J. Appl. Math. & Informatics Vol. **41***(2023), No. 2, pp. 229 - 245*
https://doi.org/10.14317/jami.2023.229

ON OPTIMAL CONTROL FOR COOPERATIVE ELLIPTIC SYSTEMS UNDER CONJUGATION CONDITIONS

H.M. SERAG, L.M. ABD-ELRHMAN AND A.A. AL-SABAN*

Abstract. In this paper, we consider cooperative elliptic systems under conjugation conditions. We first prove the existence of the state for 2×2 cooperative elliptic systems with Dirichlet and Neumann conditions, then we find the set of equations and inequalities that characterizes the optimal control of distributed type for these systems. The case of $n \times n$ cooperative systems is also established.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Cooperative elliptic systems, conjugation conditions, Dirichlet and Neumann conditions, distributed control.

1. Introduction

We consider the following elliptic system

$$-\Delta q_i = \sum_{j=1}^{n} a_{ij} q_j + f_i \qquad \text{in} \quad \Omega, \tag{1}$$

under conjugation conditions :

$$R_1 \left\{ \frac{\partial q_i}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial q_i}{\partial v_A} \right\}^+ = [q_i] + \delta \qquad \text{on} \quad \gamma, \tag{2}$$

and

$$\left[\frac{\partial q_i}{\partial v_A} \right] = \left[\sum_{i,j=1}^{n} \frac{\partial q_i}{\partial x_j} \cos(v, x_i) \right] = w_i \qquad \text{on} \quad \gamma, \quad i = 1, 2, ...n, \tag{3}$$

where Ω is a domain consists of two bounded, continuous and strictly Lipchitz domains Ω_1 and Ω_2 from R^n such that:

$$\Omega_1 \cap \Omega_2 = \phi, \text{ with boundary } \Gamma = (\partial\Omega_1 \cup \partial\Omega_2)/\gamma, \quad (\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi).$$

$$\partial\Omega_1 \cap \gamma = \gamma^-, \partial\Omega_2 \cap \gamma = \gamma^+, f_i \in L^2(\Omega),$$

$$R_1,\ R_2,\ w,\ \delta \in C(\gamma), R_1,\ R_2 \geq 0,\ R_1 + R_2 \geq R_0 > 0, \quad R_0 = \text{constant}, \tag{4}$$

Received June 21, 2021. Revised June 27, 2022. Accepted February 9, 2023. *Corresponding author.

$$\begin{aligned}&[\varphi] = \varphi^+ - \varphi^-,\\ &\varphi^+ = \{\varphi\}^+ = \varphi(x) \quad \forall\, x \in \gamma^+,\\ &\varphi^- = \{\varphi\}^- = \varphi(x) \quad \forall\, x \in \gamma^-.\end{aligned}$$

We assume that

$$a_{ij} > 0 \quad \forall \;\; i \;\neq j \tag{5}$$

Definition 1.1. System (1) is called cooperative system if (5) is satisfied, otherwise is called non -cooperative system. Such systems appear in some biological and physical problems [1].

Lions [2] discussed the optimal control problems for finite order elliptic, parabolic and hyperbolic operators with finite number of variables. Gali and serag [3] extended this discussion to cooperative systems $(a_{12}, a_{21} > 0)$ in the form

$$\begin{cases} -\Delta q_1 = a_{11}q_1 + a_{12}q_2 + f_1 & \text{in } \Omega = \Omega_1 \cup \Omega_2,\\ \\ -\Delta q_2 = a_{21}q_1 + a_{22}q_2 + f_2 & \text{in } \Omega = \Omega_1 \cup \Omega_2,\\ \\ q_1, q_2 \longrightarrow 0 & \text{as } |x| \longrightarrow \infty.\end{cases}$$

In [4, 5], the authors studied system (1), under condition

$$a_{ij} = \begin{cases} 1 & \text{i}f \quad i \geq j\\ \\ -1 & \text{i}f \quad i < j\end{cases}$$

(i.e non cooperative systems.)

Serag et.al.[6, 7, 8] generalized the above discussion to $n \times n$ cooperative systems.

Some existence results have been studied for nonlinear cooperative systems in [9, 10, 11, 12].

The control problems for infinite order hyperbolic operators have been established in [13, 14, 15, 16].

Using the theory of Sergienko and Deineka [17, 18], Serag et.al [19] introduced some control

problems for cooperative systems under conjugation conditions.

In the present work, we study the optimal control of distributed type for some cooperative elliptic systems under conjugation conditions (a complicated thin inclusion case).

2. Distributed control for 2×2 Dirichlet elliptic systems

In this section, we study the distributed control for the following 2×2 cooperative Dirichlet elliptic system:

$$
\begin{cases}
-\Delta q_1 = aq_1 + bq_2 + f_1 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\
-\Delta q_2 = cq_1 + dq_2 + f_2 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\
q_1 = q_2 = 0 & \text{on } \Gamma,
\end{cases}
\tag{6}
$$

under conjugation conditions:

$$
\begin{cases}
R_1 \left\{ \frac{\partial q_1}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial q_1}{\partial v_A} \right\}^+ = [q_1] + \delta & \text{on} \quad \gamma, \\
R_1 \left\{ \frac{\partial q_2}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial q_2}{\partial v_A} \right\}^+ = [q_2] + \delta & \text{on} \quad \gamma,
\end{cases}
\tag{7}
$$

$$
\begin{cases}
\left[\frac{\partial q_1}{\partial v_A} \right] = \left[\sum_{i,j=1}^{n} \frac{\partial q_1}{\partial x_j} \cos(v, x_i) \right] = w_1 & \text{on} \quad \gamma, \\
\left[\frac{\partial q_2}{\partial v_A} \right] = \left[\sum_{i,j=1}^{n} \frac{\partial q_2}{\partial x_j} \cos(v, x_i) \right] = w_2 & \text{on} \quad \gamma,
\end{cases}
\tag{8}
$$

where

$$
a, b, c \textit{ and } d, \text{are given numbers such that b, c>0 .} \tag{9}
$$

We first prove the existence of the state of system(6) under the following conditions:

$$
a < \mu, \;\; d < \mu, \;\; (\mu - a)(\mu - d) > bc, \tag{10}
$$

where μ is a positive constant determined by Friedrich inequality:

$$
\mu \int_{\Omega} |q|^2 \, dx \leq \int_{\Omega} |\nabla q|^2 \, dx. \tag{11}
$$

Then, we prove the existence of distributed control for this system; and we find the set of equations and inequalities that characterizes this distributed control.

Existence and uniqueness of the state. By Cartesian product, we have the following chain of Sobolev spaces:

$$
(H_0^1(\Omega))^2 \subseteq (L^2(\Omega))^2 \subseteq (H^{-1}(\Omega))^2.
$$

On $(H_0^1(\Omega))^2$, we define the bilinear form:

$$
\begin{aligned}
a(q, \psi) = & \int_{\Omega} \nabla q_1 \nabla \psi_1 dx + \int_{\Omega} \nabla q_2 \nabla \psi_2 dx - \int_{\Omega} (aq_1\psi_1 + bq_2\psi_1 + cq_1\psi_2 + dq_2\psi_2) dx \\
& + \int_{\gamma} \frac{[q_1][\psi_1]}{R_1 + R_2} d\gamma + \int_{\gamma} \frac{[q_2][\psi_2]}{R_1 + R_2} d\gamma.
\end{aligned}
\tag{12}
$$

Then, we introduce,

Lemma 2.1. *The bilinear form* (12) *is coercive on* $(H_0^1(\Omega))^2$*; that is, there exists a positive constant* r_1 *such that*

$$a(q,q) \geq r_1 \|q\|^2_{[H_0^1(\Omega)]^2} \qquad \forall q = (q_1, q_2) \in (H_0^1(\Omega))^2. \tag{13}$$

Proof. As in [1], we choose m is large enough such that a+m>0 and d+m>0. Then,

$$\begin{aligned} &a(q,q) \\ &= \frac{1}{b}\int_\Omega (|\nabla q_1|^2 + m\,|q_1|^2)dx + \frac{1}{c}\int_\Omega (|\nabla q_2|^2 + m\,|q_2|^2)dx - \frac{a+m}{b}\int_\Omega |q_1|^2\,dx \\ &- \frac{d+m}{c}\int_\Omega |q_2|^2\,dx - 2\int_\Omega q_1 q_2 dx + \int_\gamma \frac{[q_1][\psi_1]}{R_1+R_2}d\gamma + \int_\gamma \frac{[q_2][\psi_2]}{R_1+R_2}d\gamma. \end{aligned}$$

From (4), we get

$$\begin{aligned} a(q,q) \geq& \frac{1}{b}\int_\Omega (|\nabla q_1|^2 + m\,|q_1|^2)dx + \frac{1}{c}\int_\Omega (|\nabla q_2|^2 + m\,|q_2|^2)dx \\ &- \frac{a+m}{b}\int_\Omega |q_1|^2\,dx - \frac{d+m}{c}\int_\Omega |q_2|^2\,dx - 2\int_\Omega q_1 q_2 dx. \end{aligned}$$

By Cauchy Schwartz inequality

$$\begin{aligned} &a(q,q) \\ &\geq \frac{1}{b}\int_\Omega (|\nabla q_1|^2 + m\,|q_1|^2)dx + \frac{1}{c}\int_\Omega (|\nabla q_2|^2 + m\,|q_2|^2)dx - \frac{a+m}{b}\int_\Omega |q_1|^2\,dx \\ &- \frac{d+m}{c}\int_\Omega |q_2|^2\,dx - 2\left(\int_\Omega |q_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |q_2|^2\,dx\right)^{\frac{1}{2}}. \end{aligned}$$

From (11), we deduce

$$a(q,q) \geq \frac{1}{b}\left(1 - \frac{a+m}{\mu+m}\right){q_1}^2 + \frac{1}{c}\left(1 - \frac{d+m}{\mu+m}\right){q_2}^2 - \frac{2}{\mu+m}q_1 q_2.$$

Therefore (10) implies

$$\begin{aligned} a(q,q) \geq& r_1(q_1^2 + q_2^2) \\ =& r_1\|q\|^2_{[H_0^1(\Omega)]^2} \qquad \forall q \in (H_0^1(\Omega))^2. \end{aligned}$$

□

Then we have

Theorem 2.2. *For a given* $f = (f_1, f_2) \in (L^2(\Omega))^2$ *there exists a unique solution* $q = (q_1, q_2) \in (H_0^1(\Omega))^2$ *for system* (6) *with conjugation conditions* (7) *and* (8) *if conditions* (10) *are satisfied.*

Proof. Since the bilinear form $a(q,\psi)$ is continuous and coercive, then by Lax-Milgram Lemma, there exists a unique solution $q \in (H_0^1(\Omega))^2$ such that:

$$a(q,\psi) = L(\psi) \qquad \forall \psi = (\psi_1, \psi_2) \in (H_0^1(\Omega))^2, \tag{14}$$

where $L(\psi)$ is linear form defined on $(H_0^1(\Omega))^2$ by:

$$L(\psi)=\int_\Omega f_1(x)\psi_1(x)dx+\int_\Omega f_2(x)\psi_2(x)dx+\int_\gamma \frac{(R_2w_1-\delta)[\psi_1]}{R_1+R_2}\,d\gamma+ \\ \int_\gamma \frac{(R_2w_2-\delta)[\psi_2]}{R_1+R_2}\,d\gamma-\int_\gamma w_1\psi_1^+\,d\gamma-\int_\gamma w_2\psi_2^+\,d\gamma. \tag{15}$$

Now, let us multiply both sides of the first equation of (6) by $\psi_1(x)$ and the second equation by $\psi_2(x)$ and integrate over Ω, we have

$$\int_\Omega(-\Delta q_1-aq_1-bq_2)\psi_1(x)dx=\int_\Omega f_1\psi_1dx,$$
$$\int_\Omega(-\Delta q_2-cq_1-dq_2)\psi_2(x)dx=\int_\Omega f_2\psi_2dx,$$

the sum of the two equations, implies

$$\int_\Omega(-\Delta q_1-aq_1-bq_2)\psi_1(x)dx+\int_\Omega(-\Delta q_2-cq_1-dq_2)\psi_2(x)dx \\ =\int_\Omega(f_1\psi_1+f_2\psi_2)dx,$$

applying Green's formula,

$$\int_\Omega(-\Delta q_1-aq_1-bq_2)\psi_1(x)dx+\int_\Omega(-\Delta q_2-cq_1-dq_2)\psi_2(x)dx= \\ -\int_\Gamma(\frac{\partial q_1}{\partial\nu_A})\psi_1(x)d\Gamma-\int_\Gamma(\frac{\partial q_2}{\partial\nu_A})\psi_2(x)d\Gamma-\int_\gamma(\frac{\partial q_1}{\partial\nu_A})\psi_1(x)d\gamma- \\ \int_\gamma(\frac{\partial q_2}{\partial\nu_A})\psi_2(x)d\gamma+a(q,\psi)=\int_\Omega f_1\psi_1dx+\int_\Omega f_2\psi_2dx,$$

from (14), we deduce (6), which complete the proof. □

Formulation of the control problem. The space $U=(L^2(\Omega))^2$ is the space of controls, for a control $u=(u_1,u_2)\in(L^2(\Omega))^2$, the state $q(u)=(q_1(u),q_2(u))$ of the system is given by the solution of

$$\begin{cases} -\Delta q_1(u)=aq_1(u)+bq_2(u)+f_1+u_1 & \text{in } \Omega=\Omega_1\cup\Omega_2, \\ -\Delta q_2(u)=cq_1(u)+dq_2(u)+f_2+u_2 & \text{in } \Omega=\Omega_1\cup\Omega_2, \\ q_1(u)=q_2(u)=0 & \text{on } \Gamma, \end{cases} \tag{16}$$

under conjugation conditions:

$$\begin{cases} R_1\left\{\frac{\partial q_1(u)}{\partial v_A}\right\}^- + R_2\left\{\frac{\partial q_1(u)}{\partial v_A}\right\}^+ = [q_1(u)] + \delta & \text{on} \quad \gamma, \\ R_1\left\{\frac{\partial q_2(u)}{\partial v_A}\right\}^- + R_2\left\{\frac{\partial q_2(u)}{\partial v_A}\right\}^+ = [q_2(u)] + \delta & \text{on} \quad \gamma \\ \left[\frac{\partial q_1(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial q_1(u)}{\partial x_j}\cos(v, x_i)\right] = w_1 & \text{on} \quad \gamma, \\ \left[\frac{\partial q_2(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial q_2(u)}{\partial x_j}\cos(v, x_i)\right] = w_2 & \text{on} \quad \gamma. \end{cases} \tag{17}$$

Specify the observation equation by the following expression

$$z(u) = (z_1(u), z_2(u)) = K_1 q(u) = K_1(q_1(u), q_2(u)) = (q_1(u), q_2(u)).$$

Bring a value of the cost functional

$$J(v) = \|q_1(v) - z_{1d}\|^2_{L^2(\Omega)} + \|q_2(v) - z_{2d}\|^2_{L^2(\Omega)} + (Nv, v)_{(L^2(\Omega))^2}, \tag{18}$$

in correspondence with $z_d = (z_{1d}, z_{2d}) \in (L^2(\Omega))^2$, and N is a hermitian positive definite operator such that:

$$(Nu, u)_{(L^2(\Omega))^2} \geq v_0 \|u\|^2_{(L^2(\Omega))^2}, \qquad v_0 > 0, \quad \forall\ u \in U. \tag{19}$$

Then the control problem is to find u :

$$u = (u_1, u_2) \in U_{ad} \text{ such that: } J(u) = inf\ J(v) \qquad \forall v \in U_{ad}, \tag{20}$$

where the set of admissible control U_{ad} is a closed convex subset of $(L^2(\Omega))^2$. The cost functional (18) can be written as

$$J(v) = \pi(v, v) - 2f(v) + \|z_{1d} - q_1(0))\|^2_{L^2(\Omega)} + \|z_{2d} - q_2(0))\|^2_{L^2(\Omega)},$$

where

$$\begin{aligned} \pi(u, v) =& (q_1(u) - q_1(0), q_1(v) - q_1(0))_{L^2(\Omega)} + \\ & (q_2(u) - q_2(0), q_2(v) - q_2(0))_{L^2(\Omega)} + (Nu, v)_{(L^2(\Omega))^2}, \end{aligned} \tag{21}$$

is a continuous bilinear form and from (19), it is coercive, that is:

$$\pi(v, v) \geq K_2 \|v\|^2_{(L^2(\Omega))^2},$$

and

$$f(v) = (z_{1d} - q_1(0), q_1(v) - q_1(0))_{L^2(\Omega)} + (z_{2d} - q_2(0), q_2(v) - q_2(0))_{L^2(\Omega)}, \tag{22}$$

is a continuous linear form on $(L^2(\Omega))^2$. Then, using the theory of Lions [2], there exists a unique optimal control u of distributed type for problem (20); Moreover it is characterized by

Theorem 2.3. *Let us suppose that* (13) *holds and the cost functional is given by* (18), *then the distributed control* u *is characterized by*

$$\begin{cases} -\Delta p_1(u) - ap_1(u) - cp_2(u) = q_1(u) - z_{1d} & in \quad \Omega, \\ -\Delta p_2(u) - bp_1(u) - dp_2(u) = q_2(u) - z_{2d} & in \quad \Omega, \\ p_1(u) = p_2(u) = 0 & on \quad \Gamma, \\ \left[\frac{\partial p_1(u)}{\partial v_{A^*}}\right] = \left[\frac{\partial p_2(u)}{\partial v_{A^*}}\right] = 0 & on \quad \gamma, \\ \left[\frac{\partial p_1(u)}{\partial v_{A^*}}\right]^{\pm} = \frac{1}{R_1+R_2}[p_1(u)] & on \quad \gamma, \\ \left[\frac{\partial p_2(u)}{\partial v_{A^*}}\right]^{\pm} = \frac{1}{R_1+R_2}[p_2(u)] & on \quad \gamma, \\ (p_1(u), v_1 - u_1) + (p_2(u), v_2 - u_2) + (Nu, v-u)_{(L^2(\Omega))^2} \geq 0, \end{cases} \tag{23}$$

together with (16) *and* (17), *where* $p(u) = (p_1(u), p_2(u))$ *is the adjoint state.*

Proof. The optimal control $u = (u_1, u_2) \in (L^2(\Omega))^2$ is characterized by [6]:

$$\pi(u, v-u) \geq f(v-u) \qquad \forall \ v = (v_1, v_2) \in U_{ad}.$$

From (21), and (22):

$$\begin{aligned} &\pi(u, v-u) - f(v-u) = (q_1(u) - z_{1d}, q_1(v) - q_1(u))_{L^2(\Omega)} + \\ &(q_2(u) - z_{2d}, q_2(v) - q_2(u))_{L^2(\Omega)} + (Nu_1, v_1 - u_1) + (Nu_2, v_2 - u_2) \geq 0. \end{aligned} \tag{24}$$

Now since

$$(p, Aq) = (A^* p, q),$$

then

$$\begin{aligned} (p(u), Aq(u)) =& (p_1(u), -\Delta q_1(u) - aq_1(u) - bq_2(u)) \\ &+ (p_2(u), -\Delta q_2(u) - cq_1(u) - dq_2(u)) \\ =& (-\Delta p_1(u) - ap_1(u) - cp_2(u), q_1(u)) \\ &+ (-\Delta p_2(u) - bp_1(u) - dp_2(u), q_2(u)), \end{aligned}$$

and since the adjoint state is defined by (23), therefore (24) is equivalent to

$$\begin{aligned} &(-\Delta p_1(u) - ap_1(u) - cp_2(u), q_1(v) - q_1(u)) + (Nu_1, v_1 - u_1) \\ &(-\Delta p_1(u) - bp_1(u) - dp_2(u), q_2(v) - q_2(u)) + (Nu_2, v_2 - u_2) \geq 0. \end{aligned}$$

Applying Green's formula, and using equation (16), we obtain

$$\int_\Omega (p_1(u) + Nu_1)(v_1 - u_1) dx + \int_\Omega (p_2(u) + Nu_2)(v_2 - u_2) \ dx \geq 0.$$

□

3. Distributed control for 2×2 Neumann elliptic systems

We consider in this section the following Neumann elliptic system:

$$\begin{cases} -\Delta q_1 = aq_1 + bq_2 + f_1 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ -\Delta q_2 = cq_1 + dq_2 + f_2 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ \frac{\partial q_1}{\partial \nu_A} = g_1 & \text{on } \Gamma, \\ \frac{\partial q_2}{\partial \nu_A} = g_2 & \text{on } \Gamma, \end{cases} \tag{25}$$

with conjugation conditions (7), (8), where $(g_1, g_2) \in (L^2(\Gamma))^2$ are given functions. We introduce again the bilinear form (12) which is coercive on $(H^1(\Omega))^2$, since

$$(H_0^1(\Omega))^2 \subseteq (H^1(\Omega))^2.$$

Let us define

$$V_m = \left\{ q(x) \mid_{\Omega_k} \in (H^1(\Omega_k))^2, k = 1, 2 \right\}.$$

Then by Lax-Milgram lemma, there exists a unique solution $q \in V_m$ for system (25) according the equation (14), where

$$\begin{aligned} L_g(\psi) = & \int_\Omega f_1(x)\psi_1(x)dx + \int_\Omega f_2(x)\psi_2(x)dx + \\ & \int_\Gamma g_1(x)\psi_1(x)d\Gamma + \int_\Gamma g_2(x)\psi_2(x)d\Gamma + \int_\gamma \frac{(R_2 w_1 - \delta)[\psi_1]}{R_1 + R_2}\, d\gamma + \\ & \int_\gamma \frac{(R_2 w_2 - \delta)[\psi_2]}{R_1 + R_2}\, d\gamma - \int_\gamma w_1 \psi_1^+ \, d\gamma - \int_\gamma w_2 \psi_2^+ \, d\gamma, \end{aligned}$$

is a continuous linear form defined on $(H^1(\Omega))^2$.
Applying Green's formula,

$$\begin{aligned} & \int_\Omega (-\Delta q_1 - aq_1 - bq_2)\psi_1(x)dx + \int_\Omega (-\Delta q_2 - cq_1 - dq_2)\psi_2(x)dx = \\ & - \int_\Gamma \frac{\partial q_1}{\partial \nu_A}\psi_1(x)d\Gamma - \int_\Gamma \frac{\partial q_2}{\partial \nu_A}\psi_2(x)d\Gamma - \int_\gamma \frac{\partial q_1}{\partial \nu_A}\psi_1(x)d\gamma \\ & - \int_\gamma \frac{\partial q_2}{\partial \nu_A}\psi_2(x)d\gamma + a(q,\psi) = \int_\Omega f_1\psi_1 dx + \int_\Omega f_2\psi_2 dx, \end{aligned}$$

for a control $u \in (L^2(\Omega))^2$, the state $q(u) = (q_1(u), q_2(u))$ of the system is given by the solution of

$$\begin{cases} -\Delta q_1(u) = aq_1(u) + bq_2(u) + f_1 + u_1 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ -\Delta q_2(u) = cq_1(u) + dq_2(u) + f_2 + u_2 & \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ \frac{\partial q_1(u)}{\partial \nu_A} = g_1 & \text{on } \Gamma, \\ \frac{\partial q_2(u)}{\partial \nu_A} = g_2 & \text{on } \Gamma, \end{cases} \tag{26}$$

under conjugation conditions (17). For a given $z_d = (z_{1d}, z_{2d}) \in (L^2(\Omega))^2$, the cost functional is again given by (18), then there exists a unique optimal control $u \in U_{ad}$ according the equation (20), moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_1(u) - ap_1(u) - cp_2(u) = q_1(u) - z_{1d} & \text{in} \quad \Omega, \\ -\Delta p_2(u) - bp_1(u) - dp_2(u) = q_2(u) - z_{2d} & \text{in} \quad \Omega, \\ \frac{\partial p_1(u)}{\partial v_{A^*}} = \frac{\partial p_2(u)}{\partial v_{A^*}} = 0 & \text{on} \quad \Gamma, \\ \left[\frac{\partial p_1(u)}{\partial v_{A^*}}\right] = \left[\frac{\partial p_2(u)}{\partial v_{A^*}}\right] = 0 & \text{on} \quad \gamma, \\ \left[\frac{\partial p_1(u)}{\partial v_{A^*}}\right]^{\pm} = \frac{1}{R_1+R_2}[p_1(u)] & \text{on} \quad \gamma, \\ \left[\frac{\partial p_2(u)}{\partial v_{A^*}}\right]^{\pm} = \frac{1}{R_1+R_2}[p_2(u)] & \text{on} \quad \gamma, \end{cases}$$

and

$$\int_\Omega (p_1(u) + Nu_1)(v_1 - u_1)\, dx + \int_\Omega (p_2(u) + Nu_2)(v_2 - u_2) dx \ \geq 0,$$

together with (26) and (17), where $p(u) = (p_1(u), p_2(u))$ is the adjoint state, $\forall \ u = (u_1, u_2) \in U_{ad}$.

4. Distributed control for $n \times n$ systems described by Dirichlet elliptic equation under conjugation conditions

In this section, we generalized the discussion which has been introduced in section 2 to the following $n \times n$ cooperative system (1), with

$$f_i \in L^2(\Omega), \ (i = 1, 2, ..., n).$$

The homogeneous boundary Dirichlet condition

$$q_i = 0 \qquad \text{on} \quad \Gamma, \quad i = 1, 2, ..., n, \tag{27}$$

is specified in its turn on a boundary

$$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)/\gamma,\ \gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi,$$

on the section γ of the domain

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2,$$

the conjugation conditions for an imperfect contact have the form of expressions (2), (3). To study our problem, we introduce the following definition:

Definition 4.1. The matrix $(\mu I - M)$ is a non-singular M-matrix which means that all the principal minors extracted from it are positive, where, I is the identity matrix, $M = (a_{ij})$ and μ is a positive constant determined by Friedrich inequality (11).

By Cartesian product, we have the following chain of Sobolev spaces:

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H^{-1}(\Omega))^n$$

On $(H_0^1(\Omega))^n$, we define the bilinear form :

$$a(q,\psi) = \sum_{i=1}^{n} \int_\Omega \nabla q_i \nabla \psi_i dx - \sum_{i,j=1}^{n} \int_\Omega a_{ij} q_i \psi_i dx + \sum_{i=1}^{n} \int_\gamma \frac{[q_i][\psi_i]}{R_1 + R_2} d\gamma. \tag{28}$$

Lemma 4.2. *The bilinear form* (28) *is coercive on* $(H_0^1(\Omega))^n$ *if* $(\mu I - M)$ *is a non-singular M-matrix, that is, there exists a positive constant* C *such that*

$$a(q,q) \geq C\|q\|^2_{(H_0^1(\Omega))^n} \qquad \forall q = (q_1, q_2, ..., q_n) \in (H_0^1(\Omega))^n. \tag{29}$$

Proof.

$$\begin{aligned} a(q,q) = &\sum_{i=1}^{n} \int_\Omega |\nabla q_i|^2 \, dx - \sum_{i=1}^{n} a_{ij} \int_\Omega |q_i|^2 \, dx \\ &- \sum_{i=1}^{n} a_{ij} \int_\Omega q_i q_j dx + \sum_{i=1}^{n} \int_\gamma \frac{[q_i]^2}{R_1 + R_2} d\gamma \end{aligned}$$

$$\begin{aligned} = &\sum_{i=1}^{n} \int_\Omega (|\nabla q_i|^2 + m\,|q_i|^2) dx - \sum_{i=1}^{n} (a_{ij} + m) \int_\Omega |q_i|^2 \, dx \\ &- \sum_{i=1}^{n} a_{ij} \int_\Omega q_i q_j dx + \sum_{i=1}^{n} \int_\gamma \frac{[q_i]^2}{R_1 + R_2} d\gamma. \end{aligned}$$

From(4), we get,

$$a(q,q) \geq \sum_{i=1}^{n} \int_\Omega (|\nabla q_i|^2 + m\,|q_i|^2) dx - \sum_{i=1}^{n} (a_{ij} + m) \int_\Omega |q_i|^2 \, dx - \sum_{i=1}^{n} a_{ij} \int_\Omega q_i q_j dx.$$

By Cauchy Schwartz inequality

$$a(q,q) \geq \sum_{i=1}^{n} \int_{\Omega} (|\nabla q_i|^2 + m\,|q_i|^2)dx - \sum_{i=1}^{n}(a_{ij}+m)\int_{\Omega}|q_i|^2\,dx$$
$$-\sum_{i=1}^{n} a_{ij}(\int_{\Omega}|q_i|^2\,dx)^{\frac{1}{2}}(\int_{\Omega}|q_j|^2\,dx)^{\frac{1}{2}}.$$

From Friedrich inequality, we deduce

$$a(q,q) \geq \sum_{i=1}^{n}(1-\frac{a_{ij}+m}{\mu+m})\|q_i\|^2 - \sum_{i=1}^{n}(\frac{a_{ij}}{\mu+m})\|q_i\|\|q_j\|.$$

From definition 4.1, we obtain

$$a(q,q) \geq C\sum_{i=1}^{n}\|q_i\|^2 \ = C\|q\|^2_{(H_0^1(\Omega))^n}, \forall q \in (H_0^1(\Omega))^n.$$

□

Now, Let $\psi \to L(\psi)$ be a linear form defined on $(H_0^1(\Omega))^n$ by

$$L(\psi) = \sum_{i=1}^{n}\int_{\Omega} f_i(x)\psi_i(x)dx + \sum_{i=1}^{n}\int_{\gamma}\frac{(R_2 w_i - \delta)[\psi_i]}{R_1+R_2}d\gamma - \sum_{i=1}^{n}\int_{\gamma} w_i\psi_i^+ d\gamma.$$

Then by Lax-Milgram lemma, there exists a unique solution $q \in (H_0^1(\Omega))^n$ such that:

$$a(q,\psi) = L(\psi) \qquad \forall \psi = (\psi_i)_{i=1}^n \in (H_0^1(\Omega))^n.$$

Then, we have proved the following theorem

Theorem 4.3. *For $f = \{f_i\}_{i=1}^n \in (L^2(\Omega))^n$ there exists a unique solution $q \in (H_0^1(\Omega))^n$ for cooperative Dirichlet systems* (1)*and* (27) *with conjugation conditions* (2) *and* (3) *if definition* 4.1 *is satisfied.*

The space U=$(L^2(\Omega))^n$ is the space of controls. For a control $u \in (L^2(\Omega))^n$, the state $q(u)$ of the system is given by the solution of

$$\begin{cases} -\Delta q_i(u) = \sum_{j=1}^{n} a_{ij}q_j(u) + f_i(u) + u_i & \text{in} \quad \Omega, \\ q_i(u) = 0 & \text{on} \quad \Gamma, \quad i=1,2,...,n, \end{cases} \tag{30}$$

under conjugation conditions:

$$\begin{cases} R_1\left\{\frac{\partial q_i(u)}{\partial v_A}\right\}^- + R_2\left\{\frac{\partial q_i(u)}{\partial v_A}\right\}^+ = [q_i(u)] + \delta & \text{on} \quad \gamma, \\ \left[\frac{\partial q_i(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n}\frac{\partial q_i(u)}{\partial x_j}\cos(v,x_i)\right] = w_i & \text{on} \quad \gamma, \quad i=1,2,...n, \end{cases} \tag{31}$$

the observation equation z(u)= C $q(u)$= $q(u)$. For a given $z_d \in (L^2(\Omega))^n$, the cost functional

$$J(v) = \sum_{i=1}^{n} \|q_i(v) - z_{id}\|^2_{L^2(\Omega)} + (Nv, v)_{(L^2(\Omega))^n}. \tag{32}$$

N: $U \to U$ is a positive operator such that:

$$(Nv, v)_{(L^2(\Omega))^n} \geq M\|v\|^2_{(L^2(\Omega))^n}, \qquad M > 0, \quad \forall\ v \in U. \tag{33}$$

The optimal control problem is: find such an element $u \in U$ that the condition

$$J(u) = inf\ J(v) \qquad \forall v \in U_{ad}, \tag{34}$$

is met, where U_{ad} is some convex closed subset in U. The cost functional (32) can be written as

$$J(v) = \pi(v, v) - 2L(v) + \sum_{i=1}^{n} \|z_{id} - q_i(0))\|^2_{L^2(\Omega)},$$

where

$$\pi(u, v) = \sum_{i=1}^{n} (q_i(u) - q_i(0), q_i(v) - q_i(0))_{L^2(\Omega)} + (Nu, v)_{(L^2(\Omega))^n}, \tag{35}$$

is a continuous bilinear form and from (33), it is coercive, that is:

$$\pi(v, v) \geq M\|v\|^2_{(L^2(\Omega))^n},$$

and

$$L(v) = \sum_{i=1}^{n} (z_{id} - q_i(0), q_i(v) - q_i(0))_{L^2(\Omega)}, \tag{36}$$

is a continuous linear form on $(L^2(\Omega))^n$. Then, there exists a unique optimal control of problem (34); Moreover it is characterized by

Theorem 4.4. *Let us suppose that* (29) *holds and the cost functional is given by* (32)*, then the distributed control u is characterized by*

$$\begin{cases} -\Delta p_i(u) - \sum_{j=1}^{n} a_{ji} p_j(u) = q_i(u) - z_{id} & in \quad \Omega, \\ p_i(u) = 0 & on \quad \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_{A^*}}\right] = 0 & on \quad \gamma, \\ \left\{\frac{\partial p_i(u)}{\partial v_{A^*}}\right\}^{\pm} = \frac{1}{R_1+R_2}[p_i(u)] & on \quad \gamma, \\ \sum_{i=1}^{n} (p_i(u), v_i - u_i) + (Nu, v - u)_{(L^2(\Omega))^n} \geq 0 & , i = 1, 2, ..., n, \end{cases} \tag{37}$$

together with(30)*, where* $p_i(u)$ *is the adjoint state.*

Proof. The optimal control u is characterized by :

$$\pi(u, v-u) - L(v-u) \geq 0 \qquad \forall\ v \in U_{ad}.$$

From (35), and (36):

$$\begin{aligned}\pi(u, v-u) - L(v-u) = &\sum_{i=1}^{n}(q_i(u) - z_{id}, q_i(v) - q_i(u))_{L^2(\Omega)} \\ &+ \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.\end{aligned} \tag{38}$$

Since

$$Aq(u) = \sum_{i=1}^{n}(-\Delta q_i(u) - \sum_{j=1}^{n} a_{ij} q_j(u)), \quad (p, Aq) = (A^* p, q),$$

then

$$\begin{aligned}(p, Aq)_{(L^2(\Omega))^n} &= \sum_{i=1}^{n}(p_i(u), -\Delta q_i(u) - \sum_{j=1}^{n} a_{ij} q_j(u)) \\ &= \sum_{i=1}^{n}(-\Delta p_i(u) - \sum_{j=1}^{n} a_{ji} p_j(u), q_i(u))_{L^2(\Omega)},\end{aligned}$$

and since the adjoint state is defined by (37), then (38) is equivalent to

$$\sum_{i=1}^{n}(-\Delta p_i(u) - \sum_{j=1}^{n} a_{ji} p_j(u), q_i(v-u) - q_i(0))_{L^2(\Omega)} + \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.$$

Therefore

$$\sum_{i=1}^{n}(A^* p_i(u), q_i(v-u) - q_i(0))_{L^2(\Omega)} + \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0.$$

Using Green's formula

$$\sum_{i=1}^{n}(p_i(u), Aq_i(v-u) - Aq_i(0))_{L^2(\Omega)} + \sum_{i=1}^{n}(Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0,$$

and using equation (30), we obtain

$$\sum_{i=1}^{n}\int_{\Omega}(p_i(u) + Nu_i)(v_i - u_i)dx \geq 0.$$

□

5. Distributed control for $n \times n$ systems described by Neumann elliptic equation under conjugation conditions

We study the $n \times n$ cooperative Neumann elliptic system of the form

$$\begin{cases} -\Delta q_i = \sum_{j=1}^n a_{ij} q_j + f_i & \text{in} \quad \Omega, \\ \frac{\partial q_i}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \tag{39}$$

with conjugation conditions (2) and (3), where $g_i \in (L^2(\Gamma))^n$ are given functions, $i = 1, 2, ..., n$. We introduce again the bilinear form (28) which is coercive on $(H^1(\Omega))^n$, since

$$((H_0^1(\Omega))^n) \subseteq ((H^1(\Omega))^n).$$

Then by Lax-Milgram lemma, there exists a unique solution q for system (39) such that :

$$a(q, \psi) = L_N(\psi), \quad \forall \psi \in (H^1(\Omega))^n,$$

where

$$L_N(\psi) = \sum_{i=1}^n \int_\Omega f_i(x)\psi_i(x)dx + \sum_{i=1}^n \int_\Gamma g_i(x)\psi_i(x)d\Gamma + \sum_{i=1}^n \int_\gamma \frac{(R_2 w - \delta)[\psi_i]}{R_1 + R_2}\, d\gamma - \sum_{i=1}^n \int_\gamma w\psi_i^+\, d\gamma,$$

is a continuous linear form defined on $(H^1(\Omega))^n$. Let us multiply both sides of first equation of (39) by $\psi \in (H^1(\Omega))^n$ and integrate over Ω , we have

$$\sum_{i=1}^n \int_\Omega (-\Delta q_i - \sum_{j=1}^n a_{ij} q_j)\psi_i(x)dx = \sum_{i=1}^n \int_\Omega f_i \psi_i dx,$$

then

$$\sum_{i=1}^n \int_\Omega (-\Delta q_i - \sum_{j=1}^n a_{ij} q_j)\psi_i(x)dx + \sum_{i=1}^n \int_\Gamma (\frac{\partial q_i}{\partial \nu_A})\psi_i(x)d\Gamma + \sum_{i=1}^n \int_\gamma (\frac{\partial q_i}{\partial \nu_A})\psi_i(x)d\gamma + a(q, \psi) = \sum_{i=1}^n \int_\Omega f_i \psi_i dx,$$

but, from

$$a(q, \psi) = L_N(\psi),$$

we deduce the Neumann condition

$$\frac{\partial q_i}{\partial \nu_A} = g_i \quad \text{on } \Gamma.$$

In this case, the space $U = (L^2(\Omega))^n$ is the space of controls. For a control $u \in (L^2(\Omega))^n$, the state $q(u)$ of the system is given by the solution of

$$\begin{cases} -\Delta q_i(u) = \sum_{j=1}^{n} a_{ij} q_j(u) + f_i(u) + u_i & \text{in} \quad \Omega, \\ \frac{\partial q_i(u)}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \tag{40}$$

under conjugation conditions (31). For a given $z_d \in (L^2(\Omega))^n$, the cost functional is again given by (32), then there exists a unique optimal control $u \in U_{ad}$ such that:

$$J(u) = inf\ J(v),\ \forall v \in U_{ad},$$

moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_i(u) - \sum_{j=1}^{n} a_{ji} p_j(u) = q_i(u) - z_{id} & \text{in} \quad \Omega, \\ \frac{\partial p_i(u)}{\partial v_{A^*}} = 0 & \text{on} \quad \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_{A^*}} \right] = 0 & \text{on} \quad \gamma, \\ \left\{ \frac{\partial p_i(u)}{\partial v_{A^*}} \right\}^{\pm} = \frac{1}{R_1 + R_2} [p_i(u)] & \text{on} \quad \gamma, \end{cases}$$

$$\sum_{i=1}^{n} (p_i(u), v_i - u_i) + (Nu, v - u)_{(L^2(\Omega))^n} \geq 0 \quad , i = 1, 2, ..., n.$$

6. Conclusions

In this paper, we focused on optimal control problems for cooperative elliptic systems under conjugation conditions. Under some conditions on the coefficients, we proved the existence and uniqueness of the state for 2×2 Dirichlet cooperative elliptic system under conjugation conditions. Then we demonstrated the existence and uniqueness of the optimal control of distributed type for this system. We gave the set of equations and inequalities that characterizes this control. Also, we studied the problem with Neumann condition. Finally, we generalized the discussion to $n \times n$ cooperative elliptic systems under conjugation conditions.

Conflicts of interest : The authors declare no conflicts of interest regarding the publication of this paper.

Acknowledgments : The authors would like to express their gratitude to Professor Dr. I. M. Gali, Mathematics Department, Faculty of Science, Al-Azhar University, for suggesting the problem and critically reading the manuscript.

References

1. J.P. Fleckinger and H.M. Serag, *Semilinear cooperative elliptic systems on R^n*, Rend. Mat. Appl. **15** (1995), 89-108.
2. J.L. Lions, *Optimal control of a system governed by partial differential equations*, Springer-Verlag, New York, 1971.
3. I.M. Gali and H.M. Serag, *Optimal control of cooperative systems defined on R^n*, J. Egypt. Math. Soc. **3** (1995), 33-39.
4. H.A. El-Saify, H.M. Serag and B.G. Abdul-Gawad, *On optimal control for $n \times n$ elliptic systems involving operators with an infinite number of variables*, Journal of Mathematics **37** (2005), 115-128.
5. A.H. Qamlo, H.M. Serag and E.A. El-Zahrany, *Optimal control for non-cooperative parabolic system with conjugation conditions*, European Journal of Scientific Research **131** (2015), 215-226.
6. H.M. Serag, *On Optimal control for elliptic system with variable coefficients*, Revista de Matematica Aplicadas, Departamento de Ingenieria Matematica, Universidad Chile, **19** (1998), 45-49.
7. H.M. Serag, *Optimal control for systems involving Schrodinger operators*, Int. J. of Control and Intelligent Systems **32** (2004), 154-159.
8. H.M. Serag, *Distributed control for cooperative systems involving parabolic operators with an infinite number of variables*, IMA Journal of Mathematical Control and Information **24** (2007), 149-161.
9. S. Khafagy and H.M. Serag, *Stability results of positive weak solution for singular p-Laplacian nonlinear system*, J. Appl. Math. Inf. **36** (2018), 173-179
10. S. Khafagy and H.M. Serag, *On the existence of positive weak solution for nonlinear system with singular weights*, Journal of Contemporary Mathematical Analysis **55** (2020), 259-267.
11. H.M. Serag and S. Khafagy, *Existence of weak solution for $n \times n$ nonlinear systems involving different degenerated p-Laplacian operators*, New Zealand J. Math. **38** (2008), 75-86.
12. H.M. Serag and S. Khafagy, *On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear elliptic systems involving degenerated p-Laplacian operators*, Turk. J. Math. **34** (2010), 59-71.
13. I.M. Gali and H.A. El-Saify, *Control of system governed by infinite order equation of hyperbolic type*, Proceeding of the International Conference on Functional-Differential Systems and Related Topics, **3** (1983), 99-103.
14. W. Kotarski and G.M. Bahaa, *Optimal control problem for infinite order hyperbolic system with mixed control-state constraints*, European Journal of Control **11** (2005), 149-159.
15. A.H. Qamlo, *Boundary control problems for 2×2 cooperative hyperbolic systems with infinite order operators*, Open Journal of Optimization **10** (2021), 1-12.
16. A.H. Qamlo and G.M. Bedaiwi, *Distributed control for 2×2 coupled infinite order hyperbolic systems*, Advances in Differential Equations and Control Processes **18** (2017), 201-227.
17. I.V. Sergienko and V.S. Deineka, *Optimal control of an elliptic system with conjugation conditions and Neumann boundary conditions*, Cybernetics and Systems Analysis **40** (2004), 865-882.
18. I.V. Sergienko and V.S. Deineka, *Optimal control of distributed systems with conjugation conditions*, Springer, 2005.
19. H.M. Serag, S.A. EL-Zahaby and L.M. Abd Elrhman, *Distributed control for cooperative parabolic systems with conjugation conditions*, Journal of Progressive Research in Mathematics **4** (2015), 348-365.

H.M. Serag received M.Sc. from Al-Azhar University, and Ph.D. from Al-Azhar University & Paul Sabatier University-Toulouse-France(channel system). He is currently a professor at Al-Azhar University. His research interests are functional analysis, optimal control and maximum principle.

Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt.
e-mail: serraghm@yahoo.com

L.M. Abd-Elrhman received M.Sc. from Al-Azhar University, and Ph.D. from Al-Azhar University. She is currently a lecturer at Al-Azhar University. Her research interests are functional analysis.

Department of Mathematics, Faculty of Science, Al-Azhar University[for girls], Nasr City, Cairo, Egypt.
e-mail: lamia.abdelrhman3461@yahoo.com

A.A. Al-Saban received M.Sc. in pure mathematics from Helwan University, Since 2018, studying for a Ph.D. degree at Al-Azhar University. His research interests focus on functional analysis (optimal control).

Department of Mathematics, Faculty of Science, Ibb University, Ibb, Yemen.
e-mail: ahmedalsaban@yahoo.com