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HYPERBOLIC AND SPHERICAL POWER OF A CIRCLE

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Abstract. Suppose that a line passing through a given point P intersects a given circle $\mathcal{C}$ at Q and R in the Euclidean plane. It is well known that $|PQ||PR|$ is independent of the choice of the line as long as the line meets the circle at two points. It is also known that similar properties hold in the 2-sphere and in the hyperbolic plane. New proofs for the similar properties in the 2-sphere and in the hyperbolic plane are given.

1. Introduction

Suppose that a line passing through a given point P intersects a given circle $\mathcal{C}$ at Q and R in the Euclidean plane. It is well known that $|PQ||PR|$ is independent of the choice of the line as long as the line meets the circle at two points, which is called the power of the point P with respect to the circle $\mathcal{C}$. Furthermore, if a line passing through a given point P is tangent to the circle $\mathcal{C}$ at T, it follows that

$$|PT|^2 = |PQ||PR|.$$

It is interesting that a similar property holds in the round sphere $\mathbb{S}^2$ (2 dimensional simply connected Riemannian manifold with the positive constant curvature 1) and in the hyperbolic plane $\mathbb{H}^2$ (2 dimensional simply connected Riemannian manifold with the negative constant curvature -1).

Theorem 1. *Suppose that a spherical line passing through a given point P intersects a given spherical circle $\mathcal{C}_S$ at Q and R in the 2-sphere $\mathbb{S}^2$ and that P^*, the antipodal point of P, is not on $\mathcal{C}_S$. Then*

$$\tan\left(\frac{1}{2}|PQ|_S\right)\tan\left(\frac{1}{2}|PR|_S\right)$$

is independent of the choice of the line as long as the line meets the circle $\mathcal{C}_S$ at two points, where $|AB|_S$ is a spherical distance between A and B.

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If another line passing through P is tangent to $\mathcal{C}_S$ at T, then

$$\tan^2\left(\frac{1}{2}|PT|_S\right) = \tan\left(\frac{1}{2}|PQ|_S\right)\tan\left(\frac{1}{2}|PR|_S\right),$$

which can be shown by continuity argument.

Theorem 2. *Suppose that a hyperbolic line passing through a given point P intersects a given hyperbolic circle $\mathcal{C}_H$ at Q and R in the hyperbolic plane. Then*

$$\tanh\left(\frac{1}{2}|PQ|_H\right)\tanh\left(\frac{1}{2}|PR|_H\right)$$

is independent of the choice of the line as long as the line meets the circle $\mathcal{C}_H$ at two points, where $|AB|_H$ is the hyperbolic distance between A and B.

If another line passing through P is tangent to $\mathcal{C}_H$ at T, then it holds that

$$\tan^2\left(\frac{1}{2}|PT|_H\right) = \tan\left(\frac{1}{2}|PQ|_H\right)\tan\left(\frac{1}{2}|PR|_H\right),$$

which can also be shown by continuity argument.

Proofs in the literature [2, 3] use spherical or hyperbolic trigonometry. Our proofs in this note do not use these trigonometries, instead, utilize the Euclidean power of the point with respect to circles. In order to utilize the Euclidean power, we consider the stereographic projection for the proof of Theorem 1 and use the Poincare's unit disk model for the hyperbolic plane for the proof of Theorem 2. Then we can show that the same property holds not only for hyperbolic circles, but also for horocircles and hypercircles, because they retain the circular appearances - only their centers are offset.

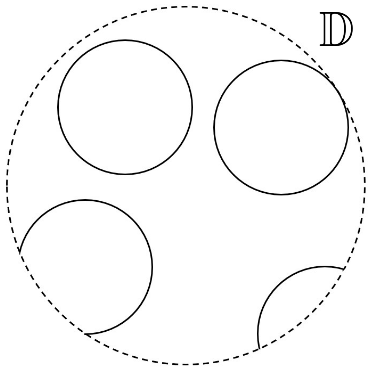

FIGURE 1. In the Poincare disk model for the hyperbolic plane, hyperbolic circles, hypercircles and horocircles are all (parts of) Euclidean circles.

Theorem 3. *Let $\mathcal{C}_H$ be a horocircle or a hypercircle. Suppose that a hyperbolic line passing through a given point P intersects $\mathcal{C}_H$ at Q and R in the hyperbolic plane. Then*

$$\tanh\left(\frac{1}{2}|PQ|_H\right)\tanh\left(\frac{1}{2}|PR|_H\right)$$

is independent of the choice of the line as long as the line meets $\mathcal{C}_H$ at two points. If another line passing through P is tangent to $\mathcal{C}_H$ at T, it follows that

$$\tanh^2\left(\frac{1}{2}|PT|_H\right) = \tanh\left(\frac{1}{2}|PQ|_H\right)\tanh\left(\frac{1}{2}|PR|_H\right).$$

Theorem 4. *Suppose that a hyperbolic line passing through a point P intersects a horocircle $\mathcal{C}_H$ at Q and R and another hyperbolic line passing through a point P intersects $\mathcal{C}_H$ transversally at U only. Then it follows that*

$$\tanh\left(\frac{1}{2}|PQ|_H\right)\tanh\left(\frac{1}{2}|PR|_H\right) = \tanh\left(\frac{1}{2}|PU|_H\right).$$

Since $|PU|_H$ is the distance from the point to the horocircle $\mathcal{C}_H$, we think this theorem is particularly interesting since it claims that the power of the point P with respect to the horocircle $\mathcal{C}_H$ determines the distance from the point P to the horocircle $\mathcal{C}_H$. Furthermore, if another line passing through P is tangent to the horocircle $\mathcal{C}_H$ at T, it follows from continuity that

$$\tanh^2\left(\frac{1}{2}|PT|_H\right) = \tanh\left(\frac{1}{2}|PQ|_H\right)\tanh\left(\frac{1}{2}|PR|_H\right)$$

which then gives

$$\tanh^2\left(\frac{1}{2}|PT|_H\right) = \tanh\left(\frac{1}{2}|PU|_H\right).$$

This equality also claims that the distance from the point P to the tangent point to the horosphere determines the distance to the horocircle $\mathcal{C}_H$.

The method to use the Poincare disk model and the Euclidean geometry to describe the power of a point with respect to the circle is not new, see, for example, [1] for the description of the spherical and hyperbolic concepts and their connections.

2. A proof of Theorem 1

We firstly remark that Theorem 1 does not hold if the point P^* is on $\mathcal{C}_S$, since every spherical line passing through P passes through P^* and hence intersects $\mathcal{C}_S$ at P^*.

Let us consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ with the center $(0,0,0)$,

$$\mathbb{S}^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

and suppose that a spherical line l_a passing through P intersects the spherical circle $\mathcal{C}_S$ at A_1', A_2' and another line l_b passing through P intersects the circle

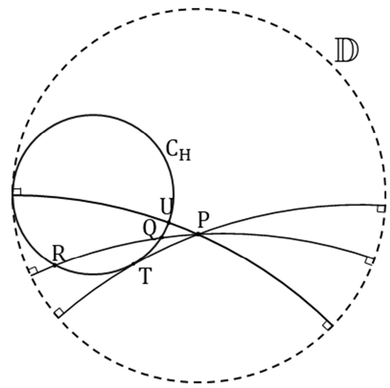

FIGURE 2. The power of the point with respect to the horocircle determines the distance from the point to the horocircle.

$\mathcal{C}_S$ at B_1', B_2'. We may assume that the coordinate of P is $(0,0,-1)$. Let $\phi : \mathbb{S}^2 \setminus (0,0,1) \to \mathbb{R}^2$ be the stereographic projection. Then we have

- $\phi(P)$ is the origin O of $\mathbb{R}^2$: $\phi(P) = (0,0)$,
- the image $\phi(\mathcal{C}_S)$ of the spherical circle $\mathcal{C}_S$ is the Euclidean circle $\mathcal{C}$,
- the images $\phi(l_a)$ and $\phi(l_b)$ of the spherical lines l_a and l_b are the Euclidean lines passing through the origin O.

Let $\phi(A_i') = A_i$, $\phi(B_i') = B_i$, $i = 1,2$. Then the Euclidean line $\phi(l_a)$ passing through the origin O intersects the Euclidean circle $\mathcal{C}$ at A_1, A_2 and the Euclidean line $\phi(l_b)$ passing through the origin O intersects the Euclidean circle $\mathcal{C}$ at B_1, B_2. Now we have, see Figure 1,

$$|OA_i| = \tan\left(\frac{1}{2}|PA_i'|_S\right), \quad |OB_i| = \tan\left(\frac{1}{2}|PB_i'|_S\right), \quad i = 1,2$$

and since it holds that

$$|OA_1||OA_2| = |OB_1||OB_2|$$

we have

$$\tan\left(\frac{1}{2}|PA_1'|_S\right)\tan\left(\frac{1}{2}|PA_2'|_S\right) = \tan\left(\frac{1}{2}|PB_1'|_S\right)\tan\left(\frac{1}{2}|PB_2'|_S\right),$$

which completes the proof of Theorem 1.

3. Proofs of Theorem 2 and Theorem 3

We use the Poincare disk model

$$\left(\mathbb{D},\ ds^2 = \frac{4(dx^2+dy^2)}{(1-(x^2+y^2)^2)^2}\right)$$

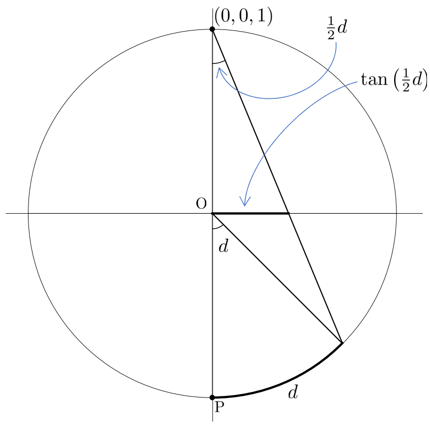

FIGURE 3. Spherical distance and Euclidean distance.

for the hyperbolic plane $\mathbb{H}^2$. Let O be the center $(0, 0) \in \mathbb{D}$. Among the reasons why we choose this model are

- every hyperbolic line through O is the Euclidean line,
- every hyperbolic circle is a Euclidean circle in $\mathbb{D}$,
- every horocircle is a Euclidean circle in $\mathbb{D}$ meeting the boundary circle $\partial\mathbb{D}$ tangentially,
- every hypercircle is a part of the Euclidean circle in $\mathbb{D}$ meeting the boundary circle $\partial\mathbb{D}$ transversally,
- for a point $Q \in \mathbb{D}$, the relation between the Euclidean distance $|OP|$ and the hyperbolic distance $|OP|_H$ is
$$|OP| = \tanh\left(\frac{1}{2}|OP|_H\right)$$
since
$$|OP|_H = \int_0^{|OP|} \frac{2}{1-r^2}dr = \ln\frac{1+|OP|}{1-|OP|}.$$

Now let us prove Theorem 2. By applying proper isometry if necessary, we may assume the given point is the center O. Suppose that a hyperbolic line passing through O intersects $\mathcal{C}_H$ at A_1, A_2 and another line passing through O intersects $\mathcal{C}_H$ at B_1, B_2. Then, since it holds that
$$|OA_1||OA_2| = |OB_1||OB_2|$$
we have
$$\tanh\left(\frac{1}{2}|OA_1|_H\right)\tanh\left(\frac{1}{2}|OA_2|_H\right) = \tanh\left(\frac{1}{2}|OB_1|_H\right)\tanh\left(\frac{1}{2}|OB_2|_H\right),$$
which completes the proof of Theorem 2 and Theorem 3.

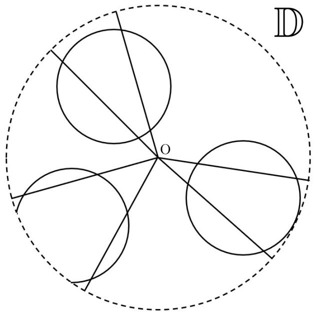

FIGURE 4. Power of a point with respect to a hyperbolic circle, hypercircle, horocircle.

4. A Proof of Theorem 4

Suppose that a hyperbolic line l, which is not tangent to the horocircle $\mathcal{C}_H$, meets the horocircle $\mathcal{C}_H$. Then the set $l \cap \mathcal{C}_H$ consists either of two points or of one point. The latter case happens only when the point B_2 in the proof of Theorem 3 is the horocenter of $\mathcal{C}_H$, which lies in $\partial\mathbb{D}$, see Figure 5.

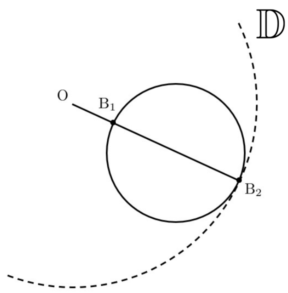

FIGURE 5. Horocenter of a horosphere.

Then, since $|OB_2| = 1$, one has

$$|OA_1||OA_2| = |OB_1|$$

and hence

$$\tanh\left(\frac{1}{2}|OA_1|_H\right)\tanh\left(\frac{1}{2}|OA_2|_H\right) = \tanh\left(\frac{1}{2}|OB_1|_H\right),$$

which completes the proof of Theorem 4. $\square$

We remark that, for a hypercircle $\mathcal{C}_H$, there are exactly two points B_1, B_2 on $\mathcal{C}_H$ so that the equality in Theorem 4 holds,

$$\tanh\left(\frac{1}{2}|OA_1|_H\right)\tanh\left(\frac{1}{2}|OA_2|_H\right) = \tanh\left(\frac{1}{2}|OB_i|_H\right),\ i = 1, 2,$$

see Figure 6.

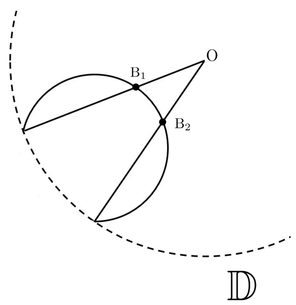

FIGURE 6. Existence of B_1, B_2.

5. An observation: Steiner's porism for horocircles

For a given hyperbolic circle $\mathcal{C}_H$, let us consider a finite set of horocircles, all of which are tangent to $\mathcal{C}_H$ and each horocircle in this set is tangent to the previous and next horocircles in this set. If such a set of horocircles exists, let us call them a *closed chain* of horocircles for the hyperbolic circle $\mathcal{C}_H$, see Figure 7.

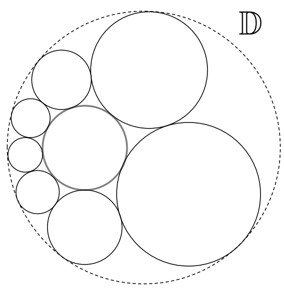

FIGURE 7. A closed chain of horocircles for a hyperbolic circle.

Now the Steiner's porism in the Euclidean plane gives the following porism:

Theorem 5. *If at least one closed chain of n horocircles exists for a given hyperbolic circle $\mathcal{C}_H$, then there is an infinite number of closed chains of n horocircles; and any horocircle tangent to $\mathcal{C}_H$ in the same way is a member of such a chain.*

References

[1] G. Horváth, *Malfatti's problem on the hyperbolic plane*, Studia Sci. Math. Hungar. **51** (2014), no. 2, 201–212. https://doi.org/10.1556/SScMath.51.2014.2.1276

[2] G. Horváth, *Hyperbolic plane geometry revisited*, J. Geom. **106** (2015), no. 2, 341–362. https://doi.org/10.1007/s00022-014-0252-0

[3] D. M. Y. Sommerville, *The Elements of Non-Euclidean Geometry*, The Open Court Publishing, Chicago, 1919.

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