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GENERATING FUNCTIONS OF (p,q)-ANALOGUE OF ALEPH-FUNCTION SATISFYING TRUESDELL'S ASCENDING AND DESCENDING $F_{p,q}$-EQUATION[†]

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ABSTRACT. In this paper we have obtained various forms of (p,q)-analogue of Aleph-Function satisfying Truesdell's ascending and descending $F_{p,q}$-equation. These structures have been employed to arrive at certain generating functions for (p,q)-analogue of Aleph-Function. Some specific instances of these outcomes as far as (p,q)-analogue of I-function, H-function and G-functions have likewise been obtained.

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1. Introduction

In the present era, great emphasis is laid on the research of another dimension of calculus which is popularly called calculus without limits or $q-$calculus. The q-calculus has its birth in the last century. Kac and Cheung's book [7] entitled "Quantum Calculus" gives the nuts and bolts of such kind of calculus. The investigations of q-integrals and q-derivatives of arbitrary order have picked up significance because of their different applications in various areas like solutions of the q-difference (differential) and q-integral equations, ordinary and fractional differential equations, q-transform analysis and many more.The $q-$deformed algebras [9] and their $(p,q)-$analogues [8]pull in much consideration recently. From these works, numerous speculations of special functions emerge.

Amid the most recent three decades, uses of quantum calculus dependent on $q-$numbers have been contemplated and examined effectively, thickly and significantly (see [11]). Related to the inspiration and motivation of these applications

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and presentation of the $(p,q)-$numbers, numerous mathematicians and physicists have broadly built up the hypothesis of post quantum calculus dependent on $(p,q)-$numbers along the conventional lines of classical and quantum calculus. Agyuz et al. [16] exhibited some novel consequences of augmentations of $(p,q)-$Bernstein polynomials and determined a few new relations with identified with $(p,q)-$Gamma and $(p,q)-$Beta functions.

Duran et al. [10] presented another class of Bernoulli, Euler and Genocchi polynomials dependent on the $(p,q)-$calculus and explored their numerous properties including addition theorems, difference equations, derivative properties, recurrence relationships etc. Sadjang [12] explored a few properties of the $(p,q)-$derivative and the $(p,q)-$integration and introduced two appropriate polynomials basis for the $(p,q)-$derivative, and after that he inferred different properties of these bases. As an application, he gave two $(p,q)-$Taylor recipes for polynomials. Post quantum calculus, or $(p,q)-$calculus, is known as an augmentation of quantum calculus that recuperates the outcomes as $p \to 1$. For some essential properties of $(p,q)-$calculus, we allude to [13]. In the $q-$case, the solutions of a $q-$Sturm-Liouville problem are $q-$orthogonal functions [14], which diminish to the $q-$classical orthogonal polynomials, show up normally [26]. All around as of late [15], another speculation of $q-$Sturm-Liouville problems, to be specific, $(p,q)-$ Sturm-Liouville problems, has been analyzed. Masjed-Jamei et al. [25] demonstrated that the $(p,q)-$difference equation is of hypergeometric type, that is, the $(p,q)-$difference of any arrangement of the equation is likewise an answer of a equation of a similar kind.

The connection between approximation hypothesis and $q-$calculus urged the mathematicians to give $q-$analogue of known outcomes (see [16]). This quick improvement of $q-$calculus has driven to the disclosure of new speculation of this hypothesis. This delivers a few focal points like that the rate of convergence of $q-$operators is more adaptable and superior to the established one. Since the $q-$calculus depends on one parameter, there is a probability of extension of $q-$calculus. Duran et al. [17] presented another speculation of the Hermite polynomials through $(p,q)-$exponential generating function and explore a few properties and relations for referenced polynomials including subsidiary property, explicit formula, recurrence relation, integral representation. They additionally defined $(p,q)-$analogue of the Bernstein polynomials and secure a few equations, at that point give a few $(p,q)-$hyperbolic representations of the $(p,q)-$ Bernstein polynomials. Relationship between $(p,q)-$ Hermite polynomials and $(p,q)-$ Bernstein polynomials have been acquired.

Sadjang [18] introduced two $(p,q)-$analogues of the Sumudu transform and proved several properties. Applications have been done to solve some $(p,q)-$difference and functional equations. For more details, we refer to [12, 19, 20, 21].

Ryoo et al. [26, 27, 28] gave some properties of $(p,q)-$cosine euler polynomials, poly-cosine tangent and poly-sine tangent polynomials. Also they gave the properties of $q-$differential equation of higher order and visualisation of Fractal using $q-$Bernoulli polynomials.

The $q-$analogue of Aleph-function have been presented by Dutta et al. [3], by utilizing $q-$Gamma function, which is a $q-$expansion of the generalized H-function and I-function prior defined by Saxena et al. [2].

In his exertion towards accomplishing unification of special functions, Truesdell [6] has advanced a hypothesis which yielded various outcomes for special functions fulfilling the alleged Truesdell's F-equation.
Agrawal [5] broadened this hypothesis further and determined outcomes for descending F-equation. He acquired different properties like Orthogonality, Rodrigue's and Schalafli's formulae for Fq-condition, which end up being special functions. Renu Jain et al. [9] acquired different types of I-function which full fill Truesdell's ascending and descending F-equation.

The paper will determine a few identities for (p, q)-analogue of Gamma function and generating functions which satisfy Truesdell's ascending and descending $F_{p,q}$-equation.
In this paper, we have stretched out Truesdell's F-equation to its $(p, q)-$analogue and named the corresponding equation as $F_{p,q}$-equation. We have additionally determined different types of $(p, q)-$analogue of Aleph-function fulfilling Truesdell's ascending and descending $F_{p,q}$-equation. These structures have been utilized to land at certain generating functions for $(p, q)-$analogue Aleph-function. Some specific instances of these outcomes which seem, by all accounts, to be new have likewise been acquired.

The paper is organized as follows: In Section 2, we recall basic notations, definitions required in subsequent sections. In Section 3, we derived some identities for the $(p, q)-$analogue of Gamma function in order to obtain further results of the coming sections. We also derived various forms of $(p, q)-$analogue of Aleph function which satisfies Truesdell's ascending $F_{p,q}$-equation. In Section 4, we gave a conclusion.

2. Mathematical Preliminaries

From the theory of basic hypergeometric series [23], some basic definitions are given below:
The $q-$shifted factorial is given by

$$(a, q)_n = \left[\begin{array}{r} 1,\ n = 0 \\ (1-a)(1-aq)...(1-aq^{n-1}), n = 1, 2, 3, ... \end{array} \right.$$

with $(a_1, a_2, ..., a_k; q)_n = (a_1; q)_n (a_2; q)_n ... (a_k; q)_n$.
Thomae first of all introduced he q-gamma function, then by Jackson. Jackson [22] defined the q-analogue of gamma function as

$$\Gamma_q(x) = \frac{(q; q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty}, 0 < q < 1.$$

Jackson gave the general definition which is given below

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt$$

$$where \int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

Jackson also defined an integral i.e.

$$\int_0^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(aq^n)q^n$$

Sadjang[12] introduced the so-called the shifted factorial as follows:

$$(x \ominus a)_{p,q}^n = (x-a)(px-aq)(p^2x-aq^2)...(xp^{n-1}-aq^{n-1})$$
$$(x \oplus a)_{p,q}^n = (x+a)(px+aq)(p^2x+aq^2)...(xp^{n-1}+aq^{n-1})$$

The further extensions of above definitions are:

$$(a \ominus b)_{p,q}^n = \prod_{k=0}^{\infty}(ap^k - bq^k)$$

$$(a \oplus b)_{p,q}^n = \prod_{k=0}^{\infty}(ap^k + bq^k)$$

Let x be a complex number, the (p,q)-Gamma function is defined by Sadjang [20]

$$\Gamma_{p,q}(x) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^x \ominus q^x)_{p,q}^{\infty}}(p-q)^{1-x}, 0 < q < p$$

If we put $p = 1$, then $\Gamma_{p,q}$ reduces to Γ_q.
The $(p,q)-$Gamma function fulfils the fundamental property given below

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x)$$

For n to be a nonnegative integer, we have

$$\Gamma_{p,q}(x+1) = [x]_{p,q}!$$

It follows from the above

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)_{p,q}^n}{(p-q)_{p,q}^n}$$

(p,q)-Beta function were also defined by Sadjang [20] as

$$B_{p,q}(x,y) = \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}$$

The $(p,q)-$derivative of the function f(x) is defined as follows [19, 20]:

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, x \neq 0$$

where $D_{p,q}f(0)=f'(0)$, provided that $f(x)$ is differentiable at $x=0$.
The authors in [12] defined the (p,q)-numbers $[n]_{p,q}$ and (p,q) factorials $[n]_{p,q}!$ as:

$$[n]_{p,q}=\frac{p^n-q^n}{(p-q)} \qquad \text{and} \qquad [n]_{p,q}!=[1]_{p,q}[2]_{p,q}...[n]_{p,q}$$

respectively. More over it is evident that $D_{p,q}(x^n)=[n]_{p,q}x^{n-1}$.
Remark 1: $D_{p,q}(x)$ reduces to Hahn Derivative $d_qf(x)$ iff $p\to 1$.
Remark 2: $[n]_{p,q}=[n]_q$ (Hahn Basic Number) iff $p\to 1$. where $[n]_q=\frac{1-q^n}{1-q}, q\neq 1$.
Dutta et al. [3] defined the q-analogue of Aleph-function in terms of Mellin-Barnes type basic contour integral as:

$$\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}$$

$$\times\int_L \frac{\prod_{j=1}^{m}G(q^{(b_j-B_js)})\prod_{j=1}^{n}G(q^{(1-a_j-A_js)})}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}G(q^{(1-b_{ji}+B_{ji}s)})\prod_{j=n+1}^{p_i}G(q^{(a_{ji}-A_{ji}s)})G(q^s)G(q^{1-s})\sin\pi s]}\pi z^s ds \tag{1}$$

where $z\neq 0, 0<|q|<1$ and $\omega=\sqrt{-1}$ Altaf et al. [24] gave the definition of $(p,q)-$analogue of Aleph-function by using $(p,q)-$Gamma function as follows:

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{2\pi i}\int_L$$

$$\frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)\sin\pi s]}ds \tag{2}$$

The function $F(z,\alpha)$ is said to satisfy the ascending F-equation if

$$D^r_zF(z,\alpha)=F(z,\alpha+r) \tag{3}$$

For $F(z,\alpha)$ satisfying ascending F-equation, Truesdell [6] has obtained following generating functions using Taylor's series

$$F(z+y,\alpha)=\sum_{n=0}^{\infty}y^n\frac{F(z,\alpha+n)}{n!} \tag{4}$$

The function $F(z,\alpha)$ is said to satisfy the descending F-equation if

$$D^r_zF(z,\alpha)=F(z,\alpha-r) \tag{5}$$

For $F(z,\alpha)$ satisfying descending F-equation, Agrawal [6] has obtained following generating functions:

$$F(z+y,\alpha)=\sum_{n=0}^{\infty} y^n \frac{F(z,\alpha-n)}{n!} \tag{6}$$

The (p,q)-derivative of equation (3) and (5) can be written respectively in the following manner:

$$D^r_{(p,q),z}F(z,\alpha)=F_{p,q}(z,\alpha+r) \tag{7}$$

$$D^r_{(p,q),z}F(z,\alpha)=F_{p,q}(z,\alpha-r) \tag{8}$$

3. Main Results

3.1. Identities involving gamma function. In this section, we first derive some identities for the $(p,q)-$analogue of the gamma function in order to obtain further results of this paper.

Lemma 3.1. *The following identity for the $(p,q)-$analogue of gamma function holds true:*

$$\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha+r+k}{m})=\frac{(p^\alpha\ominus q^\alpha)^r_{p,q}}{(p-q)^r}\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha+k}{m}) \tag{9}$$

Proof. Considering the involved ratio and simplifying in the following manner, we have

$$\begin{aligned}\frac{\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha+r+k}{m})}{\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha+k}{m})} &= \frac{\Gamma_{p,q}(\frac{\alpha+r}{m})}{\Gamma_{p,q}(\frac{\alpha}{m})}\frac{\Gamma_{p,q}(\frac{\alpha+r+1}{m})}{\Gamma_{p,q}(\frac{\alpha+1}{m})}\frac{\Gamma_{p,q}(\frac{\alpha+r+2}{m})}{\Gamma_{p,q}(\frac{\alpha+2}{m})}\cdots\frac{\Gamma_{p,q}(\frac{\alpha+r+m-1}{m})}{\Gamma_{p,q}(\frac{\alpha+m-1}{m})}\\ &= [\frac{\alpha}{m}]_{(p,q),r/m}[\frac{\alpha+1}{m}]_{(p,q),r/m}\cdots[\frac{\alpha+m-1}{m}]_{(p,q),r/m}\\ &= \frac{(p^\alpha\ominus q^\alpha)^r_{p,q}}{(p-q)^r}\end{aligned}$$

Hence the result. □

Remark 3: Similarly, by following the same type of calculations, we can also derive the following identities:

$$\prod_{k=0}^{m-1}\Gamma_{p,q}(1-\frac{\alpha+r+k}{m})=\frac{(pq)^{\frac{2r\alpha+r^2-r}{2m}}(p-q)^r}{(-1)^r(p^\alpha\ominus q^\alpha)^r_{p,q}}\prod_{k=0}^{m-1}\Gamma_{p,q}(1-\frac{\alpha+k}{m}) \tag{10}$$

$$\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha-r+k}{m})=\frac{(pq)^{\frac{r}{2m}(r-2\alpha+1)}(p-q)^r}{(-1)^r(p^{1-\alpha}\ominus q^{1-\alpha})^r_{p,q}}\prod_{k=0}^{m-1}\Gamma_{p,q}(\frac{\alpha+k}{m}) \tag{11}$$

$$\prod_{k=0}^{m-1}\Gamma_{p,q}(1-\frac{\alpha-r+k}{m})=\frac{(q^{1-\alpha}\ominus q^{1-\alpha})^r_{p,q}}{(p-q)^r}\prod_{k=0}^{m-1}\Gamma_{p,q}(1-\frac{\alpha+k}{m}) \tag{12}$$

Here and in the following, $\triangle(\mu,\alpha)$ denotes the array of μ parameters:

$$\frac{\alpha}{\mu},\frac{\alpha+1}{\mu},...,\frac{\alpha+\mu-1}{\mu};(\mu=1,2,3,...) \tag{13}$$

and

$$(\triangle(\mu,\alpha),\beta) \text{ stands for } (\frac{\alpha}{\mu},\beta),(\frac{\alpha+1}{\mu},\beta),...,(\frac{\alpha+\mu-1}{\mu},\beta) \tag{14}$$

3.2. Generating Functions satisfying Truesdell's ascending $F_{p,q}$-equation. In this section, we will derive various forms of the $(p,q)-$analogue of Aleph-function, which satisfy Truesdell's ascending $F_{p,q}$-equation.

Theorem 3.2. *The following form of the Aleph-function satisfies Truesdell's ascending $F_{p,q}$-equation:*

$$\left((pq)^{\frac{\alpha-1}{2}[\frac{\rho-1}{\rho}]}z\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\left((pq)^{\alpha h(\lambda-1)}z^{h\lambda};(p,q)\right.$$
$$\left.\left|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha),h)\\(\triangle(\lambda,\alpha),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha),h)\end{matrix}\right)\right] \tag{15}$$

Proof. Expecting that structure (15) is $A(z,\alpha)$, supplanting $(p,q)-$analogue of Aleph-function according to its definition (2) and then interchanging order of integration and differentiation, which is legitimized under the conditions of convergence [42], we see that

$$D^r_{(p,q),z}A(z,\alpha)=\frac{1}{2\pi\omega}\int_L\frac{\prod\limits_{k=0}^{\lambda-1}\Gamma_{p,q}(\frac{\alpha+k}{\lambda}-hs)\prod\limits_{j=\lambda+1}^{m}\Gamma_{p,q}(b_j-\beta_j s)}{\sum\limits_{i=1}^{l}\tau_i[\prod\limits_{j=m+1}^{q_i-\rho}\Gamma_{p,q}(1-b_{ji}+B_{ji}s)\prod\limits_{k=0}^{\rho-1}\Gamma_{p,q}(1-\frac{\alpha+k}{\rho}+hs)}$$
$$\times\frac{\prod\limits_{j=1}^{n}\Gamma_{p,q}(1-a_j-\alpha_j s)(pq)^{\frac{\alpha(\alpha-1)}{2\rho}-\frac{\alpha(\alpha-1)}{2}}(pq)^{\alpha h(\lambda-1)s}D^r_{(p,q),z}z^{h\lambda s-\alpha}\pi}{\prod\limits_{j=n+1}^{p_i-\rho}\Gamma_{p,q}(a_{ji}-\alpha_{ji}s)\prod\limits_{k=0}^{\rho-1}\Gamma_{p,q}(\frac{\alpha+k}{\rho}-hs)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)\sin\pi s]}ds \tag{16}$$

Using results (9) and (10), we see that

$$\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(\frac{\alpha+k}{\lambda}-hs)=\frac{(p-q)^r}{(p^{\alpha-h\lambda s}\ominus q^{\alpha-h\lambda s})^r_{(p,q)}}\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(\frac{\alpha+r+k}{\lambda}-hs) \tag{17}$$

$$\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(1-\frac{\alpha+k}{\lambda}+hs)=\frac{(-1)^r(p^{\alpha-h\lambda s}\ominus q^{\alpha-h\lambda s})^r_{(p,q)}}{(p-q)^r(pq)^{\frac{2r(\alpha-h\lambda s)+r^2-r}{2\lambda}}}\prod_{k=0}^{\lambda-1}\Gamma_{(p,q)}(1-\frac{\alpha+r+k}{\lambda}+hs) \tag{18}$$

Using the identities (17)-(18) in (16) we get the required Truesdell's ascending $F_{p,q}$-Eq. (7).

□

Remark 4: Similarly the following structures (19) to (23) can be shown to fulfill Truesdell's ascending $F_{p,q}$-condition:

$$(\frac{(pq)^{\frac{1}{2}[\frac{\alpha}{\lambda}+\alpha-1]}z}{(p-q)})^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\left((pq)^{\alpha h(\lambda+1)}z^{h\lambda};(p,q)\right.$$
$$\left.\left|\begin{matrix}(\triangle(\lambda,\alpha+1/2),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(2\lambda,2\alpha),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{19}$$

$$(\frac{(pq)^{\frac{1}{2}[\frac{3\alpha+1}{3\lambda}+\alpha-1]}z}{(p-q)})^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg((pq)^{\alpha h(\lambda+1)}z^{h\lambda};(p,q)$$
$$\Bigg|\begin{matrix}(\triangle(\lambda,\alpha+\frac{2}{3}),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\lambda},(\triangle(\lambda,\alpha+1/3),h)\\(\triangle(3\lambda,3\alpha),h),(b_j,\beta_j)_{3\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\Bigg)\Bigg] \tag{20}$$

$$\Bigg((pq)^{\frac{\alpha-1}{2}}z\Bigg)^{-\alpha}e^{\pi i\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg(\Bigg((pq)^{\alpha}z\Bigg)^{h\lambda};(p,q)$$
$$\Bigg|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\Bigg)\Bigg] \tag{21}$$

$$\Bigg((pq)^{\frac{\alpha-1}{2}\frac{\lambda-1}{\lambda}}z\Bigg)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg((pq)^{h\alpha(\lambda-1)}z^{h\lambda};(p,q)$$
$$\Bigg|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\Bigg)\Bigg] \tag{22}$$

$$\Bigg((pq)^{\frac{\alpha-1}{2}\frac{\lambda-1}{\lambda}+\frac{1}{\rho}}z\Bigg)^{-\alpha}e^{\pi i\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg(\Bigg((pq)^{\alpha}z\Bigg)^{h\lambda};(p,q)$$
$$\Bigg|\begin{matrix}(\triangle(\rho,\alpha),h),(a_j,\alpha_j)_{\rho+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\rho,\alpha),h),(b_j,\beta_j)_{\rho+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda}(\triangle(\lambda,\alpha),h)\end{matrix}\Bigg)\Bigg] \tag{23}$$

Next we use the above forms (15) and (19)-(23) to establish the following new generating functions for the $(p,q)-$analogue of the Aleph-functions using Truesdell's ascending $F_{p,q}$-equation technique.
For example, we establish the following generating function (24) by substituting the structure (15) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and $(pq)^{-\alpha h}y^{h\lambda}$ by x, respectively:

$$\Bigg(1+(pq)^{\alpha}\Bigg)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg(\Bigg(1+(pq)^{\alpha}\Bigg)^{h\lambda}x;(p,q)$$
$$\Bigg|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha),h)\\(\triangle(\lambda,\alpha),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha),h)\end{matrix}\Bigg)\Bigg]$$
$$=\sum_{r=0}^{\infty}\frac{\Bigg((pq)^{\frac{(-2\alpha-r+1)((\rho-1)/\rho)}{2}+2\alpha}\Bigg)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\Bigg[\Bigg((pq)^{rh(\lambda-1)}x;(p,q)$$
$$\Bigg|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha+r),h)\\(\triangle(\lambda,\alpha+r),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha+r),h)\end{matrix}\Bigg)\Bigg] \tag{24}$$

Similarly, we establish the following generating function (25) by putting the structures (19) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and $(pq)^{\alpha h}y^{h\lambda}$

by x, respectively:

$$\left(1+(pq)^{\alpha}\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\left|\begin{matrix}(\triangle(\lambda,\alpha+1/2),h)(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(2\lambda,2\alpha),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{\left((pq)^{\frac{-1}{2}(\frac{2\alpha}{\lambda}+\frac{r}{\lambda}-1)(p-q)}\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{rh(\lambda+1)}x;(p,q)\right.\right.$$
$$\left.\left.\left|\begin{matrix}(\triangle(\lambda,\alpha+r+1/2),h)(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(2\lambda,2(\alpha+r)),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{25}$$

Following the generating function (26) can be established by substituting the structures (20) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and $(pq)^{\alpha h}y^{h\lambda}$ by x, respectively:

$$\left(1+(pq)^{\alpha}\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\left|\begin{matrix}(\triangle(\lambda,\alpha+2/3),h)(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\lambda}(\triangle(\lambda,\alpha+1/3),h)\\(\triangle(3\lambda,3\alpha),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{\left((pq)^{\frac{-1}{2}(\frac{\alpha}{\lambda}+\frac{r}{2\lambda}+\frac{r}{2}+\frac{1}{6\lambda}-\frac{1}{2})(p-q)}\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{rh(\lambda+1)}x;(p,q)\right|(\triangle\right.$$
$$\left.\left.\begin{matrix}(\lambda,\alpha+r+2/3),h)(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\lambda}(\triangle(\lambda,\alpha+r+1/3),h)\\(\triangle(3\lambda,3(\alpha+r)),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{26}$$

Similarly, we establish the following generating function (27) by substituting the structures (21) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and $y^{h\lambda}$ by x, respectively:

$$\left(1+(pq)^{\alpha}\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\left|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^{r}\left((pq)^{\frac{1-r}{2}}\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{rh\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\left|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha+r),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{27}$$

To obtain (28) we put the structures (22) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and $(pq)^{-\alpha h}y^{h\lambda}$ by x, respectively:

$$\left(1+(pq)^{\alpha}\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda}(\triangle(\lambda,\alpha),h)\end{matrix}\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{\left((pq)^{\frac{(\lambda-1)}{\lambda}(-\alpha-\frac{r}{2}+\frac{1}{2})+\alpha}\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{rh(\lambda-1)}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda}(\triangle(\lambda,\alpha+r),h)\end{matrix}\right)\right] \tag{28}$$

Similarly we obtain (29) by putting the structures (23) in Truesdell's ascending $F_{p,q}$-condition (7) and supplant z by $\frac{y}{(pq)^{\alpha}}$ and$y^{h\lambda}$ by x, respectively:

$$\left(1+(pq)^{\alpha}\right)^{-\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\rho,\alpha),h),(a_j,\alpha_j)_{\rho+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\rho,\alpha),h),(b_j,\beta_j)_{\rho+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda}(\triangle(\lambda,\alpha),h)\end{matrix}\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^{r}\left((pq)^{(\frac{}{1}\lambda-\frac{1}{\rho}-1)(\alpha+\frac{r}{2}-\frac{1}{2})+\alpha}\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{rh\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\rho,\alpha+r),h),(a_j,\alpha_j)_{\rho+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\rho,\alpha+r),h),(b_j,\beta_j)_{\rho+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda}(\triangle(\lambda,\alpha+r),h)\end{matrix}\right)\right] \tag{29}$$

Remark 5: The above results yield certain special cases of the generating function for the $(p,q)-$analogue and $q-$analogue of the I-function, H-function and G-function [1, 2, 4].

3.3. Generating Functions satisfying Truesdell's descending $F_{p,q}$-eqn.

In this subsection we will derive the different forms of the $(p,q)-$analogue of the Aleph function, which satisfy Truesdell's descending $F_{p,q}$-equation.

Theorem 3.3. *The following form of the $(p,q)-$analogue of Aleph function satisfies Truesdell's descending $F_{p,q}$-equation:*

$$\left((pq)^{\frac{\alpha}{2}[\frac{1-\rho}{\rho}]}z\right)^{\alpha-1}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{\alpha h(\lambda+1)}}{z^{h\lambda}};(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\lambda,\alpha),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha),h)\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha),h)\end{matrix}\right)\right] \tag{30}$$

Proof. Assuming that the form (30) is $B(z,\alpha)$, replacing $(p,q)-$analogue of Aleph-function by its statement (2), then exchanging order of differentiation and integration, which is legitimized under the states of combination [24], we observe that

$$D^{r}_{(p,q),z}B(z,\alpha)=\frac{1}{2\pi\omega}\int_{L}\frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-\beta_j s)\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(1-\frac{\alpha+k}{\lambda}-hs)}{\sum_{i=1}^{l}\tau_i[\prod_{j=m+1}^{q_i-\rho}\Gamma_{p,q}(1-b_{ji}+B_{ji}s)\prod_{k=0}^{\rho-1}\Gamma_{p,q}(1-\frac{\alpha+k}{\rho}+hs)}\times$$

$$\frac{\prod_{j=\lambda+1}^{n}\Gamma_{p,q}(1-a_j+\alpha_j s)(pq)^{\frac{\alpha(\alpha-1)(1-\rho)}{2\rho}}(pq)^{\alpha h(\lambda+1)s}D^{r}_{(p,q),z}z^{\alpha-1-h\lambda s}\pi}{\prod_{j=n+1}^{p_i-\rho}\Gamma_{p,q}(a_{ji}-\alpha_{ji}s)\prod_{k=0}^{\rho-1}\Gamma_{p,q}(\frac{\alpha+k}{\rho}-hs)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)\sin\pi s]}ds \tag{31}$$

Using results (11) and (12), we have

$$\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(\frac{\alpha+k}{\lambda}-hs)=\frac{(-1)^r(p^{1-(\alpha-h\lambda s)}\ominus q^{1-(\alpha-h\lambda s)})^r_{(p,q)}}{(p-q)^r(pq)^{\frac{r}{2m}(r-2(\alpha-h\lambda s)+1)}}\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(\frac{\alpha-r+k}{\lambda}-hs) \tag{32}$$

$$\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(1-\frac{\alpha+k}{\lambda}+hs)=\frac{(p-q)^r}{(p^{1-(\alpha-h\lambda s)}\ominus q^{1-(\alpha-h\lambda s)})^r_{(p,q)}}\prod_{k=0}^{\lambda-1}\Gamma_{p,q}(1-\frac{\alpha-r+k}{\lambda}+hs) \tag{33}$$

Making use of identities (32) and (33) in (31) we get the required Truesdell's form of the descending $F_{p,q}$-Eq. (8). $\square$

Remark 6: In a similar fashion, the following forms (34) to (38) can be shown to satisfy the descending $F_{p,q}$-condition:

$$\frac{\left(q^{\frac{-\alpha}{2}[\frac{1+\lambda}{\lambda}]}z\right)^{\alpha-1}}{(p-q)^{\alpha}}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{\alpha h(\lambda-1)}}{z^{h\lambda}};(p,q)\,\middle|\,\begin{matrix}(\triangle(2\lambda,2\alpha),h),(a_j,\alpha_j)_{2\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha+1/2),h),(b_j,\beta_j)_{2\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{34}$$

$$\frac{\left((pq)^{\frac{-\alpha}{2(\alpha-1)}[\frac{3\alpha+1}{3\lambda}+\alpha-1]}z\right)^{\alpha-1}}{(p-q)^{\alpha}}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{\alpha h(\lambda+1)}}{z^{h\lambda}};(p,q)\,\middle|\,\begin{matrix}(\triangle(3\lambda,3\alpha),h),(a_j,\alpha_j)_{3\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\triangle(\lambda,\alpha+2/3),h),(b_j,\beta_j)_{\lambda+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\lambda},(\triangle(\lambda,\alpha+1/3),h)\end{matrix}\right)\right] \tag{35}$$

$$\left((pq)^{\frac{-\alpha}{2}}z\right)^{\alpha-1}e^{\pi i\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{\alpha h\lambda}}{z^{h\lambda}};(p,q)\,\middle|\,\begin{matrix}(\triangle(\lambda,\alpha),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{36}$$

$$\left((pq)^{\frac{\alpha}{2}[\frac{1-\lambda}{\lambda}]}z\right)^{\alpha-1}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{h\alpha(\lambda-1)}}{z^{h\lambda}};(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\lambda},(\triangle(\lambda,\alpha),h)\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{37}$$

$$\left((pq)^{\frac{\alpha}{2}[\frac{1-\lambda}{\lambda}-\frac{1}{\rho}]}z\right)^{\alpha-1}e^{\pi i\alpha}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\frac{(pq)^{\alpha h\lambda}}{z^{h\lambda}};(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\rho,\alpha),h),(a_j,\alpha_j)_{\rho+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\lambda},(\triangle(\lambda,\alpha),h),\\(\triangle(\rho,\alpha),h),(b_j,\beta_j)_{\rho+1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \tag{38}$$

Next we use the above forms (30) and (34)-(38) and establish the following generating functions for $(p,q)-$analogue of Aleph-functions by applying the descending $F_{p,q}$-equation technique.
To obtain (39) we put the structure (30) in Truesdell's descending $F_{p,q}$-Eq. (8) and replace z by $\frac{y}{(pq)^{\alpha}}$ and $\frac{(pq)^{\alpha h+2\alpha h\lambda}}{y^{h\lambda}}$ by x in progression to get the required outcomes:

$$\left(1+(pq)^{\alpha}\right)^{\alpha-1}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(\left(1+(pq)^{\alpha}\right)^{-h\lambda}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\lambda,\alpha),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha),h)\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha),h)\end{matrix}\right)\right]$$
$$=\sum_{r=0}^{\infty}\frac{\left((pq)^{(-\alpha+r/2+1/2)((1-\rho)/\rho)}+\alpha\right)^{r}}{r!}\aleph^{m,n}_{p_i,q_i;\tau_i;l}\left[\left(((pq)^{-rh(\lambda+1)}x;(p,q)\right.\right.$$
$$\left.\left.\middle|\begin{matrix}(\triangle(\lambda,\alpha-r),h),(a_j,\alpha_j)_{\lambda+1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i-\rho},(\triangle(\rho,\alpha-r),h)\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i-\rho},(\triangle(\rho,\alpha-r),h)\end{matrix}\right)\right] \tag{39}$$

Remark 7: The above results yield certain special cases of the generating function for the $(p,q)-$analogue and $q-$analogue of the I-function, H-function and G-function [2, 4].

Conclusion

This paper presents different types of $(p,q)-$analogue of Aleph-function fulfilling Truesdell's ascending and descending $F_{p,q}$-equation. These structures have been utilized to arrive at certain generating functions for $(p,q)-$analogue of Aleph-function.The outcomes demonstrated in this paper alongside their specific cases are accepted to be new. As these functions have well established as applicable functions, these outcomes are probably going to contribute fundamentally in certain utilization of the hypothesis of $(p,q)-$calculus.

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