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UNIQUENESS OF q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIAL OF MEROMORPHIC AND ENTIRE FUNCTION WITH ZERO-ORDER

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ABSTRACT. In this article, we investigate the uniqueness problem of q-shift difference polynomial of meromorphic (entire) function with zero-order. Consequently, we prove three results with significantly generalize the results of Goutam Haldar.

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1. Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For some $a \in \mathbb{C} \cup \{\infty\}$, if the zero of $f - a$ and $g - a$ have the same locations as well as same multiplicities, we say that f and g share the value a CM (counting multiplicities). If we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities). Throughout the paper the elemental and standard notations of Nevanlinna's Value Distribution Theory of meromorphic functions which are discussed in [16] have been adopted. A meromorphic function a is said to be a small with respect to f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o\{T(r, f)\}$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. Also, we use I to denote any set of infinite linear measure of $0 < r < \infty$. If $\alpha \equiv \alpha(z)$ is a small function, we define that f and g share α CM (IM) according as $f - \alpha$ and $g - \alpha$ share 0 CM (IM). The polynomial $\mathcal{Q}(\omega)$ of degree $n + m$ defined by

$$\mathcal{Q}(\omega) = a^*_{m+n}\omega^{m+n} + \ldots + a^*_1 z + a^*_0 = a^*_{m+n}\prod_{j=1}^{s}(\omega - \omega_{p_j})^{p_j}, \tag{1}$$

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where $a_j^* \in \mathbb{C}$, $(j = 0, 1, \dots, n+m)$ with $a_{m+n}^* \neq 0$, and ω_{p_j} are distinct complex numbers, and $2 \leq s \leq n+m$, $p_1, p_2, \dots, p_s$, $s \geq 2$, n, m are any non-negative integers satisfying $p_1 + p_2 + \dots + p_s = n+m$. We also suppose that $p > \max_{p \neq p_j,\ j=1,\ 2,\dots,s-1} \{p_j\}$.

Let $\mathcal{P}(\omega_1) = a_{n+m}^* \prod_{j=1}^{s-1} (\omega_1 + \omega_p - \omega_{p_j})^{p_j} = a_q \omega_1^q + a_{q-1}\omega_1^{q-1} + \dots + a_1\omega_1 + a_0$, where $a_{m+n}^* = a_q$, $\omega_1 = \omega - \omega_p$, $q = n+m-p$. Thus, we see that

$$\mathcal{Q}(\omega) = \omega_1^p \mathcal{P}(\omega_1)$$

where $\mathcal{P}(\omega_1) = a_q\omega_1^q + a_{q-1}\omega_1^{q-1} + \dots + a_0$ is a polynomial of degree q such that $p+q = n+m$ and hence for a meromorphic function f and f_1 satisfying $f = f_1 + \omega_p$, we have

$$\mathcal{Q}(f) = f_1^p \mathcal{P}(f_1) \tag{2}$$

Suppose c be a non-zero complex constant. We define the shift of $f(z)$ by $f(z+c)$ and define the difference operators by

$$\Delta_c f(z) = f(z+c) - f(z),$$

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)),\ n \in \mathbb{N},\ n \geq 2.$$

We recall a linear difference polynomial $\mathcal{L}(z,f)$ of f which is introduced in [18] as

$$\mathcal{L}(z,f) = b_t f(z+c_t) + \dots + b_1 f(z+c_1) + b_0 f(z+c_0),$$

where $b_t(\neq 0), \dots, b_1,\ b_0;\ c_t, \dots c_1,\ c_0$ are complex constants and t be a positive integer. It can be seen that $\Delta_c f$ is a particular form of $\mathcal{L}_c f = c_1 f(z+c) + c_0 f(z)$ (see [19]). In fact $\mathcal{L}_c f$ and $\Delta_c^n f(z)$ are particular form of $\mathcal{L}(z,f)$. We define a linear q-shift difference polynomial as follows,

$$\mathcal{L}(z,f) = b_t f(qz+c_t) + \dots + b_1 f(qz+c_1) + b_0 f(qz+c_0). \tag{3}$$

For $s \in \mathbb{N}$, let us define

$$\chi_{b_0} = \begin{cases} 1, & if \quad b_0 \neq 0 \\ 0, & if \quad b_0 = 0. \end{cases}$$

Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \dots + a_0$ be a non-zero polynomial of degree n, where $a_m(\neq 0),\ a_{m-1}, \dots, a_0$ are complex constants and m is a positive integer. Let m_1 be the number of distinct simple zeros and m_2 be the number of distinct multiple zeros of $P(z)$. Let $\Gamma_0 = m_1 + 2m_2$ and $\Gamma_1 = m_1 + m_2$.

In 2021, Goutam Haldar [20] proved the following results.

Theorem 1.1. *(see [20]) Let f be transcendental meromorphic (resp. entire) function of zero order, and s $(\neq 0)$, k be non-negative integers. If $m > \Gamma_1 + km_2 + 2s + (1+s)\chi_{b_0} + 2$ (resp. $n > \Gamma_1 + km_2$), then $(\mathcal{P}(f)L(z,f)^s)^{(k)} - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \in S(f) - \{0\}$.*

Theorem 1.2. [20] *Let f and g be two transcendental entire functions of zero order and n be a positive integer such that $n \geq m+5$. Let $f^n P(f)L(z,f) - p(z)$ and $g^n P(g)L(z,g) - p(z)$ share (0,2), where $p(z)$ be a non-zero polynomial such that $deg(p) < \frac{n-1}{2}$ and $g(z)$, $g(qz+c)$ share 0 CM. Then one of the following conclusions can be realized. (i) $f \equiv tg$ where t is a constant satisfying $t^d = 1$, where $d = GCD\{n+m+1,\ n+m, \ldots, n+1\}$ and $a_{q-j} \neq 0$ for some $j = 0,\ 1, \ldots, m$.(ii) f and g satisfy the algebraic equation $A(x,y) = 0$, where $A(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + \ldots + a_0)L(z,\omega_1) - \omega_2^n(a_m\omega_2^m + \ldots + a_0L(z,\omega_2)$.*

Theorem 1.3. [20] *Let f, g be two transccendental entire functions of zero order. If $E_l(1;(P(f)L(z,f))^{(k)}) = E_l(1;(P(g)g(qz+c))^{(k)})$ and l, m, n are integers satisfying one of the following conditions.*
(i) $l \geq 2$, $m > 2\Gamma_0 + 2km_2 + 1$;
(ii) $l = 1$, $m > \frac{1}{2}(\Gamma_1 + 4\Gamma_0 + 5km_2 + 3)$;
(iii) $l = 0$, $m > 3\Gamma_1 + 2\Gamma_0 + 5km_2 + 4$, then one of the following results holds.
(i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0,\ \lambda_1, \ldots, \lambda_m\}$.
(ii) f and g satisfy the algebraic equation $A(\omega_1,\omega_2) = 0$ where $A(\omega_1,\omega_2) = P(\omega_1)L(z,\omega_1) - P(\omega_2)L(z,\omega_2)$.

2. Definitions

In 2009, Lahiri [14] introduced a gradation of sharing of values or sets which is known as weighted sharing. Below we are recalling the notion.

Definition 2.1. (see [14]) Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k. We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 2.2. (see [12]) Let f and g be non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. Similarly, one can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.

Definition 2.3. (see [14], [7]) Let f, g share a value a IM. Denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly, we note that $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 2.4. (see [15]) Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.
(i) $N(r, a; f| \geq p)$ denotes the counting function of those a-points of f whose multiplicities are not less than p.
(ii) $\overline{N}(r, a; f| \geq p)$ denotes the reduced counting function of those a-points of f whose multiplicities are not less than p.
(iii) $N(r, a; f| \leq p)$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.
(iv) $\overline{N}(r, a; f| \leq p)$ denotes the reduced counting function of those a-points of f whose multiplicities are not greater than p.

3. Lemmas

In this section, we prove some Lemmas which will play an important role in proving the main results. We denote H by the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{4}$$

where F and G are two non-constant meromorphic functions.

Lemma 3.1. *(see [2]) Let f be a zero order meromorphic function, and let $c,\ q\ (\neq 0) \in \mathbb{C}$. Then*

$$m\Big(r, \frac{f(qz+c)}{f(z)}\Big) = S(r, f).$$

Lemma 3.2. *(see [4]) Let f be a zero order meromorphic function, and $c,\ q \in \mathbb{C}$. Then*

$$T(r, f(qz+c)) = T(r, f) + S(r, f).$$
$$N(r, \infty; f(qz+c)) = N(r, \infty; f(z)) + S(r, f).$$
$$N(r, 0; f(qz+c)) = N(r, 0; f(z)) + S(r, f).$$
$$\overline{N}(r, \infty; f(qz+c)) = \overline{N}(r, \infty; f(z)) + S(r, f).$$
$$\overline{N}(r, 0; f(qz+c)) = \overline{N}(r, 0; f(z)) + S(r, f).$$

on a set of logarithmic density 1.

Lemma 3.3. *(see [5]) If $N(r, 0; f^{(k)}|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to multiplicity then*

$$N(r, 0; f^{(k)}|f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f| < k) + k\overline{N}(r, 0; f| \geq k) + S(r, f).$$

Lemma 3.4. *(see [6]) Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{i=0}^{n} a_i f^i}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant co-efficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f).$$

where $d = max\{n, m\}$.

Lemma 3.5. *(see [7]) Let F and G be two non-constant meromorphic functions satisfying $E_F(1, m) = E_G(1, m)$, $0 \leq m < \infty$ with $H \not\equiv 0$, then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Similar inequality holds for G also.

Lemma 3.6. *(see [8]) Let $H \equiv 0$ and F, G share $(\infty, 0)$, then F, G share $(1, \infty)$, (∞, ∞).*

Lemma 3.7. *(see [9]) Suppose F and G share (1,0), $(\infty, 0)$. If $H \not\equiv 0$, then*

$$\begin{aligned}N(r,\infty;H) \leq{}& N(r,0;F|\geq 2) + N(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;F,G)\\ &+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G).\end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

Lemma 3.8. *(see [7]) If two non-constant meromorphic functions F, G share (1,2) then*

$$\overline{N}_0(r,0;G') + \overline{N}(r,1;G|\geq 2) + \overline{N}_*(r,1;F,G) \leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + S(r,G),$$

where $\overline{N}_0(r, 0; G')$ is the reduced counting function of those zeros of G' which are not the zeros of $G(G-1)$.

Lemma 3.9. *(see [10]) Let f and g be two non-constant meromorphic functions. Then*

$$N\Big(r,\infty;\frac{f}{g}\Big) - N\Big(r,\infty;\frac{g}{f}\Big) = N(r,\infty;f) + N(r,0;g) - N(r,\infty;g) - N(r,0;f).$$

Lemma 3.10. *Let f be a transcendental entire function of zero-order, and let $q \in \mathbb{C} - \{0\}$ and n, $s \in \mathbb{N}$. If $\phi(z) = f^n \mathcal{P}(f)\mathcal{L}(z, f)^s$, then*

$$(n + p + q + s)T(r, f) \leq T(r, \phi) - N(r, 0; \mathcal{L}(z, f)^s) + S(r, f).$$

Proof. Using first fundamental theorem of Nevanlinna and Lemmas 3.1 and 3.9, we have

$$\begin{aligned}(n+p+q+s)T(r,f) &= m(r, f^{n+p+s}P(f_1))\\ &= m\Big(r, \frac{\phi(z)f(z)^s}{\mathcal{L}(z,f)^s}\Big)\\ &\leq m(r,\phi(z)) + m\Big(r, \frac{f(z)^s}{\mathcal{L}(z,f)^s}\Big) + S(r,f)\\ &\leq m(r,\phi(z)) + N(r,0;f(z)^s) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f)\\ &\leq m(r,\phi(z)) + sT(r,f) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f).\end{aligned}$$

This implies that

$$(n+p+q+s)T(r,f) \le T(r,\phi(z)) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f).$$

□

Lemma 3.11. *Let f be a transcendental entire function of zero-order, and let $q \in \mathbb{C} - \{0\}$ and $n,\ s \in \mathbb{N}$. If $\phi(z) = \mathcal{P}(f)\mathcal{L}(z,f)^s$, then*

$$(p+q)T(r,f) \le T(r,\phi) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f).$$

Proof. Using first fundamental theorem of Nevanlinna and Lemmas 3.1 and 3.9, we have

$$\begin{aligned}(p+q+s)T(r,f) &= m(r,f^{p+s}P(f_1))\\ &\le m(r,\phi(z)) + T\Big(r,\frac{f(z)^s}{\mathcal{L}(z,f)^s}\Big) - N\Big(r,\infty;\frac{f(z)^s}{\mathcal{L}(z,f)^s}\Big) + S(r,f)\\ &\le m(r,\phi(z)) + sT(r,f) - N(r,0;L(z,f)^s) + S(r,f).\end{aligned}$$

This implies that

$$(p+q)T(r,f) \le T(r,\phi(z)) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f).$$

□

Lemma 3.12. *(see* [11]*) Let f be a non-constant meromorphic function and $p,\ k \in \mathbb{N}$. Then*

$$N_p(r,0;f^{(k)}) \le T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f).$$

$$N_p(r,0;f^{(k)}) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 3.13. *(see* [12]*) If $F,\ G$ be two non-constant meromorphic functions such that they share (1,1). Then*

$$2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)-\overline{N}_{F>2}(r,1;G) \le N(r,1;G)-\overline{N}(r,1;G).$$

Lemma 3.14. *(see* [13]*) If two non-constant meromorphic functions F, G share (1,1), then*

$$\overline{N}_{F>2}(r,1;G) \le \frac{1}{2}(\overline{N}(r,0;F) + \overline{N}(r,\infty;F) - N_0(r,0;F')) + S(r,F),$$

where $N_0(r,0;F')$ is the counting function of those zeros of F' which are not the zeros of $F(F-1)$.

Lemma 3.15. *(see* [13]*) Let F and G be two non-constant meromorphic functions sharing (1,0). Then*

$$\begin{aligned}\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) - \overline{N}_{F>1}(r,1;G)&\\ - \overline{N}_{G>1}(r,1;F) \le N(r,1;G) - \overline{N}(r,1;G).&\end{aligned}$$

Lemma 3.16. *(see [13]) If F and G share (1,0), then*

$$\overline{N}_L(r,1;F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + S(r,F).$$

$$\overline{N}_{F>1}(r,1;G) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) - N_0(r,0;F') + S(r,F).$$

Similar inequality holds for G also.

4. Main results

In the present research article, we are replacing $P(f)$ by $\mathcal{P}(f) = f_1^p P(f_1)$ and $\mathcal{L}(z,f) = b_1 f(qz+c) + b_0 f(z)$ by equation (3) and obtained the following results.

Theorem 4.1. *Let f be a transcendental meromorphic function (resp. entire) function of zero order and s ($\neq 0$), k be a positive integer. If $q > \Gamma_1 + km_2 + 2s + \chi_{b_0}(1+s) + p + 1$ (resp. $n > \Gamma_1 + p + km_2$) then $(\mathcal{P}(f)\mathcal{L}(z,f)^s)^{(k)} - \alpha(z)$ has infinitely many zeros where $\alpha(z) \in S(f) - \{0\}$.*

Proof. Suppose $F = F_1^{(k)}$ where $F_1 = \mathcal{P}(f)\mathcal{L}(z,f)^s$. Let us first suppose that f is a transcendental entire function of zero order. On the contrary, we assume that $F - \alpha(z)$ has finitely many zeros. In view of Lemmas 3.1, 3.11, 3.12 and by second fundamental theorem of Nevanlinna for small functions we get

$$\begin{aligned}(p+q)T(r,f) &\le T(r,\mathcal{P}(f)\mathcal{L}(z,f)^s) - N(r,0;\mathcal{L}(z,f)^s) + S(r,f)\\ &\le T(r,F) + N_{k+1}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)^s) - \overline{N}(r,0;F)\\ &- N(r,0;\mathcal{L}(z,f)^s) + S(r,f)\\ &\le (p+\Gamma_1+km_2)T(r,f) + S(r,f).\end{aligned}$$

which is not possible since $n \ge p + \Gamma_1 + km_2$. Suppose f is a transcendental meromorphic function of zero order. Now

$$\begin{aligned}(p+q+s)T(r,f) &= T(r,f^s\mathcal{P}(f))\\ &= T\left(r,\frac{F_1 f^s}{\mathcal{L}(z,f)^s}\right) + S(r,f)\\ &\le T(r,F_1) + 2sT(r,f) + S(r,f).\end{aligned}$$

i.e.,

$$\begin{aligned}(p+q-s)T(r,f) &\le T(r,F_1) + S(r,f)\\ &\le T(r,F) + N_{k+1}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)^s) - \overline{N}(r,0;F) + S(r,f)\\ &\le \overline{N}(r,\infty;\mathcal{P}(f)\mathcal{L}(z,f)^s) + N_{k+1}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)^s) + S(r,f)\\ &\le (1+p+\chi_{b_0})\overline{N}(r,\infty;f) + (\Gamma_1+km_2+p)\overline{N}(r,0;f)\\ &+ (1+\chi_{b_0})sT(r,f) + S(r,f).\end{aligned}$$

i.e.,

$$q \le (\Gamma_1 + km_2 + 2s + \chi_{b_0}(1+s) + p + 1)T(r,f) + S(r,f).$$

which is not possible since $q > \Gamma_1 + km_2 + 2s + \chi_{b_0}(1+s) + p + 1)T(r,f) + S(r,f)$. Hence the proof of the Theorem 4.1. □

Theorem 4.2. *Let f and g be two transcendental entire functions of zero order and n be a positive integer such that $n \geq p+q+5$. Let $f^n\mathcal{P}(f)\mathcal{L}(z,f)-p(z)$ and $g^n\mathcal{P}(g)\mathcal{L}(z,g)-p(z)$ share (0,2) where $p(z)$ be a non-zero polynomial such that $deg(p) < \frac{n-1}{2}$ and $g(z)$, $g(qz+c)$ share 0 CM. Then one of the following conclusion holds.m(i) $f \equiv tg$ where t is a constant satisfying $t^d = 1$, where $d = GCD(n+m+p+t, \ldots, n+p+t)$ and $a_{q-i} \neq 0$ for some $i = 0,\ 1, \ldots, m$. (ii) f and g satisfy algebraic difference equation $A(\omega_1,\omega_2) = 0$, where $A(\omega_1,\omega_2) = \omega_1^n\mathcal{P}(\omega_1)\mathcal{L}(z,\omega_1) = \omega_2^n\mathcal{P}(\omega_2)\mathcal{L}(z,\omega_2)$.*

Proof. Denote $F = \frac{f^n\mathcal{P}(f)\mathcal{L}(z,f)}{p(z)}$ and $G = \frac{g^n\mathcal{P}(g)\mathcal{L}(z,g)}{p(z)}$. From the given condition it follows that $F,\ G$ share (1,2) except for the zeros of $p(z)$.
Case 1. Let $H \not\equiv 0$. From (4), we obtain

$$\begin{aligned} N(r,\infty;H) \leq \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) \\ + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'). \end{aligned} \tag{5}$$

If z_0 be a simple zero of $F-1$ such that $p(z_0) \neq 0$, then z_0 is also a simple zero of $G-1$ and hence a zero of H. So

$$N(r,1;F| = 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,f) + S(r,g). \tag{6}$$

Using (5) and (6), we get

$$\begin{aligned} \overline{N}(r,1;F) &= N(r,1;F| = 1) + \overline{N}(r,1;F| \geq 2) \\ &\leq N(r,\infty;H) + \overline{N}(r,1;F| \geq 2) + S(r,f) + S(r,g) \\ &\leq \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) \\ &+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + \overline{N}(r,1;F| \geq 2) + S(r,f) + S(r,g). \end{aligned} \tag{7}$$

Now, by Lemma 3.3 we obtain

$$\begin{aligned} \overline{N}_0(r,0;G') + \overline{N}(r,1;F| \geq 2) + \overline{N}_*(r,1;F,G) &\leq N(r,0;G'|G \neq 0) \\ &\leq \overline{N}(r,0;G) + S(r,g). \end{aligned} \tag{8}$$

Since $g(z)$ and $g(qz+c)$ share 0 CM, we must have $N\Big(r,\infty;\frac{\mathcal{L}(z,g)}{g}\Big) = 0$. Hence using (7) and (8) and Lemmas 3.10, 3.12, we get from the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} (n+p+q)T(r,f) &\leq T(r,F) - N(r,0;\mathcal{L}(z,f)) + S(r,f) \\ &\leq N_2(r,0;F) + N_2(r,0;G) - N(r,0;\mathcal{L}(z,f)) + S(r,f) + S(r,g) \\ &\leq (p+q+2)T(r,f) + (p+q+2)T(r,g) + S(r,f) + S(r,g). \end{aligned}$$

i.e.,

$$nT(r,f) \leq 2T(r,f) + (p+q+2)T(r,g) + S(r,f) + S(r,g). \tag{9}$$

Since $N\Big(r,\infty;\frac{\mathcal{L}(z,g)}{g}\Big)=0$, Keeping in view of Lemmas 3.1 and 3.4 we get

$$\begin{aligned}(n+p+q+1)T(r,g)&=T(r,g^{n+1}\mathcal{P}(g))\\&\le m\Big(r,\frac{g^{n+1}\mathcal{P}(g)}{G}\Big)+m(r,G)\\&\le T(r,G)+O(log\ r).\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}(n+p+q+1)T(r,g)&\le T(r,G)+S(r,g)\\&\le \overline{N}(r,0;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;G')+S(r,g)\\&\le N_2(r,0;G)+N_2(r,1;G)+S(r,f)+S(r,g)\\&\le (p+q+3)T(r,f)+(p+q+2)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

i.e.,

$$nT(r,g)\le (p+q+3)T(r,f)+T(r,g)+S(r,f)+S(r,g). \tag{10}$$

Combining (9) and (10) we obtain

$$(n-p-q-5)T(r,f)+(n-p-q-3)T(r,g)\le S(r,f)+S(r,g),$$

which contradicts to the fact that $n\ge p+q+5$.

Case 2. Suppose $H\equiv 0$. Then by integration we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B. \tag{11}$$

where $A,\ B$ are constant with $A\neq 0$. From (11) it can be easily seen that $F,\ G$ share $(1,\infty)$. We now consider following three subcases.

Subcase 2.1. Let $B\neq 0$ and $A\neq B$. If $B=-1$, then from (11) we have $F=\frac{-A}{G-A-1}$. Therefore $\overline{N}(r,A+1;G)=\overline{N}(r,\infty;F)=N(r,0;p)=S(r,g)$. So in view of Lemma 3.10 and second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}(n+p+q)T(r,g)&\le T(r,g^n\mathcal{P}(g)\mathcal{L}(z,g))-N(r,0;\mathcal{L}(z,g))+S(r,g)\\&\le T(r,G)-N(r,0;\mathcal{L}(z,g))+S(r,g)\\&\le \overline{N}(r,0;G)+\overline{N}(r,A+1;G)-N(r,0;\mathcal{L}(z,g))+S(r,g)\\&\le (p+q+1)T(r,g)+S(r,g).\end{aligned}$$

which is a contradiction since $n\ge p+q+5$. If $B\neq -1$, then from (11) we get $F-\Big(1+\frac{1}{B}\Big)=\frac{-A}{B^2\Big(G+\frac{A-B}{B}\Big)}$. Therefore $\overline{N}\Big(r,\frac{B-A}{B};G\Big)=N(r,0;P(g))=O(log\ r)=S(r,g)$. Using Lemmas 3.12, 3.10 and the same argument as used in the case $B=-1$ we get a contradiction.

Subcase 2.2. Let $B\neq 0$ and $A=B$. If $B=-1$, then from (11) we have

$$f^n\mathcal{P}(f)\mathcal{L}(z,f)g^n\mathcal{P}(g)\mathcal{L}(z,g)\equiv p^2(z). \tag{12}$$

Keeping in view of (12) and $deg(p)<\frac{n-1}{2}$, we can say that f and g have zeros. Since f and g are of zero orders, f and g both must be contants which contradicts to our assumption. Therefore (12) is not possible. If $B\neq -1$ from (11) we have

$\frac{1}{F} = \frac{AG}{(1+A)G-1}$. Hence $\overline{N}\Big(r, \frac{1}{1+A}; G\Big) = \overline{N}(r, 0; F) + S(r, f)$. So in the view of Lemmas 3.1 and 3.10 and second fundamental theorem of Nevanlinna we get

$$\begin{aligned}(n+p+q)T(r,g) &\leq T(r, g^n\mathcal{P}(g)\mathcal{L}(z,g)) - N(r,0;\mathcal{L}(z,g)) + S(r,g)\\ &\leq \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{1}{A+1}; G\Big) - N(r,0;\mathcal{L}(z,g)) + S(r,g)\\ &\leq \overline{N}(r,0;G) + \overline{N}(r,0;\mathcal{P}(g)) + \overline{N}(r,0;F) + S(r,g)\\ &\leq (p+q+1)T(r,g) + (p+q+2+\chi_{b_0})T(r,f) + S(r,g)\end{aligned}$$

Therefore

$$nT(r,g) \leq (p+q+3+\chi_{b_0})T(r,g) + S(r,g)$$

which is a contradiction since $n \geq p+q+5$.

Subcase 2.3. Let $B = 0$, then from (11) we get

$$F = \frac{G+A-1}{A}. \tag{13}$$

If $A \neq 1$, we obtain $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Therefore, we can similarly get a contradiction as in Subcase 2.2. Hence $A = 1$ and from (13) we get $F \equiv G$, that is

$$f^n\mathcal{P}(f)\mathcal{L}(z,f) = g^n\mathcal{P}(g)\mathcal{L}(z,g). \tag{14}$$

Let $h = \frac{f}{g}$, then

$$\begin{aligned}a_m g^{n+m+p}\sum_{i=1}^{t} b_i g_i(qz+c_i)\left[h^{n+m+p}\sum_{i=1}^{t} h_i(qz+c_i) - 1\right] + \ldots\\ +a_0 g^{n+p}\sum_{i=1}^{t} b_i g_i(qz+c_i)\left[h^{n+p}\sum_{i=1}^{t} h_i(qz+c_i) - 1\right] = 0.\end{aligned}$$

Since g is a non-constant we must have $t^d = 1$, where $d = GCD(n+m+p+t+\ldots+n+p+t)$ and $a_{m-i} \neq 0$ for some $i = 0,\ 1, \ldots, m$. Hence $f = tg$ for a constant t such that $t^d = 1$, where d is mentioned above. If t is not constant then $f,\ g$ satisfy algebraic difference equation $A(\omega_1, \omega_2) = 0$, where

$$A(\omega_1,\omega_2) = \omega_1^n\mathcal{P}(\omega_1)\mathcal{L}(z,\omega_1) - \omega_2^n\mathcal{P}(\omega_2)\mathcal{L}(z,\omega_2).$$

□

Theorem 4.3. *Let f and g be any two transcendental entire functions of zero order. If $E_l(1; (\mathcal{P}(f)\mathcal{L}(z,f))^{(k)}) = E_l(1; (\mathcal{P}(g)\mathcal{L}(z,g))^{(k)})$ and $l,\ m,\ n$ are three integers satisfies one of the following conditions.*
(i) $l \geq 2$; $p+q > 2\Gamma_0 + 2km_2 + 3$.
(ii) $l = 1$; $p+q > \Gamma_0 + \frac{\Gamma_1}{2} + \frac{3}{2}km_2 + 3$.
(iii) $l = 0$; $p+q > 2\Gamma_0 + 3\Gamma_1 + 5km_2 + 9$ then one of the result holds. (i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(p+q+1, \ldots, p+q+1-i, \ldots, p+1)$.
(ii) f and g satisfy the algebraic equation $R(\omega_1,\omega_2) = 0$, where $R(\omega_1,\omega_2) = \mathcal{P}(\omega_1)\mathcal{L}(z,\omega_1) - \mathcal{P}(\omega_2)\mathcal{L}(z,\omega_2)$.

Proof. Let $F(z) = (\mathcal{P}(f)\mathcal{L}(z,f))^{(k)}$ and $G(z) = (\mathcal{P}(g)\mathcal{L}(z,g))^{(k)}$. If follows that F and G share $(1,l)$.
Case 1. Suppose $H \not\equiv 0$.
(i) Let $l \geq 2$. Using Lemma 3.5, 3.7 and 3.8 we get

$$\begin{aligned} \overline{N}(r,1;F) &= N(r,1;F|=1) + \overline{N}(r,1;F|\geq 2) \\ &\leq \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{15}$$

Hence, using (15), Lemmas 3.1, 3.11, 3.12 and from second fundamental theorem of Nevanlinna we get,

$$\begin{aligned} (p+q)T(r,f) &\leq T(r,\mathcal{P}(f)\mathcal{L}(z,f)) - N(r,0;\mathcal{L}(z,f)) + S(r,f) \\ &\leq N_2(r,0;G) + N_{k+2}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)) - N(r,0;\mathcal{L}(z,f)) \\ &+ S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)) + N_{k+2}(r,0;\mathcal{P}(g)\mathcal{L}(z,g)) \\ &- N(r,0;\mathcal{L}(z,f)) + S(r,f) + S(r,g) \\ &\leq (1+m_1+2m_2+km_2)\{T(r,f)+T(r,g)\} + T(r,g) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{16}$$

Similarly,

$$(p+q)T(r,f) \leq (1+m_1+2m_2+km_2)\{T(r,f)+T(r,g)\}+T(r,f)+S(r,f)+S(r,g). \tag{17}$$

Combining (16) and (17) we get

$$(p+q)\{T(r,f)+T(r,g)\} \leq (2\Gamma_0+2km_2+3)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

which is a contradiction as $p+q > 2\Gamma_0 + 2km_2 + 3$.
(ii) Let $l = 1$, using Lemmas 3.3, 3.5, 3.7, 3.13 and 3.14 we get

$$\begin{aligned} \overline{N}(r,1;F) &\leq N(r,1;F|=1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \\ &\leq \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\overline{N}(r,0;F) + N_2(r,0;G) + \overline{N}_0(r,0;F') \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{18}$$

Hence using (18), Lemmas 3.1, 3.11, 3.12 and second fundamental theorem of Nevanlinna we get

$$\begin{aligned}(p+q)T(r,f) &\leq T(r,\mathcal{P}(f)\mathcal{L}(z,f)) - N(r,0;\mathcal{L}(z,f)) + S(r,f)\\ &\leq N_{k+2}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)) + \frac{1}{2}\overline{N}(r,0;F) + N_2(r,0;G)\\ &- N(r,0;\mathcal{L}(z,f)) + S(r,f) + S(r,g)\\ &\leq (1+m_1+2m_2+km_2)\{T(r,f)+T(r,g)\}\\ &+ \frac{1}{2}(1+m_1+m_2+km_2)T(r,f) + T(r,g) + \frac{1}{2}T(r,f)\\ &+ S(r,f) + S(r,g).\end{aligned} \tag{19}$$

In a similar manner, we get

$$\begin{aligned}(p+q)T(r,f) &\leq (1+m_1+2m_2+km_2)\{T(r,f)+T(r,g)\}\\ &+ \frac{1}{2}(1+m_1+m_2+km_2)T(r,g)\\ &+ T(r,f) + \frac{1}{2}T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{20}$$

Combining (19) and (20) we get

$$(p+q)\{T(r,f)+T(r,g)\} \leq \left(\Gamma_0 + \frac{\Gamma_1}{2} + \frac{3}{2}km_2 + 3\right) + S(r,f) + S(r,g).$$

which is a contradiction as $p+q > \Gamma_0 + \frac{\Gamma_1}{2} + \frac{3}{2}km_2 + 3$.
(iii) Let $l = 0$. Using Lemmas 3.3, 3.5, 3.7, 3.15, 3.16 we get

$$\begin{aligned}\overline{N}(r,1;F) &\leq N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\ &\leq N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;G) + N_0(r,0;F')\\ &+ S(r,f) + S(r,g).\end{aligned} \tag{21}$$

Hence using (21), Lemmas 3.1, 3.11, 3.12 and second fundamental theorem of Nevanlinna we obtain

$$\begin{aligned}(p+q)T(r,f) &\leq T(r,\mathcal{P}(f)\mathcal{L}(z,f)) - N(r,0;\mathcal{L}(z,f)) + S(r,f)\\ &\leq N_{k+2}(r,0;\mathcal{P}(f)\mathcal{L}(z,f)) + 2\overline{N}(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;G)\\ &- N(r,0;\mathcal{L}(z,f)) + S(r,f) + S(r,g)\\ &\leq (1+m_1+2m_2+km_2+2)\{T(r,f)+T(r,g)\}\\ &+ 2(1+m_1+m_2+km_2)T(r,f)\\ &+ (1+m_1+m_2+km_2)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{22}$$

Similarly, we get

$$\begin{aligned}(p+q)T(r,g) &\le 2(1+m_1+2m_2+km_2+2)\{T(r,f)+T(r,g)\}\\ &\quad + 2(1+m_1+m_2+km_2)T(r,g)\\ &\quad + (1+m_1+m_2+km_2)T(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{23}$$

Combining (22) and (23) we obtain

$$\begin{aligned}&(p+q)\{T(r,f)+T(r,g)\}\\ &\le (2\Gamma_0+3\Gamma_1+5km_2+9)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

which is a contradiction as $p+q>2\Gamma_0+3\Gamma_1+5km_2+9$.
Case 2. Let $H\equiv 0$. By integration we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B. \tag{24}$$

where $A,\ B$ are constants with $A\neq 0$. From (24) it can be easily seen that $F,\ G$ share $(1,\infty)$. We now consider the following subcases.
Subcase 2.1. Let $B\neq 0$ and $A\neq B$. If $B=-1$, then from (24), we have $F=\frac{-A}{G-A-1}$. Therefore $\overline{N}(r,A+1;G)=\overline{N}(r,\infty;F)=S(r,f)$. Therefore, using Lemma 3.11 and second fundamental theorem of Nevanlinna we get,

$$\begin{aligned}(p+q)T(r,g) &\le T(r,\mathcal{P}(g)\mathcal{L}(z,g))-N(r,0;\mathcal{L}(z,g))+S(r,g)\\ &\le T(r,G)+N_{k+2}(r,0;\mathcal{P}(g)\mathcal{L}(z,g))-N_2(r,0;G)\\ &\quad - N(r,0;\mathcal{L}(z,g))+S(r,g)\\ &\le \overline{N}(r,0;G)+\overline{N}(r,A+1;G)+N_{k+2}(r,0;\mathcal{P}(g))-N_2(r,0;G)\\ &\quad - N(r,0;\mathcal{L}(z,g))+S(r,g)\\ &\le (2\Gamma_0+2km_2+2-m_2)T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction since $p+q>2\Gamma_0+2km_2+2$. If $B\neq -1$, then from (24), we have $F=\frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and therefore, $\overline{N}\Big(r,\frac{A-B}{B};G\Big)=\overline{N}(r,\infty;F)=S(r,f)$. Therefore, in a similar manner as done in the case $B=-1$, we arrive at a contradiction.
Subcase 2.2. Let $B\neq 0$ and $A=B$. If $B\neq -1$ then from (3.20), we have $\frac{1}{F}=\frac{BG}{(B+1)G-1}$ and therefore $\overline{N}(r,0;G)=\overline{N}(r,\infty;F)=S(r,f)$ and $\overline{N}\Big(r,\frac{1}{B+1};G\Big)=\overline{N}(r,0;F)$. Therefore using Lemma 3.11 and second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}(p+q)T(r,g) &\le T(r,\mathcal{P}(g)\mathcal{L}(z,g))-N(r,0;\mathcal{L}(z,g))+S(r,g)\\ &\le N_{k+1}(r,0;\mathcal{P}(f)\mathcal{L}(z,f))+N_{k+2}(r,0;\mathcal{P}(g))+N(r,0;\mathcal{L}(z,g))\\ &\quad - N(r,0;\mathcal{L}(z,g))+S(r,g)\\ &\le (m_1+2m_2+km_2+1)T(r,g)+(m_1+m_2+km_2+2)T(r,f)\\ &\quad + S(r,f)+S(r,g)\end{aligned}$$

Similarly,

$$\begin{aligned}&(p+q)T(r,f)\\&\leq (m_1+2m_2+km_2+1)T(r,f)+(m_1+m_2+km_2+2)T(r,g)\\&+S(r,f)+S(r,g).\end{aligned}$$

Combining above two inequalities we get

$$(p+q)\{T(r,f)+T(r,g)\} \leq (2\Gamma_0+2km_2+2-m_2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

which is a contradiction since $p+q > 2\Gamma_0+2km_2+2$. If $B=-1$, then (24) reduces to $FG \equiv 1$. This implies

$$(\mathcal{P}(f)\mathcal{L}(z,f))^{(k)}(\mathcal{P}(g)\mathcal{L}(z,g))^{(k)} \equiv 1. \tag{25}$$

Suppose $P(z)=0$ has t roots $\alpha_1,\ \alpha_2,\dots\alpha_t$ with multiplicities $u_1,\ u_2,\dots,u_t$. Then we must have $u_1+u_2+\ldots+u_t=m$. Therefore (4) can be rewritten as

$$(a_m f_1^p(f-\alpha_1)^{u_1}\ldots(f-\alpha_t)^{u_t}\mathcal{L}(z,f)^{(k)})(a_m g_1^p(g-\alpha_1)^{u_1}\ldots(g-\alpha_t)^{u_t}\mathcal{L}(z,g)^{(k)}) \equiv 1. \tag{26}$$

Since f and g are entire functions, from (26), we can say that $\alpha_1,\ \alpha_2,\dots\alpha_t$ are Picard exceptional values of f and g. Since by Picard's theorem, an entire function can have atmost one finite exceptional value, all $\alpha_j's$ are equal for $1 \leq j \leq t$. Let $P(z)=a_m(z-\alpha)^m$. Therefore (26) reduces to

$$(a_m f_1^p(f-\alpha)^m\mathcal{L}(z,f)^{(k)})(a_m g_1^p(g-\alpha)^m\mathcal{L}(z,g)^{(k)}) \equiv 1. \tag{27}$$

Equation (27) shows that α is an exceptional value of f and g. Since f is an entire function of zero order having an exceptional value α, f must be constant, which is not possible since f is assumed to be transcendental and therefore non-constant.

Subcase 2.3. Let $B=0$. Then (24) reduces to $F=\frac{G+A-1}{A}$. If $A \neq 1$, then $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$. Proceeding in a similar manner as done in Subcase 2.2. we get a contradiction. Hence $A=1$. Therefore $F \equiv G$. This implies that

$$(\mathcal{P}(f)\mathcal{L}(z,f))^{(k)} \equiv (\mathcal{P}(g)\mathcal{L}(z,g))^{(k)} \tag{28}$$

Integrating (28) k times, we get

$$\mathcal{P}(f)\mathcal{L}(z,f)=\mathcal{P}(g)\mathcal{L}(z,g)+p_1(z), \tag{29}$$

where $p_1(z)$ is a polynomial in z of degree $k-1$. Suppose $p_1(z) \not\equiv 0$. Then (29) can be written as

$$\frac{\mathcal{P}(f)\mathcal{L}(z,f)}{p_1(z)}=\frac{\mathcal{P}(g)\mathcal{L}(z,g)}{p_1(z)}+1. \tag{30}$$

Now in Lemmas 3.1, 3.11 and second fundamental theorem, we have

$$\begin{aligned}(p+q)T(r,f) &\le T(r,\mathcal{P}(f)\mathcal{L}(z,f)) - N(r,0;\mathcal{L}(z,f)) + S(r,f)\\ &\le T\left(r,\frac{\mathcal{P}(f)\mathcal{L}(z,f)}{p_1(z)}\right) - N(r,0;\mathcal{L}(z,f)) + S(r,f)\\ &\le \overline{N}\left(r,0;\frac{\mathcal{P}(f)\mathcal{L}(z,f)}{p_1(z)}\right) + \overline{N}\left(r,\infty;\frac{\mathcal{P}(f)\mathcal{L}(z,f)}{p_1(z)}\right)\\ &+ \overline{N}\left(r,1;\frac{\mathcal{P}(f)\mathcal{L}(z,f)}{p_1(z)}\right) - N(r,0;\mathcal{L}(z,f)) + S(r,f)\\ &\le (1+m_1+m_2)\{T(r,f)+T(r,g)\} + T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Similarly we obtain

$$(p+q)T(r,g) \le (1+m_1+m_2)\{T(r,f)+T(r,g)\} + T(r,f) + S(r,f) + S(r,g).$$

Combining above two inequalities we get

$$(p+q)\{T(r,f)+T(r,g)\} \le (2\Gamma_1+3)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).$$

which is a contradiction since $p+q > 2\Gamma_0 + 2km_2 + 2$. Hence $p_1(z) \equiv 0$ and (29) we have

$$\mathcal{P}(f)\mathcal{L}(z,f) \equiv \mathcal{P}(g)\mathcal{L}(z,g). \tag{31}$$

Set $h = \frac{f}{g}$. If h is non-constant from (3.27), we can get that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \mathcal{P}(\omega_1)L(z,\omega_1) - \mathcal{P}(\omega_2)L(z,\omega_2)$. If h is a constant, substituting $f = gh$ into (30) we get

$$[a_q g_1^{p+q}(h^{p+q+1}-1) + \ldots + a_0 g_1^p(h^{p+1}-1)]\mathcal{L}(z,g) = 0.$$

Then in a similar argument as done in Case 2 in the proof of Theorem 1.1 in [4], we obtain $f \equiv tg$ for a constant t such that $t^d = 1$ where $d = GCD(p+q+1,\ldots,p+q+1-i,\ldots,p+1)$ and $a_{q-i} \ne 0$ for some $i = 0,\ 1,\ldots,q$. □

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