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AUGMENTED INVERSE GRAPHS WITH RESPECT TO A GROUP

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ABSTRACT. In this paper, the Augmented graph $E_s(\tau)$ of the inverse graph $G_s(\tau)$ of a cyclic group $(\tau,\circ)$ was studied. The Augmented inverse graph was constructed by applying the method of Mycielski's construction. The dimension of Augmented inverse graph and different properties of the graph were investigated. Later the chromatic number of Augmented inverse graph was discussed and the relation between the maximum degree of the graph and the chromatic number was established. In the Mycielski's construction, the properties of the key node 'u' in $E_s(\tau)$ were established based on cardinality of the cyclic group $(\tau,\circ)$ and also proved that the Augmented inverse graph $E_s(\tau)$ was a triangle free graph.

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1. Introduction

Graph theory is one of the most interesting branch of modern Mathematics. It has wide applications in computers, classical algebraic problems and optimization problems. The study on groups and graphs are done by several authors since 19^{th} century. The construction of graphs using groups was introduced by W. B. Vasantha Kandasamy et al.[7]. The construction of the inverse graphs was introduced by Monther R. Alfuraidan, Yusuf F. Zakariya [3]. The construction of different inverse graphs using non-self-invertible elements of a Cyclic group was introduced by Lakshmi et al. [4]. Throughout this paper, simple and undirected graphs were constructed based on the Cyclic groupsLet $(\tau, \circ)$ be a cyclic group and S be a non-empty subset of τ which consists of all non-self-invertible elements of τ with respect to the binary operation $\circ$. The inverse graph associated with the group τ denoted by $G_S(\tau)$, whose vertices are the elements of the group τ and the edges are formed by joining two distinct elements $x, y \ \ of \tau$

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if and only if $x \circ y \in S$ or $y \circ x \in S$ [7]. The focus of this research is to introduce the Augmented inverse graph, its dimension, chromatic number and some basic properties.

2. Main results

Procedure to draw the Augmented inverse graph by applying the method of Mycielski's construction.
Let the vertices of the inverse graph $G_s(\tau)$ be $x_1, x_2, x_3 \ldots, x_n$. Form a new graph $E_s(\tau)$ from $G_s(\tau)$ as follows: Add n+1 vertices $x_1', x_2', x_3', \ldots, x_n', u$ (key node) and then for $1 \leq i \leq n$, join x_i' to the neighbors of x_i and to the key node u.

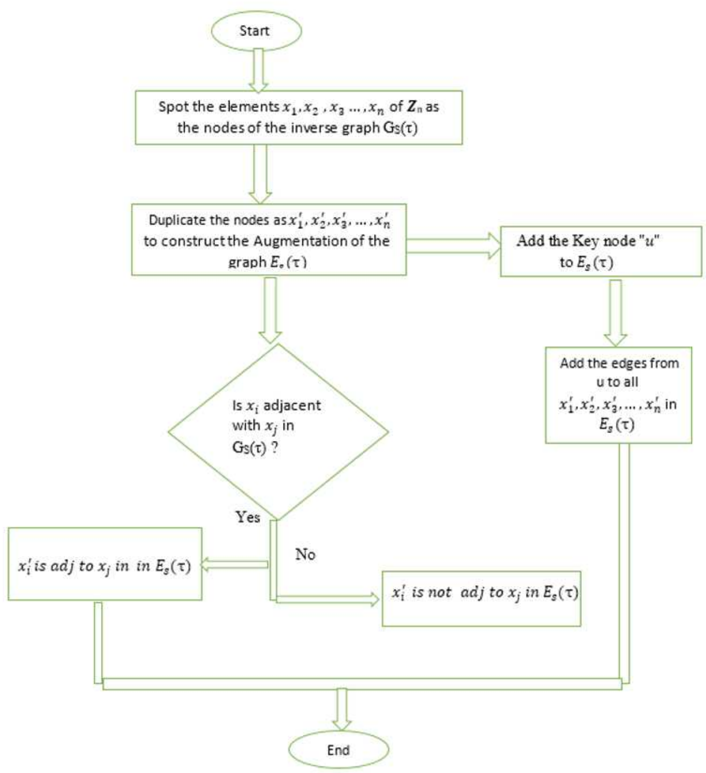

Figure 1. Algorithm to construct the inverse graph

$(Z_4, +)$ is a group. S is a non empty set of non-self invertible elements of $Z_4 = \{0, 1, 2, 3\}$ *and* S $= \{1, 3\}$.*TheAugmented inverse graph* $E_S(Z_4)$ will be represented as follows

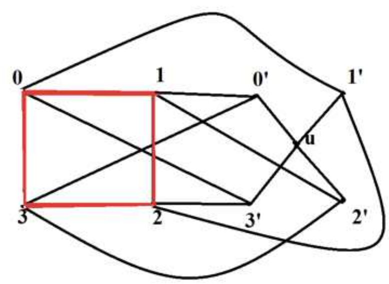

Figure 2. Augmented inverse graph

Remark 2.1. Colouring procedure of Augmented inverse graph
If the key node u is coloured with (say) colour a and the remaining n vertices $x_1^{'}, x_2^{'}, x_3^{'}, \ldots, x_n^{'}$ $of the Augmented inverse graph E_S(Z_n)$ should be coloured with the other colour (say) b . The identity node can be colored with the color of key node and the remaining vertices can be properly coloured.

3. The Foresight of Main Results

Lemma 3.1. *If the Chromatic number of the inverse graph of $G_s(\tau)$ is k then the chromatic number of the Augmented graph of $E_s(\tau)$ is k+1*

Proof. Let $\mathrm{G}_S(\tau)$ is the inverse graph of the group $(\tau, \circ)$ and S be the set of non-self invertible elements of the group $(\tau, \circ)$. Let the Augmented inverse graph of $G_S(\tau\ , \circ) denoted\ by\ E_s(\tau, \circ)$ We will prove the result by induction on the number of vertices 'n'. If n=1 or n=2 the result is trivial, hence will start the induction on n=3 If the chromatic number of the graph is 3, then the chromatic number of the inverse graph is 4.
since the augmented inverse graph is a tripartite graph with a key node u *we have to assign a new colour to that keynode.Similarlyif* $\chi(\mathrm{G}_S(\tau)) = 4$ $then \chi(\mathrm{E}_S(\tau)) = 5$ and $u \in E_s\,(\tau)$. Assume that the result is true for (k-1) i.e., If the chromatic number of the graph is k-1, then the chromatic number of the corresponding inverse graph is k Therefore by induction if he chromatic number of the graph is k, then the chromatic number of the corresponding inverse graph is k+1 =(m/2)+1 where m=the cardinality of S [4]
In the Augmented inverse graph $\mathrm{E}_S(\tau)$ as we have a key node u to assign a new color other than k colors to the key node u
Hence the chromatic number of the Augmented graph of $E_s(\tau)$ is $k + 1$ ☐

Lemma 3.2. *In the inverse graph of $G_s(\tau)$ the identity e of the group $(\tau,\circ)$ is analogous to the key node $u \in E_s\,(\tau)$ if n is odd.*

Proof. Let $G_s(\tau)$ is a inverse graph of $(\tau,\circ)$ and S be a set of non-self-invertible elements of the group $(\tau,\circ)$. Let $E_s(\tau)$ be the Augmented graph of $G_s(\tau)$.We will prove this by using contradiction
Let $'e'$ be the identity element in the group $(\tau,\circ)$. Let us assume that, n to be even, then $e \in G_s(\tau)$ is not adjacent to $(\frac{n}{2})^{th}$ node in $G_s(\tau)$.Hence $d(e) = n-2$
In the $E_s(\tau)$, the key node $u \in E_s(\tau)$ is of degree n.But by the Mycielski's construction [4] the key node u must be adjacent to all n nodes.$\therefore$ e is not analogous to u. □

i.e., $d(e) \neq d(v)$ It proves that n must be odd □

Remark 3.1. In the inverse graph $G_s(Z_n,\circ)$ the identity node is not adjacent with $(\frac{n}{2})^{th}$node and the degree of the identity node is (n -2) if n is even. Hence its Augmented graph $E_s(Z_n,\circ)$ the degree of Identity node is $2(n-2)$

Lemma 3.3. *The identity node $e \in E_s(\tau)$ is of degree $2(n-2)$ if n is even and $2(n-1)$ if n is odd*

Proof. : Let the cardinality of the group $(\tau,\circ)$ be n. Then the degree of the identity node e in the Augmented inverse graph $E_s(\tau)$ has two possibilities
Case-i: If n is even
In $G_s(\tau)$, for $e \in \tau$, e must be adjacent to all the vertices say

$$x_1,\ x_2,\ x_3 \ldots, x_{\frac{n}{2}} - 1, x_{\frac{n}{2}} + 1 \ldots, x_{n-1}.$$

Then e is adjacent with $(n-2)$ vertices i.e., $(n-2)$ elements of τ, except the self-invertible elements. The $d(e)$ in $G_s(\tau)$ is $(n-2)$
In $E_s(\tau)$, as every vertex x_i' is adjacent with e for $i = 1, 2, 3, \ldots, \frac{n}{2}-1, \frac{n}{2}+1 \ldots, n$
Hence $d(e)$ in $E_s(\tau)$ is $(n-2)+(n-2)$
Therefore $d(e) = 2(\text{n}-2)$
Case-ii: If n is odd
In $G_s(\tau)$, for $e \in \tau$, e must be adjacent to all $(n-1)$ vertices say $x_1, x_2, x_3, \ldots,$ x_{n-1}
i.e., $d(e)$ in $G_s(\tau)$ is $(n-1)$
In $E_s(\tau)$, as every vertex x_i' is adjacent with e for $i = 1, 2, 3, \ldots, (n-1)$
Hence $d(e)$ in $E_s(\tau)$ is $(n-1)+(n-1)$ Therefore is $d(e) = 2(n-1)$ □ □

Lemma 3.4. *In $E_s(\tau)$ the identity element $e \in \tau$ has the maximum degree*

Proof. Let $x_i \in \tau$ $(x_i \neq e)$ be a node in $E_s(\tau)$ which is of maximum degree

$$d(e) < d(x_i) \tag{1}$$

Case -i : If n is even

$$d(e) = 2(n-2) \tag{2}$$

We know that each x_i $(i = 1, 2, 3, ..., n) \in$ $G_s(\tau)$ is not adjacent to all $(n-1)$ vertices

$$d(x_i) \leq (n-1) \tag{3}$$

From (1),(2),(3), d(e)=2n-4

$$\Rightarrow 2n - 4 < (n-1) \Rightarrow n < 3$$

Which is absurd as $n \in N$
Hence $d(e)$ must be maximum
That is $d(x_i) < d(e) \quad \forall\ i$
Case-ii : If n is odd,

$$d(e) = 2(n-1) \tag{4}$$

As each x_i $(i = 1, 2, 3, \ldots, n) \in$ $G_s(\tau)$ is not adjacent to all $(n-1)$ vertices

$$d(x_i) \leq (n-1) \tag{5}$$

From (4) and (5) , $d(e) = 2n - 2 is\ less\ than\ or\ equal\ to\ n - 1.$
$Hence\ the\ number\ of\ vertices\ must\ be\ less\ than\ or\ equal\ to\ 1$
$Which\ is\ absurd\ as\ n\ is\ a\ natural\ number. Hence\ degree\ of\ e\ must\ be\ maximum$ □

4. Main results

Theorem 4.1. *The Augmented inverse graph $E_s(\tau)$ is a triangle free graph*

Proof. Let $G_s(\tau)$ be an inverse graph of (τ,$\circ$) and S be a set of non-self-invertible elements of the group (τ,$\circ$)
Let $x_1, x_2, x_3 \ldots, x_n$ be the nodes of $G_s(\tau)$, add n+1 vertices $x_1', x_2', x_3' \ldots, x_n', u$
Let $E_s(\tau)$ be the Augmented graph of $G_s(\tau)$ In $E_s(\tau)$ we find three disjoint sets A $= \{x_1, x_2, x_3 \ldots, x_n\}$, A$' = \left\{x_1', x_2', x_3' \ldots, x_n'\right\}$, and $U = \{u\}$ such that $A \cup A' \cup U = E_s(\tau)$ and $A \cap A' \cap U = \varphi$
$\therefore E_s(\tau)$ is a tripartite graph and it is triangle free graph □

Theorem 4.2. *If $(G_s(\tau), V, \mathrm{E})$ is the inverse graph of the cyclic group (τ,$\circ$), then the Augmented inverse graph $(E_s(\tau), V', E')$ contains $2n+1$ vertices and $3\sum d(x_i) + 2n$ edges*

Proof. Let $(G_s(\tau), V, E)$ is the inverse graph of the cyclic group (τ,$\circ$)
Let S be the set of all non-self-invertible elements of (τ,$\circ$)
By the Mycielski's construction [4] the Augmented inverse graph $E_s(\tau)$ contains $2n+1$ vertices,
$\left\{ x_1, x_2, x_3 \ldots, x_n,\ x_1', x_2', x_3' \ldots, x_n', u\right\}$ such that
$d(x_i') =$ $d(x_i)$+1 for all i= 1, 2, 3 $\ldots, n$ and $d(u) = n$
Also $\forall\ x_i \in G_s(\tau)$, $d(x_i)$ becomes twice in $E_s(\tau)$, therefore

$$\sum_{v' \in V'(E_s(\tau))} d(v') = \sum_{i=1}^{n} d(x_i) + \sum_{i=1}^{n} d(x_i^{'}) + d(u)$$

$$= 2\sum_{i=1}^{n} d(x_i) + \sum_{i=1}^{n} d(x_i) + \sum_{i=1}^{n} 1 + d(u)$$

$$= 2\sum_{i=1}^{n} d(x_i) + \sum_{i=1}^{n} d(x_i) + n + n$$

$$= 3\sum_{i=1}^{n} d(x_i) + 2n$$

Therefore $|E| = \frac{3\sum_{i=1}^{n} d(x_i)+2n}{2}$ (By the Hand shaking Property: The sum of the degrees of the vertices is equal to twice the number of edges)[5] □

Theorem 4.3. *The Chromatic number of Augmented inverse graph cannot exceed the degree of the identity element of the group*
$(\tau,\circ)$ i.e., $\chi(E_s(\tau)) \leq \Delta(E_s(\tau)) = d(e)$

Proof. Let $E_s(\tau)$ be a Augmented inverse graph and S be a non-self-invertible elements of $(\tau,\circ)$. Let us suppose that $d(e) \leq \chi(E_s(\tau))$
The identity node e has $(n-1)$ neighbours in $G_s(\tau)$ and again $(n-1)$ neighbours in $E_s(\tau)$, if n is odd
The identity node e has $(n-2)$ neighbours in $G_s(\tau)$ and again $(n-2)$ neighbours in $E_s(\tau)$, if n is even.
Hence $d(e)$ is either twice of $(n-1)$ or twice of $(n-2)$
$\Longrightarrow d(e)$ is always even
By taking the nodes other than $'e'$ in $G_s(\tau)$ as A $= \{x_1, x_2, x_3, \ldots, x_{n-1}\}$, B $= \{x_1^{'}, x_2^{'}, x_3^{'}, \ldots, x_n^{'}\}$ *where* $x_1^{'}, x_2^{'}, x_3^{'}, \ldots, x_n^{'}$ *in* $E_s(\tau)$ and $C = \{u\}$ are the disjoint sets and does not have any edge whose end vertices are in the same set.
$\therefore$ The Augmented graph $E_s(\tau)$ must be coloured with fewer colours than degree of e
This is a contradiction to the assumption that $d(e) \leq \chi(E_s(\tau))$
Hence $\chi(E_s(\tau)) \leq \Delta(E_s(\tau))$ □ □

5. Conclusion

In this paper, augmented inverse graph was constructed using Mycielski's construction. Later the properties of Augmented inverse graph were proved to establish a relation between its maximum degree and the chromatic number. This study can be extended to find the chromatic polynomial of inverse graph and its augmentation. In Future, this work can be extended to find the rainbow connected number of the augmented inverse graph, which can be applied in Networking and Coding to enhance the quality of life in various areas such as

Traffic management, Marking schedule or Timetable, Mobile radio frequency assignments, Register allocations and Map colouring etc.

Conflicts of interest : On behalf of all the authors, the corresponding author states that there is no conflict of interest.

Data availability : our article has no associated data“Not applicable” here.

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