theorem 1.3]. In the following,

$$\mathbf{I}_1^f(x,R) \coloneqq \int_0^R \left(r \fint_{B_r(x)} f \, d\tilde{x} \right) \frac{dr}{r}$$

denotes the truncated 1-Riesz potential of f.

Theorem 3.5. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be the limit of approximating solutions to the problem $OP(\psi; f)$ under assumptions* (2) *and* (7) *with $p > 2 - 1/n$. If*

$$\mathbf{I}_1^f(x,R) + \int_0^R \left(\fint_{B_r(x)} |A(D\psi^+) - (A(D\psi^+))_{B_r(x)}|^{p'} \, d\tilde{x} \right)^{\frac{1}{m}} \frac{dr}{r} < \infty$$

holds on a ball $B_R(x) \subset \Omega$, where $m \coloneqq \max\{p', 2\}$, then x is a Lebesgue point of $A(Du)$. Moreover, there exists a constant $c = c(n, p, \nu, L)$ such that the following estimate holds:

$$\begin{aligned} |Du(x)|^{p-1} \le {} & c \fint_{B_R(x)} |Du|^{p-1} \, d\tilde{x} + c\mathbf{I}_1^f(x,R) \\ & + c \left[\int_0^R \left(\fint_{B_r(x)} |A(D\psi^+) - (A(D\psi^+))_{B_r(x)}|^{p'} \, d\tilde{x} \right)^{\frac{1}{m}} \frac{dr}{r} \right]^{\frac{m}{p'}} . \end{aligned}$$

Pointwise gradient estimates via the truncated 1-Riesz potential were first obtained for nonlinear elliptic measure data problems with linear growth in

[18]. Such gradient potential estimates have been studied intensively as a universal method to obtain regularity theory for elliptic problems, for instance, C^1, $C^{1,\alpha}$ and VMO-regularity. We refer to [19] for a well-written summary of nonlinear potential estimates for solutions and their gradient, and [17] for their applications.

3.2. An application to global Calderón-Zygmund type estimates

We assume that $f \in L^{q_0}(\Omega)$ for

$$q_0 = \begin{cases} \frac{np}{np-n+p} & \text{if } p < n, \\ \frac{3}{2} & \text{if } p \geq n. \end{cases}$$

Then $f \in W^{-1,p'}(\Omega)$, and the *limit of approximating solutions* u to $OP(\psi; f)$ with $\psi \in W^{1,p}(\Omega)$ satisfying $\psi^+ \in W_0^{1,p}(\Omega)$ is the weak solution to (4) with the obstacle function ψ^+.

For obstacle problems of p-Laplacian type, Calderón-Zygmund type estimates were first proved in [4]. Later in [7], such local estimates were extended to global ones under suitable assumptions on the vector field $A(\cdot)$ and the domain Ω, which we state as follows:

Definition 2. Let $\delta \in (0, 1/8)$ and $R > 0$ be given. We say that $(A(\cdot), \Omega)$ is (δ, R)-vanishing if the following two conditions hold:

(i) Denoting

$$\theta(S)(x) := \sup_{z \in \mathbb{R}^n \setminus \{0\}} \frac{1}{|z|^{p-1}} \left| A(x, z) - \fint_S A(\tilde{x}, z)\, d\tilde{x} \right|$$

for any measurable set $S \subset \mathbb{R}^n$ and $x \in S$, we have

$$\sup_{0<r<R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} \theta(B_r(y))(x)\, dx \leq \delta.$$

(ii) For each $y \in \partial\Omega$ and $r \in (0, R]$, there exists a coordinate system $\{\tilde{y}_1, \dots, \tilde{y}_n\}$, depending on y and r, such that y is at the origin and

$$B_r(0) \cap \{\tilde{y}_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{\tilde{y}_n > -\delta r\}.$$

A domain satisfying (ii) is called a (δ, R)-Reifenberg flat domain. Note that its definition is motivated from Lipschitz domains with small Lipschitz constant. In particular, a (δ, R)-Reifenberg flat domain satisfies the following measure density conditions

$$\sup_{0<r\leq R} \sup_{x \in \Omega} \frac{|B_r(x)|}{|\Omega \cap B_r(x)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n,$$
$$\inf_{0<r\leq R} \inf_{x \in \Omega} \frac{|\Omega^c \cap B_r(x)|}{|B_r(x)|} \geq \left(\frac{1-\delta}{2}\right)^n \geq \left(\frac{7}{16}\right)^n.$$

We recall the result in [7] in the following way: let $u \in \mathcal{A}_\psi(\Omega)$ be the weak solution to

$$\int_\Omega A(x, Du) \cdot D(\phi - u)\, dx \geq \int_\Omega F \cdot D(\phi - u)\, dx \qquad \forall\ \phi \in \mathcal{A}_\psi(\Omega) \tag{10}$$

under assumptions (2) and (7), where $F \in L^{p'}(\Omega; \mathbb{R}^n)$ is a given vector field. Assume that $D\psi \in L^{pq}(\Omega; \mathbb{R}^n)$ and $F \in L^{p'q}(\Omega; \mathbb{R}^n)$ for some $q \in (1, \infty)$. Then there exists a constant $\delta_1 = \delta_1(n, p, \nu, L, q) > 0$ such that if $(A(\cdot), \Omega)$ is (δ_1, R)-vanishing, then

$$\|Du\|_{L^{pq}(\Omega)} \leq c\|D\psi\|_{L^{pq}(\Omega)} + c\|F\|_{L^{p'q}(\Omega)} \tag{11}$$

holds for a constant $c = c(n, p, \nu, L, q, R, \Omega)$. We note that it was later extended to several problems with nonstandard growth, see [1, 5] and references therein. We also refer to [15] and [10] for the extensions of (11) to obstacle problems with measurable nonlinearities and to double obstacle problems, respectively.

Theorem 3.6. *Let $u \in \mathcal{A}_\psi(\Omega)$ be the weak solution to (4) under assumptions (2) and (7). Assume that*

$$D\psi^+ \in L^{pq}(\Omega; \mathbb{R}^n) \ \text{ and } \ f \in L^{m(q)}(\Omega)$$

for some $q \in (1, \infty)$, where

$$m(q) = \max\left\{\frac{npq}{n(p-1) + pq}, 1\right\}. \tag{12}$$

Then there exists a constant $\delta = \delta(n, p, \nu, L, q) > 0$ such that if $(A(\cdot), \Omega)$ is (δ, R)-vanishing, then

$$\|Du\|_{L^{pq}(\Omega)} \leq c\|D\psi^+\|_{L^{pq}(\Omega)} + c\|f\|_{L^{m(q)}(\Omega)}$$

holds for a constant $c = c(n, p, \nu, L, q, R, \Omega)$.

Proof. We first consider the unique solution $v \in W_0^{1,1}(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\triangle v = f & \text{in } \Omega, \\ \quad\ \ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

By the Calderón-Zygmund type estimates for elliptic measure data problems, see for instance [9, Theorem 2.1], for any $\gamma > 0$ there exists a constant $\delta_2 = \delta_2(n, \gamma) > 0$ such that if $\|\mathcal{M}_1(f)\|_{L^\gamma(\Omega)} < \infty$ and Ω is (δ_2, R)-Reifenberg flat, then

$$\|Dv\|_{L^\gamma(\Omega)} \leq c\|\mathcal{M}_1(f)\|_{L^\gamma(\Omega)} \tag{14}$$

holds for a constant $c = c(n, \gamma, R, \Omega)$. Here, f is considered as defined on $\mathbb{R}^n$ by letting $f \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and $\mathcal{M}_1(f)$ is the 1-fractional maximal function of f, defined by

$$\mathcal{M}_1(f)(x) \coloneqq \sup_{r>0}\left(r \fint_{B_r(x)} f\, d\tilde{x}\right).$$

Note that the estimates in [9, Theorem 2.1] contain an additional constant term on the right-hand side due to the non-autonomous setting, which is redundant for (13).

Observe that for any $q > 1$, the exponent $m(q)$ in (12) is chosen to satisfy

$$m(q) = \begin{cases} (p'q)_* & \text{if } q \geq n'/p', \\ 1 & \text{otherwise,} \end{cases}$$

where $(p'q)_*$ is the inverse Sobolev exponent of $p'q$. We now apply the embedding property of fractional maximal operators, see for instance [14], to have

$$\|\mathcal{M}_1(f)\|_{L^{p'q}(\Omega)} \leq c\|f\|_{L^{m(q)}(\Omega)}. \tag{15}$$

In particular, we have $Dv \in L^{p'q}(\Omega; \mathbb{R}^n)$. It then follows from Corollary 3.4 that u is the weak solution to (10) with $F = Dv$ and ψ replaced by ψ^+. Finally, after choosing $\delta = \min\{\delta_1, \delta_2\}$, we combine (14) and (15) with (11) in order to obtain the desired estimate. □

Remark 3.7. In Theorem 3.6, we considered obstacle problems with a nonnegative function $f \in W^{-1,p'}(\Omega)$ in order to apply the Calderón-Zygmund type estimate (11) and the comparison principle in Corollary 3.4. For obstacle problems with L^1 or measure data, most of the regularity results were obtained under the assumption $\psi \in W^{2,1}(\Omega)$ with $\mathcal{D}\Psi := |D\psi|^{p-2}D^2\psi \in L^1(\Omega)$, which allows one to control the obstacle as another inhomogeneous term in the final estimates, see [6,21]. In this case, Corollary 3.4 gives that if one instead assumes $\psi^+ \in W^{2,1}(\Omega)$ with $\mathcal{D}\Psi^+ := |D\psi^+|^{p-2}D^2\psi^+ \in L^1(\Omega)$ and $0 \leq f \in L^1(\Omega)$, then such regularity results for $OP(\psi; f)$ can be formulated with ψ replaced by ψ^+. One may expect to extend Theorem 3.6 to inhomogeneous obstacle problems (4) with nonnegative L^1-data. Our result can also be extended to problems with general growth conditions, see for instance [8, 11].

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