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THE CATENARY DEGREE OF THE SATURATED NUMERICAL SEMIGROUPS WITH PRIME MULTIPLICITY

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Abstract. In this paper, we formulate the set of all saturated numerical semigroups with prime multiplicity. We characterize the catenary degrees of elements of the semigroups we obtained which are important invariants in factorization theory. We also give the proper characterizations of the semigroups under consideration.

1. Introduction

Researchers have been interested in two different aspects of non-unique factorization invariants. Some were concerned with the lengths of the factorizations of an element and took into account the semi-factor property in a half-factorial monoid that is allocated to all factorizations of the same length of a given element. The others were concerned with the notions of the distance between the factorizations. Considering the uses of the idea of the distance between factorizations, the main focus was on the catenary and tame degrees. In this study, we will deal with the second case. Every element of a cancellative monoid is as a linear combination of its generators with non-negative integer coefficients. But this combination is not unique. Each of these different expressions is called the factorization of that element. The catenary degree of an element in the cancellative monoid describes the connection between different factorizations and it is a powerful tool for understanding factorization theory. Besides, the maximum value of all catenary degrees of all the elements in the cancellative monoid is the catenary degree of the monoid itself.

Problems with non-unique factorizations of elements in integral domains and commutative cancellative monoids have been a hot topic in the literature for years ([15] and the citation list in [15]). Most of these studies focus on combinatorial constants which explain how these systems differ from the classical concept of unique factorization. We see the earliest studies on this subject are

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on Krull domains and monoids [3, 5, 10, 11, 13, 14, 16, 20]. The recent studies in this area evaluate these properties on numerical monoids [1, 5–8, 12, 18, 19].

In the literature, a long list of studies can be found on the analysis of one-dimensional analytically irreducible local domains via value semigroups [4]. One class of the numerical semigroups obtained with this approach is the class of saturated numerical semigroups which has an important place. After characterizing the saturated rings in terms of the value semigroups, the saturated numerical semigroups appear in [9, 17]. Even though the concept of saturated semigroups is included in the ring theory, it first attracted the attention of semigroupist [24, 28–31].

The structure of this article is as follows. In Section 2 we will include the necessary definitions and notations that we will use for the main results and proofs. In Section 3 we will find all saturated numerical semigroups with prime multiplicity and fixed conductor (Theorem 3.5). Finally, in Section 4 we will express the catenary degree of these semigroups (Theorem 4.2 and Theorem 4.3).

2. Definitions and preliminaries

Let $\mathbf{Z}$ and $\mathbf{N}$ be the set of integers and non-negative integers, respectively. Let S be a non-empty subset of $\mathbf{N}$. If S is a sub-monoid of N such that $\mathbf{N} \setminus S < \infty$, then S is called a numerical semigroup. The Frobenius number of S, denoted by $F(S)$, is the maximum element of $\mathbf{Z} \setminus S$ [21]. The least integer s that provides $s + n \in S$ for all $n \in \mathbf{N}$ is called the conductor of S, denoted by $c(S)$ (in short c). The conductor is actually 1 greater than the Frobenius number of S [4].

Let $\emptyset \neq A \subset \mathbf{N}$. The submonoid of $(\mathbf{N}, +)$ generated by A is expressed as:

$$\langle A \rangle = \{n_1 a_1 + \cdots + n_r a_r : n_1, \ldots, n_r \in \mathbf{N}, a_1, \ldots, a_r \in A, r \in \mathbf{N} \setminus \{0\}\}.$$

If $S = \langle A \rangle$, then A is called a system of generators of S. Also, if no suitable proper subset of A generates S, it is said that A is a minimal system of generators of S. It must be known that every numerical semigroup has a unique minimal generator system with a finite number of elements [4, 25]. It is additionally well known that $\gcd(A) = 1$ if and only if $S = \langle A \rangle$ is a numerical semigroup (where gcd stands for the greatest common divisor) [26]. If the minimal system of generators of S is $A = \{a_1 < a_2 < \cdots < a_r\}$, then a_1, a_2 and r called the multiplicity, the ratio and the embedding dimension of S, these are denoted by $\mu(S), R(S)$ and $e(S)$, respectively. It is a fact that $e(S) \leq \mu(S)$. When S is a numerical semigroup with embedding dimension that is equal to the multiplicity, S is said to be a MED-semigroup (where MED represents for maximal embedding dimension). For $n \in S \setminus \{0\}$, the Apéry set of n in S is defined as follows:

$$Ap(S, n) = \{x \in S : x - n \neq S\}.$$

Easily, it can be proved that

$$Ap(S,n) = \{w_0 = 0, w_1, \ldots, w_{n-1}\},$$

where $w_i = \min\{x \in S : x \equiv i \pmod{n}\}$ for $i = \{0,1,\ldots,n-1\}$ (see for instance [2,26]).

A numerical semigroup S is called an Arf semigroup if for every $s_1, s_2, s_3 \in S$ with $s_3 = \min\{s_1, s_2, s_3\}$, the element $s_1 + s_2 - s_3$ is also in S. A numerical semigroup S is said to be saturated if the following condition is satisfied: if $s, s_1, \ldots, s_r \in S$ where $s_r \le s$ for all $i \in \{1,\ldots,r\}$ and $n_1, \ldots, n_r \in \mathbf{Z}$, $s_1n_1 + \cdots + s_rn_r \ge 0$, then $s + s_1n_1 + \cdots + s_rn_r \in S$. Let A be a nonempty subset of $\mathbf{N}$ and a be a nonzero element of A, $d_A(a)$ is defined as

$$d_A(a) = \gcd\{a' \in A : a' \le a\}.$$

It is well known a numerical semigroup S is saturated if and only if $s + d_S(s) \in S$ for all $s \in S \setminus \{0\}$. Also, any saturated numerical semigroup has the Arf property due to its maximal embedding dimension [4,9].

It can easily be seen by the definition that a numerical semigroup S is saturated if and only if there exists a sequence of positive integers $s_1 < s_2 < \cdots < s_r$ such that $\gcd\{s_1, s_2, \ldots, s_r\} = 1$ and $\gcd\{s_1, s_2, \ldots, s_i\} \ne \gcd\{s_1, s_2, \ldots, s_i, s_{i+1}\}$ for all $i \in \{1,2,\ldots,r-1\}$. Then, $\{s_1, s_2, \ldots, s_r\}$ is said to be a minimal SAT-system of generators of S. In addition, if $d_i = \gcd\{s_1, s_2, \ldots, s_i\}$ for each $i \in \{1,\ldots,r\}$, S is said to be a $(d_1, d_2, \ldots, d_r)$-semigroup. A saturated sequence of length k is known as a k-tuple of positive integers $(d_1, d_2, \ldots, d_k)$ with $d_1 > d_2 > \cdots > d_k = 1$ and d_{i+1} divides d_i for all $i \in \{1,\ldots,k-1\}$. For a positive integer F, an F-saturated sequence is a saturated sequence $(d_1, d_2, \ldots, d_k)$ such that there exists at least one $(d_1, d_2, \ldots, d_k)$-semigroup with Frobenius number F [24].

Let $S = \langle a_1, \ldots, a_r\rangle$. The homomorphism

$$\varphi : \mathbf{N}^r \to S \text{ defined by } \varphi(a_1, \ldots, a_r) = n_1a_1 + \cdots + n_ra_r,$$

is the factorization homomorphism of S. Let the congruence σ be the kernel congruence of φ (where $a\sigma b$ if $\varphi(a) = \varphi(b)$). The monoid S is isomorphic to $\mathbf{N}^r/\sigma$. The set of factorizations of $s \in S$ is denoted by $Z(s)$, and it is as following:

$$Z(s) = \varphi^{-1}(s) = \{(n_1, \ldots, n_r) \in \mathbf{N}^r : n_1a_1 + \cdots + n_ra_r = s\}.$$

For a factorization $x = (x_1, \ldots, x_r)$ in $Z(s)$, the length of x is denoted by $|x|$, and it is as follows:

$$|x| = x_1 + \cdots + x_r.$$

The set of lengths of all factorizations of s is denoted by $L(s)$, and it is as following:

$$L(s) = \{|x| : x \in Z(s)\} = \{m_1, \ldots, m_l\}.$$

The set $L(s)$ has finite elements. Moreover, if $S = \mathbf{N}$, there are elements with more than one length. Let $x = (x_1, \ldots, x_r), y = (y_1, \ldots, y_r) \in \mathbf{N}^r$ be two factorizations and

$$\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_r, y_r\})$$

be the common part of x and y. The distance between them is denoted by $dist(x, y)$, and it is as follows:

$$dist(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}.$$

The support of $x \in \mathbf{N}^r$ is defined by $supp(x)$, and it is as follows:

$$supp(x) = \{i : x_i \neq 0, 1 \leq i \leq r\}.$$

Let $s \in S$ be such that $s - s_i \in S$. Then the set

$$Z^i(s) = \{x \in Z(s) : i \in supp(x)\}$$

is a non-empty set. Let $N \in \mathbf{N}$. A finite sequence $z = z_0, z_1, \ldots, z_{n-1}, z_n$ of a factorization of $s \in S$ is an N-chain if $dist(z_{n-1}, z_i) \leq N$ for each $1 \leq i \leq n$. The catenary degree of the element s is defined as to be the minimal N such that there is an N-chain between any two factorizations of s, denoted by $C(s)$. The catenary degree of the numerical semigroup S is denoted by $C(S)$, and it is as follows:

$$C(S) = \sup\{C(s) : s \in S\} \in \mathbf{N} \cup \{\infty\}.$$

A presentation ρ for S is a subset of σ if σ is the least congruence containing ρ (with respect to set inclusion). That is, a system of generators of σ. Since finitely generated commutative monoid is finitely presented, every numerical semigroup is also finitely presented [22]. Moreover, for numerical semigroups, the concepts of minimality with respect to cardinality and set inclusion of a presentation coincide. Two elements a, b in $\mathbf{N}^r$ are $\Re$-related if there exists a chain $a = z_0, z_1, \ldots, z_{n-1}, z_n = b$ such that $supp(z_{i-1}) \bigcap supp(z_i) \neq \emptyset$ for $1 \leq i \leq n$. It can easily be seen that this is an equivalence relation on $Z(s)$ for s in S. The number of factorizations of the elements of the numerical semigroup is finite, and so the number of $\Re$-classes in this set is also finite. The $\Re$-classes are important because they can construct a minimal representation of S. Let $s \in S$ and $\Re_1^s, \ldots, \Re_{n_s}^s$ be different $\Re$-classes of $Z(s)$. Set $m(s) = \max\{r_1^s, \ldots, r_{n_s}^s\}$, where $r_i^s = \min\{|z| : z \in \Re_i^s\}$. Denote by $m(S) = \max\{m(s) : s \in S \text{ and } n_s \geq 2\}$. We know that $C(S) = m(S)$ [8].

For $A, B \subset \mathbf{N}$, we set

$$A + B = \{x + y : x \in A, y \in B\} \text{ and } kA = \underbrace{A + A + \cdots + A}_{k}.$$

3. The saturated numerical semigroups with prime multiplicity

In this section, we will calculate the set of all saturated numerical semigroups with prime integer multiplicity and fixed conductor.

Lemma 3.1 ([26]). *Let $S = \langle a_1, a_2, \ldots, a_e \rangle$ be a numerical semigroup such that $a_1 < a_2 < \cdots < a_e$. For $x \in S \setminus \{0\}$ we have the following*:

1. $\sharp Ap(S, x) = x$ ($\sharp$ *stands for cardinality*).
2. $F(S) = \max(Ap(S, x)) - x$.
3. $\{0, a_2, \ldots, a_e\} \subset Ap(S, a_1)$.
4. *S is a MED-semigroup if and only if* $Ap(S, a_1) = \{0, a_2, \ldots, a_e\}$.

Lemma 3.2 ([27], Proposition 5). *Let S and T be two saturated numerical semigroups. Then $S \cap T$ is a saturated numerical semigroup.*

Given a nonempty subset A of $\mathbf{N}$ such that $\gcd(A) = 1$. It is well known that the saturated numerical semigroups containing $\langle A \rangle$ are finite. The intersection of all saturated numerical semigroups containing A is denoted by $Sat(A)$. In fact, $Sat(A)$ is the smallest saturated numerical semigroup containing A. If $Sat(A) = S$, A is called a SAT-system of generators of S. Moreover, if no proper subset of A is a SAT-system of generators of S then A is called a minimal SAT-system of generators of S.

Lemma 3.3 ([27], Theorem 6). *Let $n_1 < n_2 < \cdots < n_r$ be positive integers such that*

$$\gcd\{n_1, n_2, \ldots, n_r\} = 1.$$

For every $i \in \{1, 2, \ldots, r\}$, set $d_i = \gcd\{n_1, n_2, \ldots, n_i\}$ and for all $j \in \{1, 2, \ldots, r-1\}$ define

$$t_j = \max\{t \in \mathbf{N} : n_j + td_i < d_{j+1}\}.$$

Then

$$\begin{aligned} Sat(n_1, n_2, \ldots, n_r) = \{&0, n_1, n_1 + d_1, \ldots, n_1 + t_1 d_1, n_2, n_2 + d_2, \ldots, n_2 + t_2 d_2, \\ &\ldots, n_{r-1}, n_{r-1} + d_{r-1}, \ldots, n_{r-1} + t_{r-1} n_{r-1}, \\ &n_r, n_r + 1, \rightarrow\}. \end{aligned}$$

Lemma 3.4 ([27], Theorem 11). *Let S be a saturated numerical semigroup. Then $\{n_1, n_2, \ldots, n_r\} = \{n \in S \setminus \{0\} : d_S(n) \neq d_S(n^\star)$ for all $n^\star < n,\ n^\star \in S\}$ is the unique minimal SAT system of generators of S.*

Let S be a numerical semigroup with conductor c and multiplicity μ. It is known that $c \not\equiv 1 \pmod{\mu}$. Because, every nonnegative multiple of μ is in S, but $c - 1$ is not in S.

Theorem 3.5. *Let S be a numerical semigroup and p a prime integer. S is a saturated numerical semigroup with multiplicity p and conductor c if and only if S is one of the following*:

(1) *If $c \equiv 0 \pmod{p}$, then $\langle p, c+1, c+2, \ldots, c+p-1 \rangle$.*
(2) *If $c \equiv i \pmod{p}$, then $\langle p, c, c+1, \ldots, c+p-i-1, c+p-i+1, \ldots, c+p-1 \rangle$ for $i \in \{2, 3, \ldots, p-1\}$.*

Proof. ($\Leftarrow$) (1) Let S be the following numerical semigroup with multiplicity p and conductor c, $c \equiv 0 \pmod p$:

$$S = \langle p, c+1, c+2, \ldots, c+p-1\rangle.$$

If $c \equiv 0 \pmod p$, then $p \mid c$. Therefore, $c = kp$ for some k. Thus,

$$S = \langle p, c+1, c+2, \ldots, c+p-1\rangle = \{0, p, 2p, \ldots, (k-1)p, kp, \rightarrow\},$$

where $\rightarrow$ denotes that all integers larger than kp are in the semigroup, that is

$$\begin{aligned} S &= \{0, p, 2p, \ldots, (k-1)p, kp, \rightarrow\} \\ &= \{0, p, 2p, \ldots, (k-1)p, kp\} \cup \{kp+1, kp+2, \ldots\}. \end{aligned}$$

If $a \le c$, then $a = rp$ for some r. For $a \in S \setminus \{0\}$, we have $d_S(a) = p$ and

$$a + d_S(a) = rp + p = (r+1)\, p \in S.$$

If $a > c$, then $d_S(a) = 1$. Thus, $a + d_S(a) = a + 1 > c$ and $a + 1 \in S$. Hence, S is a saturated numerical semigroup.

(2) Let S be the following numerical semigroup with multiplicity p and conductor c, $c \equiv i \pmod p$ and $i \in \{2, 3, \ldots, p-1\}$:

$$S = \langle p, c, c+1, \ldots, c+p-i-1, c+p-i+1, \ldots, c+p-1\rangle.$$

If $c \equiv i \pmod p$, then $p \mid (c-i)$. Therefore, $c = kp + i$ for some k. Thus,

$$\begin{aligned} S &= \langle p, c, c+1, \ldots, c+p-i-1, c+p-i+1, \ldots, c+p-1\rangle \\ &= \{0, p, 2p, \ldots, kp, kp+i \rightarrow\}. \end{aligned}$$

If $a < c$, then $a = tp$ for some t. For $a \in S \setminus \{0\}$, we have $d_S(a) = p$ and

$$a + d_S(a) = tp + p = (t+1)\, p \in S.$$

If $a \ge c$, then $d_S(a) = 1$. Thus, $a + d_S(a) = a + 1 > c$ and $a + 1 \in S$. So, S is a saturated numerical semigroup.

($\Rightarrow$) Let S be a saturated numerical semigroup with multiplicity a prime integer p and conductor c. According to Theorem 3.4,

$$\{p = n_1, n_2, \ldots, n_r\} = \{n \in S \setminus \{0\} : d_S(n) \neq d_S(n^\star) \text{ for all } n^\star < n, n^\star \in S\}$$

is the unique minimal SAT system of generators of S. Since p is a prime integer, the minimal SAT system of generators of S is $\{p = n_1, n_r\}$ or $\{p = n_1, n_r + 1\}$.

(1) If the minimal SAT system of generators of S is $\{p = n_1, n_r + 1\}$, then $n_r = kp$ for some k. By Theorem 3.3, $t_1 = \max\{t \in \mathbf{N} : p + tp < kp + 1\} = k - 1$ is calculated and obtained as

$$\begin{aligned} Sat(p = n_1, n_r + 1) &= \{0, p, p+p, \ldots, p + (k-1)p, n_r + 1, \rightarrow\} \\ &= \{0, p, 2p, \ldots, kp, \rightarrow\}. \end{aligned}$$

Thus, $c = kp$ for some k, in other words, when $c \equiv 0 \pmod p$, we have $S = \{0, p, 2p, \ldots, kp, \rightarrow\} = \langle p, c+1, c+2, \ldots, c+p-1\rangle$.

(2) If the minimal SAT system of generators of S is $\{p = n_1, n_r\}$, then $n_r = kp + i$ for some k and $i \in \{1, \dots, p-1\}$. From Theorem 3.3,

$$t_1 = \max\{t \in \mathbf{N} : p + tp < kp + i\} = k - 1$$

is calculated and obtained as

$$\begin{aligned} Sat(p = n_1, n_r) &= \{0, p, p+p, \dots, p + (k-1)p, n_r, \rightarrow\} \\ &= \{0, p, 2p, \dots, kp, kp+i, \rightarrow\}. \end{aligned}$$

Hence, $c = kp + i$ for some k and $i \in \{2, 3, \dots, p-1\}$, in other words, when $c \equiv i \pmod{p}$ we have $S = \{0, p, 2p, \dots, kp, kp+i, \rightarrow\} = \langle p, c, c+1, \dots, c+p-i-1, c+p-i+1, \dots, c+p-1\rangle$. □

It is clear that by Theorem 3.5 we get the following corollary.

Corollary 3.6. *There is only one saturated numerical semigroup with prime multiplicity p and conductor c.*

4. Catenary degree of saturated numerical semigroups

In this section, we will formulate the catenary degree of the saturated numerical semigroups given in Theorem 3.5. Let $S = \langle a_1 < a_2 < \cdots < a_r\rangle$ and $s \in S$. If $Z(s)$ has more than one $\Re$-classes, then $s = w + ai$ with $w \in Ap(S, a_1) \setminus \{0\}$ and $i \in \{2, 3 \dots, r\}$ [23].

Corollary 4.1 ([7], Corollary 3)**.** *Let S be a numerical semigroup which is minimally generated by $\{a_1, a_2, \dots, a_r\}$ and $s \in S$. If s is minimal in S with the condition $C(s) = C(S)$, then $s = w + a_i$ with $w \in Ap(S, a_1) \setminus \{0\}$ and $i \in \{2, 3 \dots, r\}$.*

Henceforth in this section we will use e_i to denote a vector that has 1 as the ith component and 0's elsewhere, namely, e_i is the ith the standard unit vector in $\mathbf{N}^r$ as

$$e_i = (0, \dots, \underbrace{1}_{\text{ith component}}, \dots, 0).$$

Theorem 4.2. *Let S be a numerical semigroup and p be a prime integer. If S is a saturated numerical semigroup with multiplicity p and conductor $c \equiv 0 \pmod{p}$, then*

$$C(S) = 2h + 1,$$

where $c = ph$ for some positive integer h.

Proof. By Theorem 3.5, if S is a saturated numerical semigroup with prime multiplicity p and conductor $c \equiv 0 \pmod{p}$, then $S = \langle p, c+1, \dots, c+p-1\rangle$, where $c = ph$ for some positive integer h. Therefore, $Ap(S, p) = \{0, c+1, \dots, c+p-1\}$ by Lemma 3.1. Let $s \in S$ and a_j be a minimal generator of S. Then $s = w + a_j$ with $w \in Ap(S, p) \setminus \{0\}$ and $j \in \{2, 3, \dots, p\}$. This implies that $s = a_j + a_k$ for k and $j \in \{2, 3, \dots, p\}$ since S is a saturated numerical semigroup. Therefore, we have $a_j + a_k = 2c + (j + k - 2)$, where $a_k = c + (k-1)$ and $a_k = c + (j-1)$ by the definition of S. Thus,

$$Ap(S,p) \setminus \{0\} + \{c+1, \dots, c+p-1\} = \{2c+2, \dots, 2c+2(p-1)\}.$$

Let's consider the set of elements in the form $s = a_j + a_k$. We first want to prove that every $Z(s)$ has at least two $\Re$-classes. Assume the contrary that there is only one $\Re$-class in $Z(s)$.

(1) Let $j = k$. Then $s = a_j + a_k = 2a_j$ and $2a_j \notin Ap(S,p)$. Also, $s - p = 2a_j - p \in S$. Thus, one of the factorizations of s is $2e_j$, where e_j is the jth unit vector in $\mathbf{N}^p$. On the other hand,

$$s = a_j + a_k = 2aj = 2(c + (j-1)) = c + (c + 2(j-1))$$

and let's write hp instead of c

$$s = 2aj = hp + (hp + 2(j-1)).$$

Since $2 \leq 2(j-1) \leq 2(p-1)$, we have two cases:

(a) If $2(j-1) < p$, then $2j - 1 \neq j$ and one of the factorizations of s is $he_1 + e_{2j-1}$.

(b) If $2(j-1) > p$, then $s = 2aj = hp + (hp + 2(j-1)) = hp + (hp + pr_1 + s_1) = (h + r_1)p + (hp + s_1)$ for some positive integer r_1 and non-negative integer $s_1 < p$. Where $r_1 = 1$ and $s_1 < p - 1$ due to the values of i and j.

(i) If $s_1 = 0$, then $1 \neq j$ and one of factorizations of s is $(h+1)e_1$.

(ii) If $s_1 \neq 0$, then $s_1 + 1 \neq j$ and one of factorizations of s is $(h+1)e_1 + e_{s_1+1}$.

(2) Let $j \neq k$. Then $s = a_j + a_k$ and $a_j + a_k \notin Ap(S,p)$. Also, $s - p \in S$. Thus, one of the factorizations of s is $e_j + e_k$. We also have two cases:

(a) If $s = a_j + a_k \equiv 0 \pmod{p}$, then $s = a_j + a_k = 2c + j + k - 2 = 2hp + (j+k-2)$ and $j+k-2 \equiv 0 \pmod{p}$. Therefore $j+k-2 = pr_1$ for some positive integer s_1. Since $2 < j + k - 2 < 2p - 4$, where $r_1 = 1$ due to the values of i and k. Thus, one of the factorizations of s is $2he_1$.

(b) If $s = a_j + a_k \equiv s_2 \pmod{p}$, then $s = a_j + a_k = 2c + j + k - 2 = 2hp + (j + k - 2)$ and $j + k - 2 \equiv s_2 \pmod{p}$. Therefore, $j + k - 2 = pr_2 + s_2$ for some positive integers r_2 and s_2. We now have two cases:

(i) If $j+k-2 < p$, then $s = a_j + a_k = 2c+j+k-2 = 2hp+s_2$ and one of the factorizations of s is $he_1 + e_{s_2}$.

(ii) If $j+k-2 > p$, then $s = a_j + a_k = 2c+j+k-2 = 2hp+s_2$ since $j + k - 2 < 2p - 4$. One of the factorizations of s is $(h+1)e_1 + e_{s_2+2}$.

It is known that every element in the semigroup involved in one of its minimal presentations has a set of factorizations with at least two $\Re$-classes.

According to the above, $Z(s)$ has at least two $\Re$-classes. Namely, for every $x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p)$ in $Z(s)$ we can write $supp(x) \bigcap supp(y) = \emptyset$. Thus, $\gcd(x, y) = (0, \ldots, 0)$. This means that $dist(x, y) = \max\{|x|, |y|\}$. We obtain the catenary degree of S with the maximum of the lengths of these factorizations.

We finally conclude that the largest factorization length in $Z(s)$ is obtained when $s = a_j + a_k \equiv 0 \pmod p$ for $j \neq k$. Then the factorization of s is $(2h+1)e_1$. Since the catenary degree is the length of this factorization, $C(S) = 2h + 1$ by Corollary 4.1. □

Theorem 4.3. *Let S be a numerical semigroup and p a prime integer. If S is a saturated numerical semigroup with multiplicity p and conductor $c \equiv i \pmod p$ for $i \in \{2, 3, \ldots, p-1\}$, then*

$$C(S) = \begin{cases} 2h+2 & \text{if} \quad i < \frac{p+2}{2}, \\ 2h+3 & \text{if} \quad i > \frac{p+2}{2}, \end{cases}$$

where $c = ph + i$ for some positive integer h.

Proof. If S is a saturated numerical semigroup with prime multiplicity p and conductor $c \equiv i \pmod p$ for $i \in \{2, 3, \ldots, p-1\}$, then $S = \langle p, c, c+1, \ldots, c+p-i-1, c+p-i+1, \ldots, c+p-1\rangle$, where $c = ph+i$ for some positive integer h by Theorem 3.5. Therefore, $Ap(S, p) = \{0, c, c+1, \ldots, c+p-i-1, c+p-i+1, \ldots, c+p-1\}$ by Lemma 3.1. Let $s \in S$ and a_j be a minimal generator of S, $s = w + a_j$ with $w \in Ap(S, p) \setminus \{0\}$ and $j \in \{2, 3 \ldots, p\}$. Then $s = a_j + a_k$ for $k, j \in \{2, 3 \ldots, p\}$, since S is a saturated numerical semigroup. Therefore, we have

$$a_j + a_k = \begin{cases} 2c + (j+k) - 4 & \text{if } 2 \le j, k \le p-i+1, \\ 2c + (j+k) - 3 & \text{if } (2 \le j \le p-i+1 \text{ and } p-i+2 \le k \le p) \\ & \qquad \text{or} \\ & \quad (2 \le k \le p-i+1 \text{ and } p-i+2 \le j \le p), \\ 2c + (j+k) - 2 & \text{if } p-i+2 \le j, k \le p, \end{cases}$$

where

$$a_k = \begin{cases} c+k-2 & \text{if } 2 \le k \le p-i+1, \\ c+k-1 & \text{if } p-i+2 \le k \le p, \end{cases}$$

and

$$a_j = \begin{cases} c+j-2 & \text{if } 2 \le j \le p-i+1, \\ c+j-1 & \text{if } p-i+2 \le j \le p, \end{cases}$$

by the definition of S. Let's consider the set of elements in the form $s = a_j + a_k$. We first want to prove that every $Z(s)$ has at least two $\Re$-classes. Assume the contrary that $Z(s)$ has only one $\Re$-class.

(1) Let $j = k$. Then $s = a_j + a_k = 2a_j$ and $2a_j \notin Ap(S, p)$. Also, $s - p = 2a_j - p \in S$. Thus, one of the factorizations of s is $2e_j$. On the

other hand,

$$s = a_j + a_k = 2a_j = \begin{cases} 2c + 2j - 4 & \text{if } 2 \le j \le p - i + 1, \\ 2c + 2j - 2 & \text{if } p - i + 2 \le j \le p, \end{cases}$$

and let's write $c = hp + i$ instead of c

$$s = 2a_j = \begin{cases} hp + (hp + i) + (i + 2j - 4) & \text{if } 2 \le j \le p - i + 1, \\ hp + (hp + i) + (i + 2j - 2) & \text{if } p - i + 2 \le j \le p. \end{cases}$$

(a) If $2 \le j \le p - i + 1$, then we have three cases.
   (i) If $2 \le i + 2j - 4 \le p - i - 1$, then $i + 2j - 2 \ne j$ and one of the factorizations of s is $he_1 + e_{i+2j-2}$.
   (ii) If $p - i - 1 \le i + 2j - 4 \le p - 1$, then $i + 2j - 3 \ne j$ and one of the factorizations of s is $he_1 + e_{i+2j-3}$.
   (iii) If $i + 2j - 4 \ge p$, then $s = 2a_j = hp + (hp + i) + (i + 2j - 4) = hp + (hp + i) + pr_1 + s_1 = (h + r_1)p + (hp + i) + s_1$ for some positive integer r_1 and non-negative integer $s_1 < p$. Since $\max(2a_j) = 2c + 2p - 2i - 2 = hp + (hp + i) + p + (p - i - 2)$ for $2 \le j \le p - i + 1$, we have $r_1 = 1$ and $s_1 < p - i - 2$ due to the values of i and j. Since $0 \le s_1 \le p - i - 2 \le p - i - 1$ and $s_1 + 2 \ne j$, one of the factorizations of s is $(h + 1)e_1 + e_{s_1+2}$.

(b) If $p - i + 2 \le j \le p$, then $i + 2j - 2 > p$. Thus, $s = 2a_j = hp + (hp + i) + (i + 2j - 2) = hp + (hp + i) + pr_2 + s_2 = (h + r_2)p + (hp + i) + s_2$ for some positive integer r_2 and non-negative integer s_2 with $s_2 < p$. Since $\max(2a_j) = 2c + 2p - 2 = hp + (hp + i) + (2p + i - 2)$ for $p - i + 2 \le j \le p$, we have $r_2 = 1$ or $r_2 = 2$ and $s_2 < p$ due to the values of i and j. Then we have two cases.
   (i) If $0 \le s_2 \le p - i + 1$, then $s_2 + 2 \ne j$ and one of the factorizations of s is $(h + r_2)e_1 + e_{s_2+2}$.
   (ii) If $p - i + 2 \le s_2 \le p$, then $s_2 + 1 \ne j$ and one of the factorizations of s is $(h + r_2)e_1 + e_{s_2+1}$.

When $s = a_j + a_k = 2a_j$ for $j = k$, other factorizations of s are different from $2e_j$. These factorizations and $2e_j$ have different $\Re$-classes in $Z(s)$. In particular, this means that there is a factorization $(s_1, \ldots, s_p)$ of s different from $2e_j$ such that

$$supp((s_1, \ldots, s_p)) \bigcap supp(2e_j) = \emptyset.$$

This contradicts with our assumption.

(2) Let $j \ne k$. Then $s = a_j + a_k$ and $a_j + a_k \notin Ap(S, p)$. Also, $s - p \in S$. Thus, one of the factorizations of s is $e_j + e_k$. Then we have two cases.
   (a) If $s = a_j + a_k \equiv 0 \pmod p$, then $a_j + a_k = pr_3$ for some positive integer r_3. Since $2c + 2p - 2i \le a_j + a_k \le 2c + (2p - 3)$, we have $r_3 = 2h + 2$ or $r_3 = 2h + 3$ due to the values of i and j. Thus, one of the factorizations of s is r_3e_1.

(b) If $s = a_j + a_k \equiv s_4 \pmod{p}$, then $a_j + a_k = pr_4 + s_4$ for some positive integers r_4 and s_4. Since $2c+1 \leq a_j+a_k \leq 2c+(2p-3)$, we have $2h \leq r_4 \leq 2h+2$ and $1 \leq s_4 \leq p-1$ due to the values of i and j. We can write $a_j + a_k = pr_4 + s_4 = (r_4 - h)p + (hp+i) + (s_4 - i)$. Therefore, we have three cases.

  (i) If $0 \leq s_4 - i \leq p - i + 1$, then $s_4 - i + 2 \neq k, j$ and one of the factorizations of s is $(r_4 - h)e_1 + e_{s_4 - i+2}$.

  (ii) If $p - i + 2 \leq s_4 - i \leq p$, then $s_4 - i + 1 \neq k, j$ and one of the factorizations of s is $(r_4 - h)e_1 + e_{s_4 - i+1}$.

  (iii) If $s_4 - i \leq 0$, then $a_j + a_k = pr_4 + s_4 = (r_4 - h - 1)p + (hp + i) + (p + s_4 - i)$. Since $p - i + 1 \leq p + s_4 - i \leq p$ and $p + s_4 - i + 1 \neq k, j$ one of the factorizations of s is $(r_4 - h - 1)e_1 + e_{p+s_4-i+1}$.

When $s = a_j + a_k$ for $j \neq k$, other factorizations of s are different from $e_j + e_k$. These factorizations and $e_j + e_k$ have different $\Re$-classes in $Z(s)$. This means in particular that there is a factorization $(s_1, \ldots, s_p)$ of s that is different from $e_j + e_k$ such that

$$supp((s_1, \ldots, s_p)) \bigcap supp(e_j + e_k) = \emptyset.$$

This contradicts with our assumption.

It is known that every element in the semigroup involved in one of its minimal presentations has a set of factorizations with at least two $\Re$-classes. According to the above, $Z(s)$ has at least two $\Re$-classes. Namely, for every $x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p) \in Z(s)$ we can write $supp(x) \bigcap supp(y) = \emptyset$. Thus, $\gcd(x, y) = (0, \ldots, 0)$. This implies in particular that $dist(x, y) = \max\{|x|, |y|\}$. Therefore, the maximum of the lengths of these factorizations gives the catenary degree of S.

We now conclude that the largest factorization length in $Z(s)$ is obtained when $s = a_j + a_k \equiv 0 \pmod{p}$ for $j \neq k$. Since $p < c < c + 1 < \cdots < c + p - i - 1 < c + p - i + 1 < \cdots < c + p - 1$, the smallest s that meets these conditions $\min(s) = \min(a_j + a_k) = 2c + 2p - 2i$. But we can find another element s larger than $2c + 2p - 2i$. Namely, there is an element s in $Z(s)$ with $2c + 2p - 2i < s < 2c + 2(p - 1)$. Since $s = a_j + a_k \equiv 0 \pmod{p}$ for $j \neq k$, if there is, then $s = a_j + a_k = 2c + 2p - 2i + pk < 2c + 2(p - 1)$ for some positive integers k. When we make the necessary cancellations, we get the inequality $i > \frac{pk+2}{2}$, and so $k = 0$ or $k = 1$ due to the values of i. Thus, we have two cases:

(1) If $i > \frac{pk+2}{2}$, then $\max(s) = \max(a_j + a_k) = 2c + 2p - 2i + p = 2hp+2i+3p-2i = 2h+3$, and the factorization of s is $(2h+3)e_1$. Since the catenary degree is the length of this factorization, $C(S) = 2h + 3$ by Corollary 4.1.

(2) In the other cases, namely $i < \frac{p+2}{2}$, then $\max(s) = \max(a_j + a_k) = 2c + 2p - 2i = 2hp + 2i + 2p - 2i = 2h + 2$, and the factorization of s is

$(2h+2)e_1$. Since the catenary degree is the length of this factorization, $C(S) = 2h+2$ by Corollary 4.1. □

Example 4.4. Consider the saturated numerical semigroup S with the multiplicity 5 and the conductor 33. Thus,

$$S = \langle 5, 33, 34, 36, 37\rangle = \{0, 5, 10, 15, 20, 25, 30, 33, \rightarrow\},$$

where $p = 5$, $i = 3$ and $h = 6$. When we consider the numerical semigroup S, $n \in \{33, 34, 36, 37\}$ and $w \in Ap(S, 5) \setminus \{0\} = \{33, 34, 36, 37\}$. Thus, $w+n$ is in $\{66, 67, 68, 69, 70, 71, 72, 73, 74\}$. Then the factorizations of these elements are as follows:

$$\begin{aligned}
Z(66) &= \{(0,2,0,0,0)\,,(6,0,0,1,0)\}\,,\\
Z(67) &= \{(0,1,1,0,0)\,,(6,0,0,0,1)\}\,,\\
Z(68) &= \{(0,0,2,0,0)\,,(7,1,0,0,0)\}\,,\\
Z(69) &= \{(0,1,0,1,0)\,,(7,0,1,0,0)\}\,,\\
Z(70) &= \{(14,0,0,0,0)\,,(0,1,0,0,1)\,,(0,0,1,1,0)\}\,,\\
Z(71) &= \{(0,0,1,0,1)\,,(7,0,0,1,0)\}\,,\\
Z(72) &= \{(0,0,0,2,0)\,,(7,0,0,0,1)\}\,,\\
Z(73) &= \{(0,0,0,1,1)\,,(8,1,0,0,0)\}\,,\\
Z(74) &= \{(0,0,0,0,2)\,,(8,0,1,0,0)\}\,.
\end{aligned}$$

Each element of $Z(s)$ is in the different $\Re$-classes. We get the catenary degree of S at 70. Then the catenary degree of S is 14. Moreover, since $i = 3 < \frac{5+2}{2} = \frac{p+2}{2}$ and $h = 6$, it can easily be found that $C(S) = 2h+2 = (2\cdot 6)+2 = 14$ by Theorem 4.3.

References

[1] F. Aguiló-Gost and P. A. García-Sánchez, *Factorization and catenary degree in 3-generated numerical semigroups*, in European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009), 157–161, Electron. Notes Discrete Math., 34, Elsevier Sci. B. V., Amsterdam, 2009. https://doi.org/10.1016/j.endm.2009.07.026

[2] A. Assi and P. A. García-Sánchez, *Numerical Semigroups and Applications*, RSME Springer Series, 1, Springer, 2016. https://doi.org/10.1007/978-3-319-41330-3

[3] P. Baginski, S. T. Chapman, R. Rodriguez, G. J. Schaeffer, and Y. She, *On the Delta set and catenary degree of Krull monoids with infinite cyclic divisor class group*, J. Pure Appl. Algebra **214** (2010), no. 8, 1334–1339. https://doi.org/10.1016/j.jpaa.2009.10.015

[4] V. Barucci, D. E. Dobbs, and M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. **125** (1997), no. 598, x+78 pp. https://doi.org/10.1090/memo/0598

[5] V. Blanco, P. A. García-Sánchez, and A. Geroldinger, *Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids*, Illinois J. Math. **55** (2011), no. 4, 1385–1414 (2013). http://projecteuclid.org/euclid.ijm/1373636689

[6] S. T. Chapman, M. Corrales, A. Miller, C. Miller, and D. Patel, *The catenary and tame degrees on a numerical monoid are eventually periodic*, J. Aust. Math. Soc. **97** (2014), no. 3, 289–300. https://doi.org/10.1017/S1446788714000330

[7] S. T. Chapman, P. A. García-Sánchez, and D. Llena, *The catenary and tame degree of numerical monoids*, Forum Math. **21** (2009), no. 1, 117–129. https://doi.org/10.1515/FORUM.2009.006

[8] S. T. Chapman, P. A. García-Sánchez, D. Llena, V. Ponomarenko, and J. C. Rosales, *The catenary and tame degree in finitely generated commutative cancellative monoids*, Manuscripta Math. **120** (2006), no. 3, 253–264. https://doi.org/10.1007/s00229-006-0008-8

[9] F. Delgado de la Mata and C. A. Núñez Jiménez, *Monomial rings and saturated rings*, in Géométrie algébrique et applications, I (La Rábida, 1984), 23–34, Travaux en Cours, 22, Hermann, Paris, 1987.

[10] Y. Fan and A. Geroldinger, *Minimal relations and catenary degrees in Krull monoids*, J. Commut. Algebra **11** (2019), no. 1, 29–47. https://doi.org/10.1216/jca-2019-11-1-29

[11] A. Foroutan, *Monotone chains of factorizations*, in Focus on commutative rings research, 107–130, Nova Sci. Publ., New York, 2006.

[12] A. Foroutan and A. Geroldinger, *Monotone chains of factorizations in C-monoids*, in Arithmetical properties of commutative rings and monoids, 99–113, Lect. Notes Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[13] A. Geroldinger, *The catenary degree and tameness of factorizations in weakly Krull domains*, in Factorization in integral domains (Iowa City, IA, 1996), 113–153, Lecture Notes in Pure and Appl. Math., 189, Dekker, New York, 1997.

[14] A. Geroldinger, D. J. Grynkiewicz, and W. A. Schmid, *The catenary degree of Krull monoids I*, J. Théor. Nombres Bordeaux **23** (2011), no. 1, 137–169.

[15] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations*, Pure and Applied Mathematics (Boca Raton), 278, Chapman & Hall/CRC, Boca Raton, FL, 2006. https://doi.org/10.1201/9781420003208

[16] A. Geroldinger and P. Yuan, *The monotone catenary degree of Krull monoids*, Results Math. **63** (2013), no. 3-4, 999–1031. https://doi.org/10.1007/s00025-012-0250-1

[17] A. Núñez, *Algebro-geometric properties of saturated rings*, J. Pure Appl. Algebra **59** (1989), no. 2, 201–214. https://doi.org/10.1016/0022-4049(89)90135-7

[18] M. Omidali, *The catenary and tame degree of numerical monoids generated by generalized arithmetic sequences*, Forum Math. **24** (2012), no. 3, 627–640. https://doi.org/10.1515/form.2011.078

[19] C. O'Neill and R. Pelayo, *Realisable sets of catenary degrees of numerical monoids*, Bull. Aust. Math. Soc. **97** (2018), no. 2, 240–245. https://doi.org/10.1017/S0004972717000995

[20] A. Philipp, *A characterization of arithmetical invariants by the monoid of relations*, Semigroup Forum **81** (2010), no. 3, 424–434. https://doi.org/10.1007/s00233-010-9218-1

[21] J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications, 30, Oxford University Press, Oxford, 2005. https://doi.org/10.1093/acprof:oso/9780198568209.001.0001

[22] L. Rédei, *The Theory of Finitely Generated Commutative Semigroups*, translation edited by N. Reilly, Pergamon Press, Oxford, 1965.

[23] J. C. Rosales, *An algorithmic method to compute a minimal relation for any numerical semigroup*, Internat. J. Algebra Comput. **6** (1996), no. 4, 441–455. https://doi.org/10.1142/S021819679600026X

[24] J. C. Rosales, M. B. Branco, and D. Torrão, *On the enumeration of the set of saturated numerical semigroups with fixed Frobenius number*, Appl. Math. Comput. **236** (2014), 471–479. https://doi.org/10.1016/j.amc.2014.03.058

[25] J. C. Rosales and P. A. García-Sánchez, *Finitely Generated Commutative Monoids*, Nova Science Publishers, Inc., Commack, NY, 1999.

[26] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Developments in Mathematics, 20, Springer, New York, 2009. https://doi.org/10.1007/978-1-4419-0160-6

[27] J. C. Rosales, P. A. García-Sánchez, J. I. García-García, and M. B. Branco, *Saturated numerical semigroups*, Houston J. Math. **30** (2004), no. 2, 321–330.

[28] J. C. Rosales and P. Vasco, *The Frobenius variety of the saturated numerical semigroups*, Houston J. Math. **36** (2010), no. 2, 357–365.

[29] O. Zariski, *General theory of saturation and of saturated local rings. I. Saturation of complete local domains of dimension one having arbitrary coefficient fields* (*of characteristic zero*), Amer. J. Math. **93** (1971), 573–648. https://doi.org/10.2307/2373462

[30] O. Zariski, *General theory of saturation and of saturated local rings. II. Saturated local rings of dimension* 1, Amer. J. Math. **93** (1971), 872–964. https://doi.org/10.2307/2373741

[31] O. Zariski, *General theory of saturation and of saturated local rings. III. Saturation in arbitrary dimension and, in particular, saturation of algebroid hypersurfaces*, Amer. J. Math. **97** (1975), 415–502. https://doi.org/10.2307/2373720

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