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ON LOCAL SPECTRAL PROPERTIES OF RIESZ OPERATORS

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ABSTRACT. In this paper we show that if $T \in L(X)$ and $S \in L(X)$ is a Riesz operator commuting with T and $X_S(F) \in Lat(S)$, where $F = \{0\}$ or $F \subseteq \mathbb{C} \setminus \{0\}$ is closed then $T|X_S(F)$ and $T|X_T(F) + S|X_S(F)$ share the local spectral properties such as SVEP, Dunford's property (C), Bishop's property (β), decomopsition property (δ) and decomposability. As a corollary, if $T \in L(X)$ and $Q \in L(X)$ is a quasinilpotent operator commuting with T then T is Riesz if and only if $T + Q$ is Riesz. We also study some spectral properties of Riesz operators acting on Banach spaces. We show that if $T, S \in L(X)$ such that $TS = ST$, and $Y \in Lat(S)$ is a hyperinvarinat subspace of X for which $\sigma(S|Y) = \{0\}$ then $\sigma_*(T|Y + S|Y) = \sigma_*(T|Y)$ for $\sigma_* \in \{\sigma, \sigma_{loc}, \sigma_{sur}, \sigma_{ap}\}$. Finally, we show that if $T \in L(X)$ and $S \in L(Y)$ on the Banach spaces X and Y and T is similar to S then T is Riesz if and only if S is Riesz.

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1. Introduction

Throughout this paper, let $L(X, Y)$ denote the set of all bounded linear operators from Banach space X to Banach space Y, and $L(X) := L(X, X)$. As usual, given $T \in L(X)$, let $ker(T)$ and $T(X)$ stand for the kernel and range of T, the spectrum of T is denoted by $\sigma(T)$ and the spectral radius of T is denoted by $r(T)$. For an operator $T \in L(X)$, we denote by $Lat(T)$ the lattice of all closed $T-$invariant subspaces of X and $M \in Lat(T)$, let $T|M \in L(M)$ be the restriction of T to M. We say that a linear subspace M of X is said to be $T-$*hyperinavriant* if $SM \subseteq M$ for every bounded linear operator $S \in L(X)$ that commutes with T.

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Definition 1.1. An operator $T \in L(X)$ on a complex Banach space X is a *Riesz operator* if for each $\lambda \in \mathbb{C} \setminus \{0\}$, the spaces $ker(T - \lambda I)$ and $X/(T - \lambda I)(X)$ are both of finite dimension.

It is well known that $T \in L(X)$ is a Riesz operator if and only if $T - \lambda I$ is a Fredholm operator for every $\lambda \in \mathbb{C} \setminus \{0\}$, i.e. dim $\ker(T - \lambda I) < \infty$ and $\mathrm{codim}(T - \lambda I)(X) < \infty$. The spectrum $\sigma(T)$ of a Riesz operator is at most countable and has no non-zero cluster point. Furthermore, each non-zero element of the spectrum is an eigenvalue. Moreover, the spectral subspaces associated with non-zero elements of the spectrum are finite dimensional. The classical Riesz-Schauder theory of compact operators establishes that every compact operator is Riesz. Examples of Riesz operators are quasinilpotent operators and compact operators, see [21].

In this note we show that if $T \in L(X)$ and $S \in L(X)$ is a Riesz operator commuting with T and $X_S(F) \in Lat(S)$, where $F = \{0\}$ or $F \subseteq \mathbb{C} \setminus \{0\}$ is closed then $T|X_S(F)$ and $T|X_T(F) + S|X_S(F)$ share the local spectral properties such as SVEP, Dunford's property (C), Bishop's property (β), decomopsition property (δ) and decomposability. We also study spectral properties of Riesz operators.

The following localized version of single valued extension property was introduced by Finch [17]. The single valued extension property has now developed into one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces, see more details [1], [2], [23], [27], [28].

Definition 1.2. An operator $T \in L(X)$ is said to have the *single valued extension property* at a point $\lambda \in \mathbb{C}$ (for brevity, SVEP at λ) provided that, for every open disc $U \subseteq \mathbb{C}$ centered at λ, the only analytic function $f : U \to X$ that satisfies the equation

$$(\mu I - T)f(\mu) = 0 \quad \text{for all } \mu \in U$$

is the constant function $f \equiv 0$ on U. Moreover, T is said to have SVEP if an operator $T \in L(X)$ has SVEP at every point $\lambda \in \mathbb{C}$.

It is clear that $T \in L(X)$ has SVEP at every point of the resolvent set $\rho(T)$. Moreover, from the identity theorem for analytic function it is easily seen that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, $T \in L(X)$ has SVEP at every isolated point of $\sigma(T)$.

For $T \in L(X)$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ and an analytic function $f : U \to X$ such that

$$(\mu I - T)f(\mu) = x \quad \text{for all } \mu \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. The local analytic solutions occuring in the definition of the local resolvent set will be unique for all $x \in X$ if and only if T has SVEP. It is clear that $\sigma_T(x)$ is a closed subset of $\sigma(T)$ and it may be empty. For every subset F of $\mathbb{C}$, we define the *local spectral subspace* of T associated with F is the set

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

It is clear from the definition that $X_T(F)$ is a hyperinvariant subspace of X, but not always closed. An operator $T \in L(X)$ is said to have *Dunford's property (C)* (for brevity, *property* (C)) if the local spectral subspace $X_T(F)$ is closed for every closed subset F of $\mathbb{C}$.

For every closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ that there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

Clearly, $\mathcal{X}_T(F)$ is a hyperinvariant subspace of X and $\mathcal{X}_T(F) \subseteq X_T(F)$. Moreover, $\mathcal{X}_T(F) = X_T(F)$ holds for all closed subsets F of $\mathbb{C}$ precisely when T has SVEP, see Proposition 3.3.2 of [23]. Recall that an operator $T \in L(X)$ is said to have the *decomposition property* (δ) (for brevity, *property* (δ)) if, $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$.

Let $O(U, X)$ denote the Fréch͏t algebra of all $X-$valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $T \in L(X)$ is said to have *Bishop's property* (β) (for brevity, *property* (β)) if for every open subset U of $\mathbb{C}$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subseteq O(U, X)$, $\lim_{n\to\infty}(\mu I - T)f_n(\mu) = 0$ in $O(U, X)$ implies $\lim_{n\to\infty} f_n(\mu) = 0$ in $O(U, X)$. Note that the property (β) implies that T has SVEP, while the property (δ) implies SVEP for T^*, see [1], [7], [8]. We say that an operator $T \in L(X)$ is said to be *decomposable* if, for every open cover $\{U, V\}$ of $\mathbb{C}$, there exist $Y, Z \in Lat(T)$ for which

$$X = Y + Z, \quad \sigma(T|Y) \subseteq U \text{ and } \sigma(T|Z) \subseteq V.$$

Examples of decomposable operators are normal operators, generalized scalar operators and spectral operators. Also, operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable, see [1], [19], [23]. It is clear that every decomposable operator has property (δ). It is well known that that $T \in L(X)$ has property (β) if and only if its adjoint $T^* \in L(X^*)$ on the topological dual space X^* has property (δ), and the same equivalence holds when the roles of (β) and (δ) are interchanged. It is well known that T is decomposable if and only if T satisfies both properties (β) and (δ), see [8] and [23].

Lemma 1.3. *Let $T \in L(X)$ and $\lambda \in \mathbb{C}$, and let $S = T + \lambda I$. Then the following assertions hold:*
(a) T has SVEP if and only if S has SVEP.

(b) T has property (C) if and only if S has property (C).
(c) T has property (β) if and only if S has property (β).
(d) T has property (δ) if and only if S has property (δ).
(e) T is decomposable if and only if S is decomposable.

Proof. (a) Suppose that T has SVEP. Let $\mu_0 \in \mathbb{C}$ be arbitrary and let U be an open neighborhood of μ_0. Assume that $f : U \to X$ is an analytic function such that $(\mu I - S)f(\mu) = 0$ for all $\mu \in U$. Then $U + \lambda$ is a neighborhood of $\mu_0 + \lambda$, where $U + \lambda = \{\mu + \lambda : \mu \in U\}$. We define $g : U + \lambda \to X$ by

$$g(\zeta) := f(\zeta - \lambda) \text{ for all } \zeta \in U + \lambda.$$

Then clearly g is analytic and $(\zeta I - T)g(\zeta) = 0$ for all $\zeta \in U + \lambda$. Since T has SVEP, we have $g \equiv 0$ on $U + \lambda$ and hence $f \equiv 0$ on U, so that S has SVEP. Conversely, suppose that S has SVEP. Let $\xi_0 \in \mathbb{C}$ and let V be an open neighborhood of ξ_0. Assume that $h : V \to X$ is an analytic function such that $(\mu I - T)h(\mu) = 0$ for all $\mu \in V$. Then clearly $h(\omega - \lambda)$ is ananlytic and

$$(\omega I - S)h(\omega - \lambda) = 0 \text{ for all } \omega \in V + \lambda.$$

Since S has SVEP, we have $h \equiv 0$ on V and hence T has SVEP.

(b) We first prove that $X_S(F) = X_T(F-\lambda)$ for every subset F of $\mathbb{C}$. It suffices to show that $\sigma_S(x) \subseteq F$ if and only if $\sigma_T(x) \subseteq F - \lambda$. Suppose that $\sigma_S(x) \subseteq F$. If $\mu \notin F - \lambda$ then $\mu + \lambda \notin F$ and hence $\mu + \lambda \in \rho_S(x)$. Thus there exist a neighborhood U of $\mu + \lambda$ and an analytic function $f : U \to X$ satisfying

$$(\omega I - S)f(\omega) = x \text{ for all } \omega \in U.$$

We define $g : U - \lambda \to X$ by

$$g(\zeta) := f(\zeta + \lambda) \text{ for all } \zeta \in U - \lambda.$$

Then clearly g is analytic satisfying $(\zeta I - T)g(\zeta) = x$ for all $\zeta \in U - \lambda$, so that $\mu \in \rho_T(x)$. We conclude that $\sigma_T(x) \subseteq F - \lambda$. Conversely, suppose that $\sigma_T(x) \subseteq F - \lambda$. If $\mu \notin F$ then $\mu - \lambda \in \rho_T(x)$. Thus there exist a neighborhood W of $\mu - \lambda$ and an analytic function $h : W \to X$ satisfying $(\omega I - T)h(\omega) = x$ for all $\omega \in W$. Then $W + \lambda$ is a neighborhood of μ. We define $k : W + \lambda \to X$ by

$$k(\zeta) := h(\zeta - \lambda) \text{ for all } \zeta \in W + \lambda.$$

Then k is analytic such that $(\zeta I - S)k(\zeta) = x$ for all $\zeta \in W + \lambda$, so that $\mu \in \rho_S(x)$ and hence $\sigma_S(x) \subseteq F$. We conclude that $X_S(F) = X_T(F - \lambda)$ for all $F \subseteq \mathbb{C}$.

(c) Suppose that T has property (β). Let U be an open subset of $\mathbb{C}$, and let $\{f_n\}_{n=1}^{\infty} \subseteq O(U, X)$ such that $\lim_{n\to\infty}(\mu I - S)f_n(\mu) = 0$ in $O(U, X)$. We define $g_n : U - \lambda \to X$ by $g_n(\zeta) := f_n(\zeta + \lambda)$ for all $\zeta \in U - \lambda$. Then clearly $\{g_n\} \subseteq O(U - \lambda, X)$ and

$$\lim_{n\to\infty} (\zeta I - T)g_n(\zeta) = 0 \text{ in } O(U - \lambda, X).$$

Since T has property (β), we have $\lim_{n\to\infty} g_n(\zeta) = 0$ in $O(U - \lambda, X)$, and hence $\lim_{n\to\infty} f_n(\mu) = 0$ in $O(U, X)$. This shows that S has property (β). The reverse implication is similar.

(d) It is well known that (β) and (δ) are complete dual. This assertion follows from (c) by duality.

(e) It is well known that an operator is decomposable if and only if it has both properties (β) and (δ). This assertion follows from (c) and (d). $\square$

It is clear that if $T \in L(X)$ and $S \in L(Y)$ then $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$, where $X \oplus Y = \{x \oplus y : x \in X, y \in Y\}$ and $\|x \oplus y\| = (\|x\|^2 + \|y\|^2)^{1/2}$.

Lemma 1.4. *Let $T \in L(X)$ and $S \in L(Y)$ on the Banach spaces X and Y. Then $T \oplus S \in L(X \oplus Y)$ has SVEP if and only if both T and S have SVEP. Moreover, $\sigma_{T\oplus S}(x \oplus y) = \sigma_T(x) \cup \sigma_S(y)$ for all $x \oplus y \in X \oplus Y$.*

Proof. Suppose that $T \oplus S$ has SVEP. Let $\mu \in \mathbb{C}$ and let U be an arbitrary open neighborhood of μ. Assume that $f : U \to X$ is an analytic function such that $(\mu I - T)f(\lambda) = 0$ for all $\lambda \in U$, and $g : U \to Y$ is an analytic function such that $(\lambda I - S)g(\lambda) = 0$ for all $\lambda \in U$. Then we have

$$(\lambda I - (T \oplus S))(f(\lambda) \oplus g(\lambda)) = (\lambda I - T)f(\lambda) + (\lambda I - S)g(\lambda) = 0$$

on U. It follows from the SVEP of $T \oplus S$ that

$$f \oplus g \equiv 0 \quad \text{on } U.$$

Thus $f \equiv 0$ and $g \equiv 0$ on U, and hence both T and S have the SVEP. Conversely, suppose that T and S have the SVEP. Let $\mu \in \mathbb{C}$ and let V be an arbitrary open neighborhood of μ. If $h = f \oplus g : V \to X \oplus Y$ is an analytic function such that

$$(\lambda I - (T \oplus S))h(\lambda) = 0 \quad \text{on } V.$$

Then clearly, $(\lambda I - T)f(\lambda) = 0$ and $(\lambda I - S)g(\lambda) = 0$ on V. By the SVEP of T and S, we have $f \equiv 0$ and $g \equiv 0$ on V. Hence $T \oplus S$ has SVEP.

Finally, we show that $\sigma_{T\oplus S}(x \oplus y) = \sigma_T(x) \cup \sigma_S(y)$ for all $x \oplus y \in X \oplus Y$. Let $\lambda \notin \sigma_{T\oplus S}(x \oplus y)$. Then there exist a neighborhood N of λ and an analytic function $k = f \oplus g : N \to X \oplus Y$ such that

$$(\lambda I - T)f(\lambda) \oplus (\lambda I - S)g(\lambda) = (\lambda I - (T \oplus S))k(\lambda) = x \oplus y$$

for all $\lambda \in N$. Thus $(\lambda I - T)f(\lambda) = x$ and $(\lambda I - S)g(\lambda) = y$ for all $\lambda \in N$, and hence $\lambda \in \rho_T(x) \cap \rho_S(y)$. It follows that $\sigma_T(x) \cup \sigma_S(y) \subseteq \sigma_{T\oplus S}(x \oplus y)$.

On the other hand, if $\lambda \in \rho_T(x) \cap \rho_S(y)$ then there exist a neighborhood W of λ and an analytic function $f : W \to X$ such that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in W$, and an analytic function $g : W \to Y$ such that $(\lambda I - S)g(\lambda) = y$ for all $\lambda \in W$. We define $f \oplus g : W \to X \oplus Y$ by

$$(f \oplus g)(\lambda) = f(\lambda) \oplus g(\lambda)$$

for all $\lambda \in W$. Then clearly, $f \oplus g$ is analytic and

$$(\lambda I - (T \oplus S))(f \oplus g)(\lambda) = (\lambda I - T)f(\lambda) \oplus (\lambda I - S)g(\lambda) = x \oplus y$$

for all $\lambda \in W$ and hence $\lambda \in \rho_{T\oplus S}(x \oplus y)$. It follows that

$$\sigma_{T\oplus S}(x \oplus y) \subseteq \sigma_T(x) \cup \sigma_S(y).$$

□

Lemma 1.5. *Let $T \in L(X)$ and $S \in L(Y)$ on the Banach spaces X and Y. Then $T \oplus S \in L(X \oplus Y)$ has property (C) if and only if both T and S have properety (C).*

Proof. By Lemma 1.4, $\sigma_{T\oplus S}(x \oplus y) = \sigma_T(x) \cup \sigma_S(y)$ for all $x \oplus y \in X \oplus Y$. It suffices to show that $(X \oplus Y)_{T\oplus S}(F) = X_T(F) \oplus Y_S(F)$ for every closed subset F of $\mathbb{C}$. For every closed subset F of $\mathbb{C}$,

$$\begin{aligned} x \oplus y \in (X \oplus Y)_{T\oplus S}(F) &\Longleftrightarrow \sigma_{T\oplus S}(x \oplus y) = \sigma_T(x) \cup \sigma_S(y) \subseteq F \\ &\Longleftrightarrow \sigma_T(x) \subseteq F \text{ and } \sigma_S(y) \subseteq F \\ &\Longleftrightarrow x \in X_T(F) \text{ and } y \in Y_S(F). \end{aligned}$$

It follows that $(X \oplus Y)_{T\oplus S}(F) = X_T(F) \oplus Y_S(F)$ for every closed subset F of $\mathbb{C}$. Suppose that T and S have property (C). Then for every closed subset F of $\mathbb{C}$, $X_T(F)$ and $Y_S(F)$ are closed, and hence $(X \oplus Y)_{T\oplus S}(F)$ is closed. It follows that $T \oplus S$ has property (C). Conversely, suppose that $T \oplus S$ has property (C). By Proposition 1.2.21 of [23], it then follows that T and S have property (C). □

Lemma 1.6. *Let $T \in L(X)$ and $S \in L(Y)$ on the Banach spaces X and Y. Then $T \oplus S \in L(X \oplus Y)$ has property (β) if and only if both T and S have property (β). Dually, $T \oplus S \in L(X \oplus Y)$ has property (δ) if and only if both T and S have that property.*

Proof. Suppose that both T and S have property (β). Let $P_1 : X \oplus Y \to X$ be the projection and $P_2 : X \oplus Y \to Y$ be the projection. Let U be an arbitrary open subset of $\mathbb{C}$ and let $\{f_n\}_{n=1}^{\infty} \subseteq O(U, X \oplus Y)$ be any sequence such that

$$(\lambda I - T \oplus S)f_n(\lambda) = 0 \text{ in } O(U, X \oplus Y).$$

Then clearly, $\{P_1 f_n\} \subseteq O(U, X)$ and $\{P_2 f_n\} \subseteq O(U, Y)$ satisfying

$$(\lambda I - T)P_1 f_n(\lambda) = 0 \text{ in } O(U, X) \text{ and } (\lambda I - S)P_2 f_n(\lambda) = 0 \text{ in } O(U, Y).$$

Since T and S have property (β), we have $P_1 f_n \equiv 0$ in $O(U, X)$ and $P_2 f_n \equiv 0$ in $O(U, Y)$. It follows that $f_n = P_1 f_n + P_2 f_n \equiv 0$ on $O(U, X \oplus Y)$. Hence $T \oplus S$ has property (β).

Conversely, suppose that $T \oplus S$ has property (β). Let V be an arbitrary open subset of $\mathbb{C}$ and let $\{f_n\}_{n=1}^{\infty} \subseteq O(V, X)$ and $\{g_n\}_{n=1}^{\infty} \subseteq O(V, Y)$ such that

$$(\lambda I - T)f_n(\lambda) = 0 \text{ in } O(V, X) \text{ and } (\lambda I - S)g_n(\lambda) = 0 \text{ in } O(V, Y).$$

We define $f_n \oplus g_n : V \to X \oplus Y$ by

$$(f_n \oplus g_n)(\lambda) = f_n(\lambda) \oplus g_n(\lambda) \text{ for all } \lambda \in V.$$

Then $\{f_n \oplus g_n\}_{n=1}^{\infty} \subseteq O(V, X \oplus Y)$ and

$$(\lambda I - (T \oplus S))(f_n(\lambda) \oplus g_n(\lambda)) = 0 \text{ in } U(V, X \oplus Y).$$

Since $T \oplus S$ has property (β), we obtain

$$f_n \oplus g_n \equiv 0 \quad \text{in} \quad O(V, X \oplus Y).$$

It follows that $f_n \equiv 0$ in $O(V, X)$ and $g_n \equiv 0$ in $O(V, Y)$. We conclude that T and S have property (β). Finally, suppose that $T \oplus S$ has property (δ). Then $(T \oplus S)^* = T^* \oplus S^*$ has property (β), and hence T^* and S^* have property (β). It follows from Theorem 2.5.18 of [23] that T and S have property (δ). The reverse implication is similar. □

The *surjective spectrum* $\sigma_{sur}(T)$ of $T \in L(X)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(T - \lambda I)(X) \neq X$. It is clear that $\sigma_{sur}(T)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(T)$. The *approximate point spectrum* $\sigma_{ap}(T)$ of T is defined as the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not bounded below. It is well known that $\sigma_{sur}(T) = \sigma_{ap}(T^*)$ and $\sigma_{ap}(T) = \sigma_{sur}(T^*)$. For $T \in L(X)$, the *localizable spectrum* $\sigma_{loc}(T)$ of T will be defined as the set of all $\lambda \in \mathbb{C}$ such that $X_T(\overline{V}) \neq \{0\}$ for each open neighborhood V of λ. It is well known that $\sigma_{loc}(T)$ is a closed subset of $\sigma(T)$ and that $\sigma_{loc}(T)$ contains the point spectrum and is included in the aproximate point spectrum of T, see [18]. As shown by Eschmeier and Prunaru [18], the localizable spectrum plays an important role in the theory of invariant subspaces; see also [18] and [24]. The following property is stable under commuting quasinilpotent perturbations: SVEP, property (C), property (β), property (δ), decomposability.

Theorem 1.7. *Let $T \in L(X)$ and $Q \in L(X)$ be a quasinilpotent operator commuting with T. Then $\sigma_T(x) = \sigma_{T+Q}(x)$ for all $x \in X$. Moreover, $\sigma_*(T+Q) = \sigma_*(T)$ for $\sigma_* \in \{\sigma, \sigma_{loc}, \sigma_{sur}, \sigma_{ap}\}$. Furthermore, the following assertions hold:*
(a) T has SVEP if and only if $T + Q$ has SVEP.
(b) T has property (C) if and only if $T + Q$ has property (C).
(c) T has property (β) if and only if $T + Q$ has property (β).
(d) T has property (δ) if and only if $T + Q$ has property (δ).
(e) T is decomposable if and only if $T + Q$ is decomposable.

Proof. Theorem 2.2, Corollary 2.4, Corollary 2.6, Corollary 2.7, Corollary 2.8 of [30]. □

Proposition 1.8. *Let $T \in L(X)$ be a Riesz operator on a complex Banach space X. Suppose that U is any open subset of $\mathbb{C}$ such that $\sigma(T) \cap U \neq \phi$ then there is a nonzero $Y \in Lat(T)$ such that $\sigma(T|Y) \subseteq U$.*

Proof. Let V be another open subset of $\mathbb{C}$ such that $\sigma(T) \not\subseteq V$ and $\{U, V\}$ be an open covering of $\mathbb{C}$. By Theorem 1.4.7 of [23], T is decomposable. Thus there exist $Y, Z \in Lat(T)$ satisfying

$$X = Y + Z, \quad \sigma(T|Y) \subseteq U \text{ and } \sigma(T|Z) \subseteq V.$$

If $Y = \{0\}$ then $\sigma(T|Z) = \sigma(T) \subseteq V$, which is impossible by the choice of V. It follows that $\{0\} \neq Y \in Lat(T)$ and $\sigma(T|Y) \subseteq U$.. □

Corollary 1.9. *Let $T \in L(X)$ be a Riesz operator on a complex Banach space X. Then $\sigma(T) = \sigma_{ap}(T) = \sigma_{sur}(T)$.*

Proof. Suppose that $\sigma(T) \neq \sigma_{ap}(T)$. Let $U = \mathbb{C} \setminus \sigma_{ap}(T)$. Then U is an open subset of $\mathbb{C}$ and $U \cap \sigma(T) \neq \phi$. Thus there exist $Y \in Lat(T)$ such that $\sigma(T|Y) \subset U$ by Proposition 1.8. It is clear that $\partial\sigma(T|Y)$ is nonempty. Thus there exists $\mu \in U$ such that

$$\mu \in \partial\sigma(T|Y) \subseteq \sigma_{ap}(T|Y) \subseteq \sigma_{ap}(T).$$

This is a contradiction, we have $\sigma(T) = \sigma_{ap}(T)$. It is clear that

$$\sigma(T) = \sigma(T^*) = \sigma_{ap}(T^*) = \sigma_{sur}(T).$$

□

2. Main result

Let M be a subset of a Banach space X. The *annihilator* of M is the closed subspace of X^* defined by $M^\perp := \{f \in X^* : f(x) = 0 \text{ for every } x \in X\}$, while the *pre-annihilator* of a subset W of X^* is the closed subspace of X defined by

$${}^\perp W := \{x \in X : f(x) = 0 \text{ for every } f \in W\}.$$

It is clear that if M is closed then ${}^\perp(M^\perp) = M$.

Theorem 2.1. *Let $T, S \in L(X)$, where S is a Riesz operator such that $TS = ST$. Let $Y \in Lat(S)$ be a hyperinvarinat subspace of X for which $\sigma(S) = \sigma(S|Y)$, and let $T_1 := T|Y \in L(Y)$ and $S_1 := S|Y \in L(Y)$. If T_1 has SVEP then $T_1 + S_1$ has SVEP.*

Proof. We claim that S_1 is a Riesz operator. By Theorem 3.17 of [11], it suffices to show that each spectral point $\lambda \neq 0$ is isolated and the spectral projection associated with $\{\lambda\}$ is finite-dimensional. We first show that $(\lambda I - S)(Y) = Y$ for all $\lambda \in \rho(S)$. It is clear that $(\lambda I - S)(Y) \subseteq Y$. Let $\lambda \in \mathbb{C}$ such that $r(S) < |\lambda|$ and let $R_\lambda := (\lambda I - S)^{-1}$. Obviously, $R_\lambda = \sum_{n=0}^{\infty} \lambda^{-n-1} S^n$. It follows that $R_\lambda(Y) \subseteq Y$. For every $f \in Y^\perp$ and $y \in Y$, we define $g : \rho(T) \to \mathbb{C}$ by

$$g(\lambda) = f(R_\lambda y) \quad \text{for all} \quad \lambda \in \rho(T).$$

Then g is analytic and vanishs outside the spectral disk of S. Since $\rho(S)$ is connected, it follows from the identity theorem that $f(R_\lambda y) = 0$ for all $\lambda \in \rho(S)$. Therefore $R_\lambda y \in Y^{\perp\perp} = Y$ and so $y = (\lambda I - S)R_\lambda y \in (\lambda I - S)(Y)$. Hence $Y \subseteq (\lambda I - S)(Y)$, and we have $(\lambda I - S)(Y) = Y$ for all $\lambda \in \rho(S)$. Obviously, $\lambda I - S$ is injective for all $\lambda \in \rho(S)$. Thus $\rho(S) \subseteq \rho(S_1)$, and hence $\sigma(S_1) \subseteq \sigma(S)$. Let μ be an isolated spectral point of S, and hence an isolated point of $\sigma(S_1)$. Let P be the spectral projection associated with $\{\mu\}$ and S, and let P_1 be the spectral projection associated with $\{\mu\}$ and S_1. Then $Px = P_1x$ for all $x \in Y$. Hence P_1 is the restriction of P to Y. Since P is finite-dimensional, P_1 is finite-dimensional. We conclude that S_1 is also Riesz. Since T_1 has SVEP, it follows from Theorem 0.3 of [2] that $T_1 + S_1$ has SVEP. $\square$

Let $T, S \in L(X)$ such that $TS = ST$, and let $Y \in Lat(S)$ be a hyperinvarinat subspace of X for which $\sigma(S|Y) = \{0\}$. Then cleraly, $S|Y$ is a quasinilpotent operator. We have the following.

Theorem 2.2. *Let $T, S \in L(X)$ such that $TS = ST$, and let $Y \in Lat(S)$ be a hyperinvarinat subspace of X for which $\sigma(S|Y) = \{0\}$. Let $T_1 := T|Y \in L(Y)$ and $S_1 := S|Y \in L(Y)$. Then the following assertions hold:*
(a) T_1 has SVEP if and only if so does $T_1 + S_1$.
(b) T_1 has property (C) if and only if so does $T_1 + S_1$.
(c) T_1 has property (β) if and only if so does $T_1 + S_1$.
(d) T_1 has property (δ) if and only if so does $T_1 + S_1$.
(e) T_1 is decomposable if and only if so does $T_1 + S_1$.

Proof. Note that S_1 is quasinilpotent and $T_1S_1 = S_1T_1$. So Theorem 1.7 applies. $\square$

Corollary 2.3. *Let $T, S \in L(X)$ such that $TS = ST$, and let $Y \in Lat(S)$ be a hyperinvarinat subspace of X for which $\sigma(S|Y) = \{0\}$. Let $T_1 := T|Y \in L(Y)$ and $S_1 := S|Y \in L(Y)$. Then $\sigma_{T_1+S_1}(x) = \sigma_{T_1}(x)$ for all $x \in Y$. Moreover, $\sigma_*(T_1 + S_1) = \sigma_*(T_1)$ for $\sigma_* \in \{\sigma, \sigma_{loc}, \sigma_{sur}, \sigma_{ap}\}$.*

Theorem 2.4. *(*[23]*) Let $T \in L(X)$ be an operator on a Banach space X. Then T is a Riesz operator if and only if T is decomposable and all the spaces $X_T(F)$, where $F \subseteq \mathbb{C} \setminus \{0\}$ is closed, are finite dimensional.*

Corollary 2.5. *Let $T \in L(X)$ and $Q \in L(X)$ be a quasinilpotent operator commuting with T. Then T is a Riesz operator if and only if so does $T + Q$.*

Proof. By Theorem 1.7, $\sigma_T(x) = \sigma_{T+Q}(x)$ for all $x \in X$, we conclude that $X_T(F) = X_{T+Q}(F)$ for all closed $F \subseteq \mathbb{C} \setminus F$. So Theorem 1.7 and Theorem 2.2 applies. $\square$

Theorem 2.6. *Let $T \in L(X)$ be a Riesz operator on a complex Banach space X. Suppose that 0 is an isolated point of the spectrum $\sigma(T)$ then T is the sum of an invertible and quasinilpotent operator.*

Proof. Assume that 0 is an isolated point of the spectrum $\sigma(T)$. Then there is a positive integer $n \in \mathbb{N}$ such that $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \frac{1}{n}\} \subseteq \rho(T)$. Let $U := \{\lambda \in \mathbb{C} : \frac{1}{n+1} < |\lambda|\}$ and $V := \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{n}\}$. Then $\{U, V\}$ be an open cover of $\mathbb{C}$. By Thorem 2.4, T is decomposable, and hence there exist $Y, Z \in Lat(T)$ such that

$$X = Y + Z, \quad \sigma(T|Y) \subseteq U \quad \text{and} \quad \sigma(T|Z) \subseteq V.$$

Let $A = T|Y$ and $B = T|Z$. Then clearly, $T = A+B$. Since $0 \notin U$ and $\sigma(A) \subseteq U$, we have $0 \in \rho(A)$ and hence A is invertible. Since $\sigma(B) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{n}\}$, we obtain $\sigma(B) = \{0\}$. It follows that B is quasinilpotent. □

Theorem 2.7. *Let $T \in L(X)$ and $S \in L(Y)$ on the Banach spaces X and Y. If T is similar to S then T is a Riesz operator if and only if S is a Riesz operator.*

Proof. Let $A \in L(X, Y)$ be a bounded invertible operator for which $AT = SA$. Then clearly, $\sigma(T) = \sigma(S)$ and T is decomposable by Theorem 2.4. We first show that S is decomposable. Let $\{U_1, U_2\}$ be an open covering of $\mathbb{C}$. Then there exist $X_1, X_2 \in Lat(T)$ such that $X = X_1 + X_2$ and $\sigma(T|X_i) \subseteq U_i$ for $i = 1, 2$. Let $Y_i = AX_i$ for $i = 1, 2$. Then clearly $Y_i \in Lat(S)$ and $Y = Y_1 + Y_2$. Since $S|Y_i$ is similar to $T|X_i$ under the invertible restrictin $A|X_i$, We have

$$\sigma(S|Y_i) = \sigma(T|X_i) \subseteq U_i \quad \text{for } i = 1, 2.$$

It follows that S is decomposable. Finally, we show that all the spces $Y_S(F)$, where $F \subseteq \mathbb{C} \setminus \{0\}$ is closed, are finite dimensional. Since T is Riesz, $X_T(F)$ is closed and $\dim X_T(F) < \infty$. By Proposition 1.2.17 of [23], we have the inclusions

$$\sigma_S(Ax) \subseteq \sigma_T(x) \quad \text{for all } \; x \in X \text{ and } \sigma_T(A^{-1}y) \subseteq \sigma_S(y) \quad \text{for all } \; y \in Y.$$

It suffices to show that $AX_T(F) = Y_S(F)$. If $y \in Y_S(F)$ then

$$\sigma_T(A^{-1}y) \subseteq \sigma_S(y) \subseteq F,$$

and therefore $Y_s(F) \subseteq AX_T(F)$. On the other hand, if $y = Ax$ for some $x \in X_T(F)$ then

$$\sigma_S(Ax) \subseteq \sigma_T(x) \subseteq F$$

and so $y = Ax \in Y_s(F)$. This implies that $AX_T(F) \subseteq Y_S(F)$ and hence $AX_T(F) = Y_S(F)$. Since $\dim X_T(F) < \infty$, we obtain $\dim Y_S(F) < \infty$. We conclude that S is a Riesz operator by Theorem 2.4. The reverse implication is similar. □

It is well known that for every $F \subseteq \mathbb{C}$, $X_T(F)$ is a hyperinvariant subspace of X. The spectrum $\sigma(S)$ of Riesz operator is at most countable and has no non-zero cluster point. Let $T \in L(X)$ and $S \in L(X)$ be a quasinilpotent operator commuting with T, and let $Y := X_S(\{0\})$. Then, by Proposition 1.2.16 and Proposition 1.2.20 of [23], Y is a hyperinvariant subspace of X and $\sigma(S|X_S(\{0\})) \subseteq \{0\}$ and hence $X = X_S(\{0\})$. We have the following.

Theorem 2.8. *Let $T \in L(X)$ and let $S \in L(X)$ be a Riesz operator commuting with T. Let $T_1 := T|X_S(\{0\})$ and $S_1 := S|X_S(\{0\})$. Then the following assertions are hold:*
(a) T_1 has SVEP if and only if so does $T_1 + S_1$.
(b) T_1 has property (C) if and only if so does $T_1 + S_1$.
(c) T_1 has property (β) if and only if so does $T_1 + S_1$.
(d) T_1 has property (δ) if and only if so does $T_1 + S_1$.
(e) T_1 is decomposable if and only if so does $T_1 + S_1$.

Proof. It suffices to show that S_1 is a quasinilpotent operator commuting with T_1. Clearly, $T_1S_1 = S_1T_1$. It follows from Theorem 2.4 that S is decomposable and hence S has property (C). By Proposition 1.2.16 of [23], $X_S(\{0\})$ is a closed hyperinvariant subspace of X, and hence, by Proposition 1.2.20 of [23],

$$\sigma(S_1) = \sigma(S|X_S(\{0\})) \subseteq \{0\}.$$

We infer that S_1 is a quasinilpotent operator commuting with T_1. So Theorem 1.7 applies. □

Theorem 2.9. *Let $T \in L(X)$ and let $S \in L(X)$ be a Riesz operator commuting with T. Let $F \subseteq \mathbb{C} \setminus \{0\}$ be a closed, and let $T_1 := T|X_S(F)$ and $S_1 := S|X_S(F)$. Then the following assertions are hold:*
(a) If T_1 has SVEP then $T_1 + S_1$ has SVEP.
(b) If T_1 has property (C) then $T_1 + S_1$ has propety (C).
(c) If T_1 has property (β) then $T_1 + S_1$ has property (β).
(d) If T_1 has property (δ) then $T_1 + S_1$ has property (δ).
(e) If T_1 is decomposable then $T_1 + S_1$ is decomposable.

Proof. It follows from Theorem 2.4 that S is decomposable and hence S has property (C). By Proposition 1.2.16 of [23], $X_S(F)$ is a closed hyperinvariant subspace of X, and hence, by Proposition 1.4.7 of [23], $X_S(F)$ is finite-dimensional. Thus $\sigma(S_1)$ is finite, say $\sigma(S_1) = \{\mu_1, \mu_2, \cdots, \mu_n\}$. For $i = 1, 2, \cdots, n$ let $P_i \in L(X)$ denote the spectral projection associated with S_1 and with the spectral set $\{\mu_i\}$, and let $X_i := P_i(X)$. From standard spectral theory it is known that $P_1 + P_2 + \cdots + P_n = I$, that $X_1, X_2, \cdots, X_n$ are closed linear subspaces of X which are each invariant under both T_1 and S_1, and that $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. For $i = 1, 2, \cdots, n$ let $A_i := T_1|X_i \in L(X_i)$ and $B_i := S_1|X_i \in L(X_i)$. Then clearly, $A_iB_i = B_iA_i$ and

$$T_1 + S_1 = (A_1 + B_1) \oplus (A_2 + B_2) \oplus \cdots \oplus (A_n + B_n).$$

Obviously, $\sigma(B_i) = \sigma(B|X_i) = \{\mu_i\}$ and hence $\sigma(B_i - \mu_i I) = \{0\}$ i.e. $B_i - \mu_i I$ is a quasinilpotent operator, for all $i = 1, 2, \cdots, n$.

(a) Suppose that T_1 has SVEP. Since SVEP is inherited by the restrictions to closed invariant subspaces, then A_i has SVEP, we conclude that $A_i + \mu_i I$ has SVEP by Lemma 1.3. Since $A_i + B_i = (A_i + \mu_i I) + (B_i - \mu_i I)$ and $B_i - \mu_i I$ is quasinilpotent, then, by Theorem 1.7, $A_i + B_i$ has SVEP, and hence $T_1 + S_1 = (A_1 + B_1) \oplus \cdots \oplus (A_n + B_n)$ has SVEP by Lemma 1.4.

(b) Suppose that T_1 has property (C). Then, by Propposition 1.2.21 of [23], A_i has property (C), and hence $A_i + \mu_i I$ has property (C) by Lemma 1.3. Since $B_i - \mu_i I$ is a quasinilpotent operator, which shows that, by Theorem 1.7, $A_i + B_i = (A_i + \mu_i I) + (B_i - \mu_i I)$ has property (C). By Lemma 1.5, it then follows that $T_1 + S_1 = (A_1 + B_1) \oplus \cdots \oplus (A_n + B_n)$ has property (C), as desired.

(c) Suppose that T_1 has property (β). Since the restriction of an operator with property (β) to a closed invariant subspace certainly inherits this property, A_i has property (β) and we have $A_i + \mu_i I$ has property (β) by Lemma 1.3. We conclude that, by Theorem 1.7, $A_i + B_i = (A_i + \mu_i I) + (B_i - \mu_i I)$ has property (β), so that Lemma 1.6 ensures that $T_1 + S_1$ has property (β).

(d) Note that (β) and (δ) are complete dual. This assertion follows from (c) by duality.

(e) Note that T_1 is decomposable if and only if it has both property (β) and (δ). This assertion follows from (c) and (d). □

It is well known from Corollary 2.2 of [20] that if $S \in L(X)$ is compact and $\sigma(S) = \{0, \lambda_1, \lambda_2, \cdots\}$ then the space $X_S(\mathbb{C} \setminus \{0\})$ is not closed.

Corollary 2.10. *Let $T \in L(X)$ and suppose that $S \in L(X)$ is compact which commutes with T. Suppose that $X_S(\mathbb{C} \setminus \{0\})$ is closed. Let $T_1 := T|X_S(\mathbb{C} \setminus \{0\})$ and $S_1 := S|X_S(\mathbb{C} \setminus \{0\})$. Then T_1 and $T_1 + S_1$ share the local spectral properties such as SVEP, Dunford's property (C), Bishop's property (β), decomopsition property (δ) and decomposability.*

Proof. Theorem 2.8 and Theorem 2.1 of [20]. □

We say that an operator $S \in L(X)$ is *polynomially Riesz* if there exists a non-zero complex polynomial $p(z)$ such that $p(S)$ is Riesz.

Corollary 2.11. *Let $T \in L(X)$, $S \in L(X)$ be a polynomially Riesz operator commuting with T. Let $T_1 := T|X_{p(S)}(\{0\})$ and $S_1 := S|X_{p(S)}(\{0\})$ for some non-zero complex polynomial $p(z)$. Then T_1 and $T_1 + S_1$ share the local spectral properties such as SVEP, Dunford's property (C), Bishop's property (β), decomopsition property (δ) and decomposability.*

Corollary 2.12. *Let $T \in L(X)$, $S \in L(X)$ be a polynomially Riesz operator commuting with T and let $F \subseteq \mathbb{C} \setminus \{0\}$ be a closed. Let $T_1 := T|X_{p(S)}(F)$ and $S_1 := S|X_{p(S)}(F)$ for some non-zero complex polynomial $p(z)$. Then T_1 and $T_1 + S_1$ share the local spectral properties such as SVEP, Dunford's property (C), Bishop's property (β), decomopsition property (δ) and decomposability.*

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