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(inf, sup)-HESITANT FUZZY BI-IDEALS OF SEMIGROUPS†

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Abstract. In this paper, we introduce the concepts of (inf, sup)-hesitant fuzzy subsemigroups and (inf, sup)-hesitant fuzzy (generalized) bi-ideals of semigroups, and investigate their properties. The concepts are established in terms of sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets, and bipolar fuzzy sets. Moreover, some characterizations of bi-ideals, fuzzy bi-ideals, anti-fuzzy bi-ideals, negative fuzzy bi-ideals, Pythagorean fuzzy bi-ideals, and bipolar fuzzy bi-ideals of semigroups are given in terms of the (inf, sup)-type of hesitant fuzzy sets. Also, we characterize a semigroup which is completely regular, a group and a semilattice of groups by (inf, sup)-hesitant fuzzy bi-ideals.

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1. Introduction

The concept of fuzzy sets introduced by Zadeh [48] laid the foundation for the development of fuzzy mathematics and has been successfully applied in many branches such as logic theory, real analysis, topology, group theory, graph theory, semigroup theory, computer science, finite state machine, control engineering, automata theory and robotics. Moreover, in the literature, a number of types of fuzzy sets and their generalizations and extensions have been introduced and studied, for instance, anti-type of fuzzy sets [29, 39], negative fuzzy sets [23], intuitionistic fuzzy sets [3, 11, 12], interval-valued fuzzy sets [49], Pythagorean

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fuzzy sets [45, 46], bipolar fuzzy sets [13, 10, 31, 50], hesitant fuzzy sets [1, 43, 44, 47], cubic sets [24, 25] and hybrid sets [2, 28].

The concept of interval-valued fuzzy sets was first introduced by Zadeh [49] as an extension of fuzzy sets, and has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Narayanan and Manikantan [35], and Thillaigovindan and Chinnadurai [42] applied interval-valued fuzzy sets to semigroups. Cheong and Hur [4] studied some properties of interval-valued fuzzy bi-ideals of semigroups and characterized a semigroup which is completely regular, a group and a semilattice of groups in terms of interval-valued fuzzy bi-ideals. Lee et al. [32] introduced interval-valued fuzzy generalized bi-ideals of semigroups and characterized semigroups by interval-valued fuzzy generalized bi-ideals. As an extension of fuzzy sets and a generalization of interval-valued fuzzy sets, Torra [43] introduced the concept of hesitant fuzzy sets. Jun et al. [26] introduced hesitant fuzzy (generalized) bi-ideals of semigroups and discussed their properties. In general, an interval-valued fuzzy (generalized) bi-ideal of a semigroup is not a hesitant fuzzy (generalized) bi-ideal and a hesitant fuzzy (generalized) bi-ideal of a semigroup is not an interval-valued fuzzy (generalized) bi-ideal. Julatha et al. [17] introduced a sup-hesitant fuzzy (generalized) bi-ideal which is a generalization of an interval-valued fuzzy (generalized) bi-ideal of a semigroup and studied its characterizations in terms of sets, fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets, hesitant fuzzy sets, cubic sets and hybrid sets. Many researchers have studied hesitant fuzzy sets on semigroups and algebraic structures (see [2, 9, 16, 18, 20, 34, 37, 40, 41, 47]).

Recently, Ratchakhwan et al. [38] introduced an $(\inf, \sup)$-hesitant fuzzy ideal, which is a generalization of an interval-valued fuzzy ideal, of a BCK/BCI-algebra and investigated its related properties via sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, bipolar fuzzy sets and hesitant fuzzy sets. Chunsee et al. [6] introduced an $(\inf, \sup)$-hesitant fuzzy subalgebra, which is a general concept of an interval-valued fuzzy subalgebra, of a BCK/BCI-algebra and characterized it in terms of sets, fuzzy sets, hesitant fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, negative fuzzy sets and bipolar fuzzy sets. Abbasi et al. [1] introduced the notions of hesitant fuzzy left (resp., right and two-sided) ideals, hesitant fuzzy bi-ideals, and hesitant fuzzy interior ideals in Γ-semigroups and characterized simple Γ-semigroups by means of hesitant fuzzy simple Γ-semigroups. In this year, Yiarayong [47] introduced the notions of (α, β)-hesitant fuzzy subsemigroups and (α, β)-hesitant fuzzy ideals in semigroups.

As previously stated, it motivated us to study the $(\inf, \sup)$-type of hesitant fuzzy sets on semigroups. We introduce $(\inf, \sup)$-hesitant fuzzy subsemigroups, $(\inf, \sup)$-hesitant fuzzy generalized bi-ideals and $(\inf, \sup)$-hesitant fuzzy bi-ideals of semigroups and look into their associated characteristics. It is shown that every interval-valued fuzzy bi-ideal (resp., subsemigroup, generalized bi-ideal) of a semigroup is an $(\inf, \sup)$-hesitant fuzzy bi-ideal (resp., subsemigroup, generalized bi-ideal), but the converse is not true. Characterizations of (inf, sup)-hesitant fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals) of semigroups are established in terms of sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets and bipolar fuzzy sets. Also, we characterize bi-ideals (resp., subsemigroups, generalized bi-ideals), fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals), anti-fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals), negative fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals), Pythagorean fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals) and bipolar fuzzy bi-ideals (resp., subsemigroups, generalized bi-ideals) of semigroups by (inf, sup)-type of hesitant fuzzy sets. Finally, we characterize a semigroup which is completely regular, a group and a semilattice of groups in terms of (inf, sup)-hesitant fuzzy bi-ideals.

2. Preliminaries

We begin by providing some fundamental concepts and results that will be applied throughout this study.

In what follows, unless otherwise specified, let $\mathcal{R}$ be the set of all real numbers, $\mathcal{N}$ be the set of all positive integers, $\mathcal{A}$ be a semigroup, $\mathcal{X}$ be a nonempty set, $\wp(\mathcal{X})$ be the power set of $\mathcal{X}$ and $\Pi, \Psi \in \wp([0,1])$. For an arbitrary element r of $\mathcal{R}$ and arbitrary functions φ and δ from $\mathcal{X}$ into $\mathcal{R}$, we define

$$r\varphi : \mathcal{X} \to \mathcal{R}, p \mapsto r\varphi(p), \tag{1}$$

$$r + \varphi : \mathcal{X} \to \mathcal{R}, p \mapsto r + \varphi(p), \tag{2}$$

$$\varphi - r : \mathcal{X} \to \mathcal{R}, p \mapsto \varphi(p) - r, \tag{3}$$

$$\varphi \leq \delta \Leftrightarrow (\forall p \in \mathcal{X})(\varphi(p) \leq \delta(p)). \tag{4}$$

We denote $-\varphi$, $r - \varphi$ and $\frac{\varphi}{r}$ for $(-1)\varphi$, $r + (-\varphi)$ and $(\frac{1}{r})\varphi$ (when $r \neq 0$), respectively.

A nonempty subset $\mathcal{X}$ of $\mathcal{A}$ is called a *subsemigroup* (SS) (resp., *generalized bi-ideal* (GBI)) of $\mathcal{A}$ if $\mathcal{X}\mathcal{X} \subseteq \mathcal{X}$ (resp., $\mathcal{X}\mathcal{A}\mathcal{X} \subseteq \mathcal{X}$) and a *bi-ideal* (BI) of $\mathcal{A}$ if $\mathcal{X}$ is both a SS and a GBI of $\mathcal{A}$. A *fuzzy subset* (FS) [48] of $\mathcal{X}$ is defined to be a mapping $\varphi : \mathcal{X} \to [0,1]$. A FS φ of $\mathcal{A}$ is called

(1) a *fuzzy subsemigroup* (FSS) [33] of $\mathcal{A}$ if $\min\{\varphi(p), \varphi(q)\} \leq \varphi(pq)$ for all $p, q \in \mathcal{A}$,
(2) a *fuzzy generalized bi-ideal* (FGBI) [33] of $\mathcal{A}$ if $\min\{\varphi(p), \varphi(q)\} \leq \varphi(pzq)$ for all $p, q, z \in \mathcal{A}$,
(3) a *fuzzy bi-ideal* (FBI) [33] of $\mathcal{A}$ if it is both a FSS and a FGBI of $\mathcal{A}$,
(4) an *anti-fuzzy subsemigroup* (AFSS) [22, 39] of $\mathcal{A}$ if $\varphi(pq) \leq \max\{\varphi(p), \varphi(q)\}$ for all $p, q \in \mathcal{A}$,
(5) an *anti-fuzzy generalized bi-ideal* (AFGBI) [22, 39] of $\mathcal{A}$ if $\varphi(pzq) \leq \max\{\varphi(p), \varphi(q)\}$ for all $p, q, z \in \mathcal{A}$,
(6) an *anti-fuzzy bi-ideal* (AFBI) [22, 39] of $\mathcal{A}$ if it is both an AFSS and an AFGBI of $\mathcal{A}$.

A *Pythagorean fuzzy set* (PFS) [45, 46] in $\mathcal{X}$ is an object having the form $P = \{(p, \varphi(p), \delta(p)) \,|\, p \in \mathcal{X}\}$ when the functions $\varphi : \mathcal{X} \to [0,1]$ denote the degree of membership and $\delta : \mathcal{X} \to [0,1]$ denote the degree of nonmembership, and $0 \leq (\varphi(p))^2 + (\delta(p))^2 \leq 1$ for all $p \in \mathcal{X}$. We denote (φ, δ) for the PFS $\{(p, \varphi(p), \delta(p)) \mid p \in \mathcal{X}\}$. Note that the concept of PFSs is an extension of the concept of FSs. A PFS (φ, δ) in $\mathcal{A}$ is called

(1) a *Pythagorean fuzzy subsemigroup* (PFSS) [5, 17] of $\mathcal{A}$ if φ is a FSS and δ is an AFSS of $\mathcal{A}$,
(2) a *Pythagorean fuzzy generalized bi-ideal* (PFGBI) [17] of $\mathcal{A}$ if φ is a FGBI and δ is an AFGBI of $\mathcal{A}$,
(3) a *Pythagorean fuzzy bi-ideal* (PFBI) [5, 17] of $\mathcal{A}$ if it is both a PFSS and a PFGBI of $\mathcal{A}$, that is, φ is a FBI and δ is an AFBI of $\mathcal{A}$.

By an interval number $\breve{a}$ we mean an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. We denote $\mathcal{D}[0,1]$ for the set of all interval numbers. For $\breve{a} = [a^-, a^+]$, $\breve{b} = [b^-, b^+] \in \mathcal{D}[0,1]$, define the operations $\preceq$, $=$, $\prec$ and rmin as follows:

(1) $\breve{a} \preceq \breve{b} \Leftrightarrow a^- \leq b^-$ and $a^+ \leq b^+$,
(2) $\breve{a} = \breve{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$,
(3) $\breve{a} \prec \breve{b} \Leftrightarrow \breve{a} \preceq \breve{b}$ and $\breve{a} \neq \breve{b}$,
(4) $\mathrm{rmin}\{\breve{a}, \breve{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$.

An *interval-valued fuzzy set* (IvFS) [49] on $\mathcal{X}$ is defined to be a mapping $\breve{\varpi} : \mathcal{X} \to \mathcal{D}[0,1]$, where $\breve{\varpi}(p) = [\breve{\varpi}^-(p), \breve{\varpi}^+(p)]$ for all $p \in \mathcal{X}$, $\breve{\varpi}^-$ and $\breve{\varpi}^+$ are FSs of $\mathcal{X}$ such that $\breve{\varpi}^- \leq \breve{\varpi}^+$. Thus the concept of IvFSs is an extension of the concept of FSs. An IvFS $\breve{\varpi} = [\breve{\varpi}^-, \breve{\varpi}^+]$ on $\mathcal{A}$ is called

(1) an *interval-valued fuzzy subsemigroup* (IvFSS) [42] of $\mathcal{A}$ if $\mathrm{rmin}\{\breve{\varpi}(p), \breve{\varpi}(q)\} \preceq \breve{\varpi}(pq)$ for all $p, q \in \mathcal{A}$, that is, $\breve{\varpi}^-$ and $\breve{\varpi}^+$ are FSSs of $\mathcal{A}$,
(2) an *interval-valued fuzzy generalized bi-ideal* (IvFGBI) [42] of $\mathcal{A}$ if $\mathrm{rmin}\{\breve{\varpi}(p), \breve{\varpi}(q)\} \preceq \breve{\varpi}(pzq)$ for all $p, q, z \in \mathcal{A}$, that is, $\breve{\varpi}^-$ and $\breve{\varpi}^+$ are FGBIs of $\mathcal{A}$,
(3) an *interval-valued fuzzy bi-ideal* (IvFBI) [42] of $\mathcal{A}$ if it is both an IvFSS and an IvFGBI of $\mathcal{A}$, that is, $\breve{\varpi}^-$ and $\breve{\varpi}^+$ are FBIs of $\mathcal{A}$.

A *negative fuzzy subset* (NFS)[27] of $\mathcal{X}$ is a mapping from $\mathcal{X}$ into $[-1, 0]$. A NFS φ of $\mathcal{A}$ is called

(1) a *negative fuzzy subsemigroup* (NFSS) of $\mathcal{A}$ if $\varphi(pq) \leq \max\{\varphi(p), \varphi(q)\}$ for all $p, q \in \mathcal{A}$,
(2) a *negative fuzzy generalized bi-ideal* (NFGBI) of $\mathcal{A}$ if $\varphi(pzq) \leq \max\{\varphi(p), \varphi(q)\}$ for all $p, q, z \in \mathcal{A}$,
(3) a *negative fuzzy bi-ideal* (NFBI) of $\mathcal{A}$ if it is both a NFSS and a NFGBI of $\mathcal{A}$.

A *bipolar fuzzy set* (BFS) [50] in $\mathcal{X}$ is an object having the form $\{(p, \varphi(p), \delta(p)) \mid p \in \mathcal{X}\}$, where φ is a NFS and δ is a FS of $\mathcal{X}$. We denote $\langle\varphi, \delta\rangle$ for the

BFS $\{(p, \varphi(p), \delta(p)) \mid p \in \mathcal{X}\}$. Note that the concept of BFSs is an extension of the concepts of FSs and NFSs. A BFS $\langle \varphi, \delta \rangle$ in $\mathcal{A}$ is called

(1) a *bipolar fuzzy subsemigroup* (BFSS) [30] of $\mathcal{A}$ if φ is a NFSS and δ is a FSS of $\mathcal{A}$,
(2) a *bipolar fuzzy generalized bi-ideal* (BFGBI) [30] of $\mathcal{A}$ if φ is a NFGBI and δ is a FGBI of $\mathcal{A}$,
(3) a *bipolar fuzzy bi-ideal* (BFBI) [30] of $\mathcal{A}$ if it is both a BFSS and a BFGBI of $\mathcal{A}$.

A *hesitant fuzzy set* (HFS) [43, 44] on $\mathcal{X}$ is defined to be a mapping $\widetilde{\varepsilon} : \mathcal{X} \to \wp([0,1])$. Note that every IvFS on $\mathcal{X}$ is a HFS on $\mathcal{X}$. A HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ is called

(1) a *hesitant fuzzy subsemigroup* (HFSS) [26] of $\mathcal{A}$ if $\widetilde{\varepsilon}(p) \cap \widetilde{\varepsilon}(q) \subseteq \widetilde{\varepsilon}(pq)$ for all $p, q \in \mathcal{A}$,
(2) a *hesitant fuzzy generalized bi-ideal* (HFGBI)[26] of $\mathcal{A}$ if $\widetilde{\varepsilon}(p) \cap \widetilde{\varepsilon}(q) \subseteq \widetilde{\varepsilon}(pzq)$ for all $p, q, z \in \mathcal{A}$,
(3) a *hesitant fuzzy bi-ideal* (HFBI) [26] of $\mathcal{A}$ if it is both a HFSS and a HFGBI of $\mathcal{A}$.

Remark 2.1. The following conditions are true.

(1) If φ is a FS of $\mathcal{X}$, then $\varphi - 1$ and $-\varphi$ are NFSs of $\mathcal{X}$.
(2) If φ is a NFS of $\mathcal{X}$, then $\varphi + 1$ and $-\varphi$ are FSs of $\mathcal{X}$.
(3) $\langle \varphi - 1, \varphi \rangle$ is a BFS in $\mathcal{X}$ for all FS φ of $\mathcal{X}$.
(4) $\langle \varphi - 1, \delta \rangle$ and $\langle \delta - 1, \varphi \rangle$ are BFSs in $\mathcal{X}$ for all PFS (φ, δ) in $\mathcal{X}$.
(5) $(\frac{1+\varphi}{1+n}, \frac{\delta}{1+n})$ and $(\frac{\delta}{1+n}, \frac{1+\varphi}{1+n})$ are PFSs in $\mathcal{X}$ for all BFS $\langle \varphi, \delta \rangle$ in $\mathcal{X}$ and $n \in \mathcal{N}$.
(6) $(\varphi, 1 - \varphi)$, $(\frac{\varphi}{1+n}, \frac{\delta}{1+n})$, $(\frac{n+\varphi}{1+2m}, \frac{n+\delta}{1+2m})$ and $(\frac{\varphi}{1+n}, \frac{\varphi}{1+n})$ are PFSs in $\mathcal{X}$ for each FSs φ and δ of $\mathcal{X}$ and $n, m \in \mathcal{N}$ such that $n \leq m$.

3. (inf, sup)-hesitant fuzzy bi-ideals

In this section, we introduce concepts of (inf, sup)-hesitant fuzzy subsemigroups, (inf, sup)-hesitant fuzzy generalized bi-ideals and (inf, sup)-hesitant fuzzy bi-ideals of semigroups and investigate some of their properties via sets, FSs, NFSs, IvFSs, PFSs, HFSs and BFSs.

For each element $\Pi \in \wp([0,1])$ and HFS $\widetilde{\varepsilon}$ on $\mathcal{X}$, define the elements $\mathrm{SUP}\,\Pi$ [16, 21] and $\mathrm{INF}\,\Pi$ [14, 15] of $[0,1]$ and the subset $[\mathcal{X}, \widetilde{\varepsilon}, \Pi]$ [6, 38] of $\mathcal{X}$ as follows:

$$\mathrm{SUP}\,\Pi = \begin{cases} \sup \Pi & \text{if } \Pi \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathrm{INF}\,\Pi = \begin{cases} \inf \Pi & \text{if } \Pi \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$[\mathcal{X}, \widetilde{\varepsilon}, \Pi] = \{p \in \mathcal{X} \mid \mathrm{SUP}\,\widetilde{\varepsilon}(p) \geq \mathrm{SUP}\,\Pi, \mathrm{INF}\,\widetilde{\varepsilon}(p) \geq \mathrm{INF}\,\Pi\}.$$

Definition 3.1. A HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ is said to be

(1) an *(*$\inf$*,* $\sup$*)-hesitant fuzzy subsemigroup* (($\inf$, $\sup$)-HFSS) of $\mathcal{A}$ if $[\mathcal{A}, \widetilde{\varepsilon}, \Pi]$ is an empty set or a SS of $\mathcal{A}$ for all $\Pi \in \wp([0,1])$.
(2) an *(*$\inf$*,* $\sup$*)-hesitant fuzzy generalized bi-ideal* (($\inf$, $\sup$)-HFGBI) of $\mathcal{A}$ if $[\mathcal{A}, \widetilde{\varepsilon}, \Pi]$ is an empty set or a GBI of $\mathcal{A}$ for all $\Pi \in \wp([0,1])$.
(3) an *(*$\inf$*,* $\sup$*)-hesitant fuzzy bi-ideal* (($\inf$, $\sup$)-HFBI) of $\mathcal{A}$ if it is both an ($\inf$, $\sup$)-HFSS and an ($\inf$, $\sup$)-HFGBI of $\mathcal{A}$.

For any HFS $\widetilde{\varepsilon}$ on $\mathcal{X}$, define the FSs $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ of $\mathcal{X}$ by $\mathrm{F}^{\widetilde{\varepsilon}}(p) = \mathrm{SUP}\,\widetilde{\varepsilon}(p)$, and $\mathrm{F}_{\widetilde{\varepsilon}}(p) = \mathrm{INF}\,\widetilde{\varepsilon}(p)$ for all $p \in \mathcal{X}$. A HFS $\widetilde{\kappa}$ on $\mathcal{X}$ is called a *supremum complement* [6, 38] of $\widetilde{\varepsilon}$ on $\mathcal{X}$ if $\mathrm{SUP}\,\widetilde{\kappa}(p) = (1 - \mathrm{F}^{\widetilde{\varepsilon}})(p)$ for all $p \in \mathcal{X}$ and called an *infimum complement* [15, 20] of $\widetilde{\varepsilon}$ on $\mathcal{X}$ if $\mathrm{INF}\,\widetilde{\kappa}(p) = (1 - \mathrm{F}_{\widetilde{\varepsilon}})(p)$ for all $p \in \mathcal{X}$. The set of all supremum complements of $\widetilde{\varepsilon}$ is denoted by $\mathrm{SC}(\widetilde{\varepsilon})$ and the set of all infimum complements of $\widetilde{\varepsilon}$ is denoted by $\mathrm{IC}(\widetilde{\varepsilon})$. Define the HFSs $\widetilde{\varepsilon}^{\pm}$ and $\widetilde{\varepsilon}^{\mp}$ on $\mathcal{X}$ by $\widetilde{\varepsilon}^{\pm}(p) = \{(1 - \mathrm{F}_{\widetilde{\varepsilon}})(p)\}$ and $\widetilde{\varepsilon}^{\mp}(p) = \{(1 - \mathrm{F}^{\widetilde{\varepsilon}})(p)\}$ for all $p \in \mathcal{X}$. Then $\widetilde{\varepsilon}^{\pm} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\varepsilon}^{\mp} \in \mathrm{SC}(\widetilde{\varepsilon})$. Moreover, $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}} = \mathrm{F}_{\widetilde{\kappa}} = 1 - \mathrm{F}_{\widetilde{\varepsilon}}$ for each $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\mathrm{F}^{\widetilde{\varepsilon}^{\mp}} = \mathrm{F}^{\widetilde{\tau}} = 1 - \mathrm{F}^{\widetilde{\varepsilon}}$ for each $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$. Next, characterizations of $(\inf,\sup)$-HFSSs, $(\inf,\sup)$-HFGBIs and $(\inf,\sup)$-HFBIs of semigroups are discussed via sets, FSs, NFSs and anti-type of FSs.

Lemma 3.2. *Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\widetilde{\varepsilon}$ is an $(\inf,\sup)$-HFGBI (resp., $(\inf,\sup)$-HFSS, $(\inf,\sup)$-HFBI) of $\mathcal{A}$.*
(2) *$\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs (resp., FSSs, FBIs) of $\mathcal{A}$.*
(3) *$-\mathrm{F}^{\widetilde{\varepsilon}}$ and $-\mathrm{F}_{\widetilde{\varepsilon}}$ are NFGBIs (resp., NFSSs, NFBIs) of $\mathcal{A}$.*
(4) *$\mathrm{F}^{\widetilde{\varepsilon}^{\mp}}$ and $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}$ are AFGBIs (resp., AFSSs, AFBIs) of $\mathcal{A}$.*
(5) *$\mathrm{F}^{\widetilde{\varepsilon}^{\mp}} - 1$ and $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}} - 1$ are NFGBIs (resp., NFSSs, NFBIs) of $\mathcal{A}$.*
(6) *$[\mathcal{A}, \widetilde{\varepsilon}, \breve{a}]$ is an empty set or a GBI (resp., SS, BI) of $\mathcal{A}$ for each $\breve{a} \in \mathcal{D}([0,1])$.*
(7) *$\mathrm{F}^{\widetilde{\tau}}$ and $\mathrm{F}_{\widetilde{\kappa}}$ are AFGBIs (resp., AFSSs, AFBIs) of $\mathcal{A}$ for each $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$.*
(8) *$\mathrm{F}^{\widetilde{\tau}} - 1$ and $\mathrm{F}_{\widetilde{\kappa}} - 1$ are NFGBIs (resp., NFSSs, NFBIs) of $\mathcal{A}$ for each $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$.*

Proof. (5) $\Leftrightarrow$ (3), (1) $\Rightarrow$ (6), (7) $\Rightarrow$ (4) and (8) $\Rightarrow$ (5). They are clear.

(3) $\Rightarrow$ (2). Assume that $-\mathrm{F}^{\widetilde{\varepsilon}}$ and $-\mathrm{F}_{\widetilde{\varepsilon}}$ are NFGBIs of $\mathcal{A}$. Let $p, q, z \in \mathcal{A}$. Then $-\mathrm{F}_{\widetilde{\varepsilon}}(pzq) \leq \max\{-\mathrm{F}_{\widetilde{\varepsilon}}(p), -\mathrm{F}_{\widetilde{\varepsilon}}(q)\}$ and $-\mathrm{F}^{\widetilde{\varepsilon}}(pzq) \leq \max\{-\mathrm{F}^{\widetilde{\varepsilon}}(p), -\mathrm{F}^{\widetilde{\varepsilon}}(q)\}$. Thus, we have

$$
\begin{aligned}
\min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} &= \min\{-(-\mathrm{F}_{\widetilde{\varepsilon}}(p)), -(-\mathrm{F}_{\widetilde{\varepsilon}}(q))\} \\
&= -(\max\{-\mathrm{F}_{\widetilde{\varepsilon}}(p), -\mathrm{F}_{\widetilde{\varepsilon}}(q)\}) \\
&\leq -(-\mathrm{F}_{\widetilde{\varepsilon}}(pzq)) \\
&= \mathrm{F}_{\widetilde{\varepsilon}}(pzq), \\
\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} &= \min\{-(-\mathrm{F}^{\widetilde{\varepsilon}}(p)), -(-\mathrm{F}^{\widetilde{\varepsilon}}(q))\} \\
&= -(\max\{-\mathrm{F}^{\widetilde{\varepsilon}}(p), -\mathrm{F}^{\widetilde{\varepsilon}}(q)\})
\end{aligned}
$$

$$\begin{aligned}&\leq -(-\mathrm{F}^{\widetilde{\varepsilon}}(pzq))\\&= \mathrm{F}^{\widetilde{\varepsilon}}(pzq).\end{aligned}$$

Hence $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$.

(2) $\Rightarrow$ (7). Assume that $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$. Let $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$, $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $p, q, z \in \mathcal{A}$. Then $\min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}_{\widetilde{\varepsilon}}(pzq)$ and $\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}^{\widetilde{\varepsilon}}(pzq)$. Thus

$$\begin{aligned}\max\{\mathrm{F}_{\widetilde{\kappa}}(p), \mathrm{F}_{\widetilde{\kappa}}(q)\} &= \max\{1 - \mathrm{F}_{\widetilde{\varepsilon}}(p), 1 - \mathrm{F}_{\widetilde{\varepsilon}}(q)\}\\&= 1 - \min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\}\\&\geq 1 - \mathrm{F}_{\widetilde{\varepsilon}}(pzq)\\&= \mathrm{F}_{\widetilde{\kappa}}(pzq),\\ \max\{\mathrm{F}^{\widetilde{\tau}}(p), \mathrm{F}^{\widetilde{\tau}}(q)\} &= \max\{1 - \mathrm{F}^{\widetilde{\varepsilon}}(p), 1 - \mathrm{F}^{\widetilde{\varepsilon}}(q)\}\\&= 1 - \min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\}\\&\geq 1 - \mathrm{F}^{\widetilde{\varepsilon}}(pzq)\\&= \mathrm{F}^{\widetilde{\tau}}(pzq).\end{aligned}$$

Therefore, we obtain that $\mathrm{F}^{\widetilde{\tau}}$ and $\mathrm{F}_{\widetilde{\kappa}}$ are AFGBIs of $\mathcal{A}$.

(7) $\Rightarrow$ (8). Assume that $\mathrm{F}^{\widetilde{\tau}}$ and $\mathrm{F}_{\widetilde{\kappa}}$ are AFGBIs of $\mathcal{A}$ for each $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$. Then $\mathrm{F}_{\widetilde{\kappa}}(pzq) \leq \max\{\mathrm{F}_{\widetilde{\kappa}}(p), \mathrm{F}_{\widetilde{\kappa}}(q)\}$ and $\mathrm{F}^{\widetilde{\tau}}(pzq) \leq \max\{\mathrm{F}^{\widetilde{\tau}}(p), \mathrm{F}^{\widetilde{\tau}}(q)\}$ for all $p, q, z \in \mathcal{A}$, $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$. Thus

$$\begin{aligned}\max\{(\mathrm{F}_{\widetilde{\kappa}} - 1)(p), (\mathrm{F}_{\widetilde{\kappa}} - 1)(q)\} &= \max\{\mathrm{F}_{\widetilde{\kappa}}(p) - 1, \mathrm{F}_{\widetilde{\kappa}}(q) - 1\}\\&= \max\{\mathrm{F}_{\widetilde{\kappa}}(p), \mathrm{F}_{\widetilde{\kappa}}(q)\} - 1\\&\geq \mathrm{F}_{\widetilde{\kappa}}(pzq) - 1\\&= (\mathrm{F}_{\widetilde{\kappa}} - 1)(pzq),\\ \max\{(\mathrm{F}^{\widetilde{\tau}} - 1)(p), (\mathrm{F}^{\widetilde{\tau}} - 1)(q)\} &= \max\{\mathrm{F}^{\widetilde{\tau}}(p) - 1, \mathrm{F}^{\widetilde{\tau}}(q) - 1\}\\&= \max\{\mathrm{F}^{\widetilde{\tau}}(p), \mathrm{F}^{\widetilde{\tau}}(q)\} - 1\\&\geq \mathrm{F}^{\widetilde{\tau}}(pzq) - 1\\&= (\mathrm{F}^{\widetilde{\tau}} - 1)(pzq),\end{aligned}$$

for all $p, q, z \in \mathcal{A}$, $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$. Therefore, we get that $\mathrm{F}^{\widetilde{\tau}} - 1$ and $\mathrm{F}_{\widetilde{\kappa}} - 1$ are NFGBIs of $\mathcal{A}$ for all $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$.

(4) $\Rightarrow$ (5). It is similar to prove (7) $\Rightarrow$ (8) and we omit the details.

(6) $\Rightarrow$ (2). Assume that $[\mathcal{A}, \widetilde{\varepsilon}, \breve{a}]$ is an empty set or a GBI of $\mathcal{A}$ for each $\breve{a} \in \mathcal{D}([0,1])$. Let $p, q, z \in \mathcal{A}$ and choose

$$\breve{a} := [\min\{\mathrm{INF}\,\widetilde{\varepsilon}(p), \mathrm{INF}\,\widetilde{\varepsilon}(q)\}, \min\{\mathrm{SUP}\,\widetilde{\varepsilon}(p), \mathrm{SUP}\,\widetilde{\varepsilon}(q)\}].$$

Then, we have $p, q \in [\mathcal{A}, \widetilde{\varepsilon}, \breve{a}]$ and $\breve{a} \in \mathcal{D}([0,1])$. Thus the set $[\mathcal{A}, \widetilde{\varepsilon}, \breve{a}]$ is a GBI of $\mathcal{A}$ which implies that $pzq \in [\mathcal{A}, \widetilde{\varepsilon}, \breve{a}]$. Hence

$$\min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} = \min\{\mathrm{INF}\,\widetilde{\varepsilon}(p), \mathrm{INF}\,\widetilde{\varepsilon}(q)\} \leq \mathrm{INF}\,\widetilde{\varepsilon}(pzq) = \mathrm{F}_{\widetilde{\varepsilon}}(pzq),$$

$$\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} = \min\{\mathrm{SUP}\,\widetilde{\varepsilon}(p), \mathrm{SUP}\,\widetilde{\varepsilon}(q)\} \leq \mathrm{SUP}\,\widetilde{\varepsilon}(pzq) = \mathrm{F}^{\widetilde{\varepsilon}}(pzq).$$

Therefore, we conclude that $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$.

(2) $\Rightarrow$ (1). Assume that $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$ and $\Pi \in \wp([0,1])$. Let $z \in \mathcal{A}$ and $p, q \in [\mathcal{A}, \widetilde{\varepsilon}, \Pi]$. Then $\mathrm{SUP}\,\Pi \leq \min\{\mathrm{SUP}\,\widetilde{\varepsilon}(p), \mathrm{SUP}\,\widetilde{\varepsilon}(q)\}$ and $\mathrm{INF}\,\Pi \leq \min\{\mathrm{INF}\,\widetilde{\varepsilon}(p), \mathrm{INF}\,\widetilde{\varepsilon}(q)\}$. By the assumption, we get

$$\mathrm{INF}\,\Pi \leq \min\{\mathrm{INF}\,\widetilde{\varepsilon}(p), \mathrm{INF}\,\widetilde{\varepsilon}(q)\} = \min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}_{\widetilde{\varepsilon}}(pzq) = \mathrm{INF}\,\widetilde{\varepsilon}(pzq),$$

$$\mathrm{SUP}\,\Pi \leq \min\{\mathrm{SUP}\,\widetilde{\varepsilon}(p), \mathrm{SUP}\,\widetilde{\varepsilon}(q)\} = \min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}^{\widetilde{\varepsilon}}(pzq) = \mathrm{SUP}\,\widetilde{\varepsilon}(pzq).$$

Thus $pzq \in [\mathcal{A}, \widetilde{\varepsilon}, \Pi]$. Hence $[\mathcal{A}, \widetilde{\varepsilon}, \Pi]$ is a GBI of $\mathcal{A}$. □

Example 3.3. Let $\mathcal{A} = \{p, q, z\}$ and define a binary operation $\cdot$ on $\mathcal{A}$ by

$\cdot$	p	q	z
p	p	p	p
q	p	q	q
z	p	q	z

Then $\mathcal{A}$ is a semigroup under the binary operation $\cdot$ [33]. We define a HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ by

$$\widetilde{\varepsilon}(p) = \emptyset,\ \widetilde{\varepsilon}(q) = \{0.2, 0.4, 0.6, 0.8\} \text{ and } \widetilde{\varepsilon}(z) = [0.5, 0.9].$$

Thus the following conditions are true.

(1) $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFSS of $\mathcal{A}$.
(2) $\widetilde{\varepsilon}$ is not an IvFSS of $\mathcal{A}$ because it is not an IvFS.
(3) $\widetilde{\varepsilon}$ is not a HFSS of $\mathcal{A}$ because $\widetilde{\varepsilon}(q) \cap \widetilde{\varepsilon}(z) = \{0.6, 0.8\} \nsubseteq \emptyset = \widetilde{\varepsilon}(q \cdot p \cdot z)$.
(4) $\widetilde{\varepsilon}$ is not an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ because the nonempty subset $[\mathcal{A}, \widetilde{\varepsilon}, \widetilde{\varepsilon}(q)]$ of $\mathcal{A}$ is not a GBI of $\mathcal{A}$.

Example 3.4. Let $\mathcal{A} = \{w, p, q, z\}$ and define a binary operation $*$ on $\mathcal{A}$ by

$*$	w	p	q	z
w	w	w	w	w
p	w	w	w	w
q	w	w	p	w
z	w	w	p	p

Then $\mathcal{A}$ is a semigroup under the binary operation $*$ [33]. We define a HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ by

$$\widetilde{\varepsilon}(w) = \{0.6, 0.8, 1\},\ \widetilde{\varepsilon}(p) = \{0\},\ \widetilde{\varepsilon}(q) = [0.3, 0.5], \text{ and } \widetilde{\varepsilon}(z) = \emptyset.$$

The following conditions are true.

(1) $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$.
(2) $\widetilde{\varepsilon}$ is not an IvFGBI of $\mathcal{A}$ because it is not an IvFS.
(3) $\widetilde{\varepsilon}$ is not a HFGBI of $\mathcal{A}$ because $\widetilde{\varepsilon}(q) \cap \widetilde{\varepsilon}(q) = [0.3, 0.5] \nsubseteq \{0\} = \widetilde{\varepsilon}(q * q)$.
(4) $\widetilde{\varepsilon}$ is not an $(\inf, \sup)$-HFSS of $\mathcal{A}$ because the nonempty subset $[\mathcal{A}, \widetilde{\varepsilon}, \widetilde{\varepsilon}(q)]$ of $\mathcal{A}$ is not a SS of $\mathcal{A}$.

Example 3.5. Let $\mathcal{A} = \{w, p, q, z\}$ and define a binary operation $\circ$ on $\mathcal{A}$ by

$\circ$	w	p	q	z
w	w	w	w	w
p	w	p	q	w
q	w	w	w	w
z	w	z	w	w

Then $\mathcal{A}$ is a semigroup under the binary operation $\circ$ [33]. We define a HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ by

$$\widetilde{\varepsilon}(w) = (0.6, 0.9),\ \widetilde{\varepsilon}(p) = \{0.6, 0.9\},\ \widetilde{\varepsilon}(q) = \emptyset \text{ and } \widetilde{\varepsilon}(z) = \{0\}.$$

It is routine to verify that $\mathrm{F}_{\widetilde{\varepsilon}}$ and $\mathrm{F}^{\widetilde{\varepsilon}}$ are FBIs of $\mathcal{A}$. By using Lemma 3.2, we get that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFBI of $\mathcal{A}$. However, $\widetilde{\varepsilon}$ is not a HFBI of $\mathcal{A}$ because $\widetilde{\varepsilon}(z) \cap \widetilde{\varepsilon}(z) = \{0\} \nsubseteq (0.6, 0.9) = \widetilde{\varepsilon}(z \circ z)$, and $\widetilde{\varepsilon}$ is not an IvFBI of $\mathcal{A}$ because it is not an IvFS.

Proposition 3.6. *Every IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$.*

Proof. Assume that $\breve{\varpi}$ is an IvFGBI of $\mathcal{A}$. Then $\breve{\varpi}^+$ and $\breve{\varpi}^-$ are FGBIs of $\mathcal{A}$. Thus, it follows from Lemma 3.2 that $\breve{\varpi}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. $\square$

The converse of the above proposition is not true, generally, as we see in Examples 3.3, 3.4 and 3.5. Then by Proposition 3.6 and Examples 3.3, 3.4 and 3.5, we have that an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of a semigroup $\mathcal{A}$ is a general concept of an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$.

Theorem 3.7. *Let $\breve{\varpi}$ be an IvFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\breve{\varpi}$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$.*

(2) *$\breve{\varpi}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$.*

(3) *$[\mathcal{A}, \breve{\varpi}, \breve{a}]$ is an empty set or a GBI (resp., SS, BI) of $\mathcal{A}$ for each $\breve{a} \in \mathcal{D}([0, 1])$.*

Proof. It follows from Lemma 3.2 and Proposition 3.6. $\square$

Theorem 3.8. *Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$.*

(2) *If $\breve{\varpi}$ is an IvFS on $\mathcal{A}$ such that $\breve{\varpi}^- = F_{\widetilde{\varepsilon}}$ and $\breve{\varpi}^+ = \mathrm{F}^{\widetilde{\varepsilon}}$, then $\breve{\varpi}$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$.*

(3) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$ for each HFS $\widetilde{\kappa}$ on $\mathcal{A}$ such that $F_{\widetilde{\kappa}} = F_{\widetilde{\varepsilon}}$ and $\mathrm{F}^{\widetilde{\kappa}} = \mathrm{F}^{\widetilde{\varepsilon}}$.*

Proof. It follows from Lemma 3.2 and Theorem 3.7. $\square$

For each HFS $\widetilde{\varepsilon}$ on $\mathcal{X}$ and element $\Pi \in \wp([0, 1])$, define the HFS $\mathcal{H}_\Pi^{\widetilde{\varepsilon}}$ [6, 38] on $\mathcal{X}$ by

$$\mathcal{H}_\Pi^{\widetilde{\varepsilon}} : \mathcal{X} \to \wp([0, 1]), p \mapsto \{m \in \Pi \mid \tfrac{\mathrm{F}_{\widetilde{\varepsilon}} \pm}{2}(p) \leq m \leq \tfrac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p)\}.$$

We denote $\mathcal{H}^{\widetilde{\varepsilon}}$ for $\mathcal{H}^{\widetilde{\varepsilon}}_{[0,1]}$. Then, we obtain that $\mathcal{H}^{\widetilde{\varepsilon}}(p) \neq \emptyset$ and $\mathcal{H}^{\widetilde{\varepsilon}}_{\Pi}(p) \subseteq \mathcal{H}^{\widetilde{\varepsilon}}_{\Psi}(p) \subseteq \mathcal{H}^{\widetilde{\varepsilon}}(p)$ when $p \in \mathcal{X}$ and $\Pi \subseteq \Psi \subseteq [0,1]$. Next, we characterize $(\inf, \sup)$-HFGBIs, $(\inf, \sup)$-HFSSs and $(\inf, \sup)$-HFBIs of semigroups via HFSs.

Theorem 3.9. *Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$.*
(2) *$\mathcal{H}^{\widetilde{\varepsilon}}$ is a HFGBI (resp., HFSS, HFBI) of $\mathcal{A}$.*
(3) *$\mathcal{H}^{\widetilde{\varepsilon}}_{\Pi}$ is a HFGBI (resp., HFSS, HFBI) of $\mathcal{A}$ for all $\Pi \in \wp([0,1])$.*

Proof. (2) $\Rightarrow$ (1). Assume that $\mathcal{H}^{\widetilde{\varepsilon}}$ is a HFGBI of $\mathcal{A}$ and $p, q, z \in \mathcal{A}$. Then $\max\{\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p), \frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q)\} \in \mathcal{H}^{\widetilde{\varepsilon}}(p) \cap \mathcal{H}^{\widetilde{\varepsilon}}(q)$ and $\min\{\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p), \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)\} \in \mathcal{H}^{\widetilde{\varepsilon}}(p) \cap \mathcal{H}^{\widetilde{\varepsilon}}(q)$. Since $\mathcal{H}^{\widetilde{\varepsilon}}$ is a HFGBI of $\mathcal{A}$, we get

$$\max\{\tfrac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p), \tfrac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q)\}, \min\{\tfrac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p), \tfrac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)\} \in \mathcal{H}^{\widetilde{\varepsilon}}(p) \cap \mathcal{H}^{\widetilde{\varepsilon}}(q) \subseteq \mathcal{H}^{\widetilde{\varepsilon}}(pzq).$$

Thus $\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(pzq) \leq \max\{\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p), \frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q)\}$ and $\min\{\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p), \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)\} \leq \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(pzq)$. Since $\mathrm{F}_{\widetilde{\varepsilon}} = 1 - 2(\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2})$ and $\mathrm{F}^{\widetilde{\varepsilon}} = 2(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}) - 1$, we obtain that

$$\begin{aligned}
\mathrm{F}_{\widetilde{\varepsilon}}(pzq) &= 1 - 2(\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(pzq)) \\
&\geq 1 - 2(\max\{\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p), \frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q)\}) \\
&= \min\{1 - 2(\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p)), 1 - 2(\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q))\} \\
&= \min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\}, \\
\mathrm{F}^{\widetilde{\varepsilon}}(pzq) &= 2(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(pzq)) - 1 \\
&\geq 2(\min\{\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p), \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)\}) - 1 \\
&= \min\{2(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p)) - 1, 2(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)) - 1\} \\
&= \min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\}.
\end{aligned}$$

Hence $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$. Therefore, it follows from Lemma 3.2 that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$.

(1) $\Rightarrow$ (3). Assume that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. Let $m \in [0,1]$, $\Pi \in \wp([0,1])$ and $p, q, z \in \mathcal{A}$ such that $m \in \mathcal{H}^{\widetilde{\varepsilon}}_{\Pi}(p) \cap \mathcal{H}^{\widetilde{\varepsilon}}_{\Pi}(q)$. Then

$$\tfrac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p) \leq m \leq \tfrac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p),\ \tfrac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q) \leq m \leq \tfrac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q) \text{ and } m \in \Pi.$$

By $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ and Lemma 3.2, we have $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(pzq) \leq \max\{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(p), \mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(q)\}$ and $\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}^{\widetilde{\varepsilon}}(pzq)$. Thus

$$\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(pzq) \leq \max\{\frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(p), \frac{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{2}(q)\} \leq m \leq \min\{\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(p), \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(q)\} \leq \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{2}(pzq)$$

and so $m \in \mathcal{H}_{\Pi}^{\widetilde{\varepsilon}}(pzq)$. Hence $\mathcal{H}_{\Pi}^{\widetilde{\varepsilon}}(p) \cap \mathcal{H}_{\Pi}^{\widetilde{\varepsilon}}(q) \subseteq \mathcal{H}_{\Pi}^{\widetilde{\varepsilon}}(pzq)$. Therefore, we obtain that $\mathcal{H}_{\Pi}^{\widetilde{\varepsilon}}$ is a HFGBI of $\mathcal{A}$ for each $\Pi \in \wp([0,1])$.

(3) $\Rightarrow$ (2). It is clear. $\square$

Lemma 3.10. *Let φ be a FS of $\mathcal{A}$. Then φ is an AFGBI (resp., AFSS, AFBI) of $\mathcal{A}$ if and only if $1-\varphi$ is a FGBI (resp., FSS, FBI) of $\mathcal{A}$.*

Proof. Assume that φ is an AFGBI of $\mathcal{A}$ and $p, q, z \in \mathcal{A}$. Then

$$\min\{1-\varphi(p), 1-\varphi(q)\} = 1 - \max\{\varphi(p), \varphi(q)\} \leq 1 - \varphi(pzq).$$

Hence $1-\varphi$ is a FGBI of $\mathcal{A}$. Conversely, assume that $1-\varphi$ is a FGBI of $\mathcal{A}$ and $p, q, z \in \mathcal{A}$. Then

$$\begin{aligned}
\max\{\varphi(p), \varphi(q)\} &= \max\{1-(1-\varphi)(p), 1-(1-\varphi)(q)\} \\
&= 1 - \min\{(1-\varphi)(p), (1-\varphi)(q)\} \\
&\geq 1 - (1-\varphi)(pzq) \\
&= \varphi(pzq).
\end{aligned}$$

Thus φ is an AFGBI of $\mathcal{A}$. $\square$

Next, we characterize (inf, sup)-HFGBIs, (inf, sup)-HFSSs and (inf, sup)-HFBIs of semigroups via PFSs.

Theorem 3.11. *Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI (resp., (inf, sup)-HFSS, (inf, sup)-HFBI) of $\mathcal{A}$.*

(2) *$(\mathrm{F}_{\widetilde{\varepsilon}}, \mathrm{F}^{\widetilde{\varepsilon}^{\mp}})$ is a PFGBI (resp., PFSS, PFBI) of $\mathcal{A}$.*

(3) *$(\mathrm{F}_{\widetilde{\varepsilon}}, \mathrm{F}^{\widetilde{\tau}})$ is a PFGBI (resp., PFSS, PFBI) of $\mathcal{A}$ for all $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$.*

(4) *$(\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}}{1+2m}, \frac{n+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{1+2m})$ is a PFGBI (resp., PFSS, PFBI) of $\mathcal{A}$ for all $n, m \in \mathcal{N}$ such that $n \leq m$.*

(5) *$(\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}}{1+2m}, \frac{n+\mathrm{F}_{\widetilde{\kappa}}}{1+2m})$ is a PFGBI (resp., PFSS, PFBI) of $\mathcal{A}$ for all $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $n, m \in \mathcal{N}$ such that $n \leq m$.*

Proof. (1) $\Rightarrow$ (3). It follows from Lemma 3.2.

(3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4). They are obvious.

(2) $\Rightarrow$ (1). Assume that $(\mathrm{F}_{\widetilde{\varepsilon}}, \mathrm{F}^{\widetilde{\varepsilon}^{\mp}})$ is a PFGBI of $\mathcal{A}$. Then $\mathrm{F}^{\widetilde{\varepsilon}^{\mp}}$ is an AFGBI and $\mathrm{F}_{\widetilde{\varepsilon}}$ is a FGBI of $\mathcal{A}$. By Lemma 3.10, we get that $\mathrm{F}^{\widetilde{\varepsilon}} = 1 - \mathrm{F}^{\widetilde{\varepsilon}^{\mp}}$ is a FGBI of $\mathcal{A}$. Thus $\mathrm{F}_{\widetilde{\varepsilon}}$ and $\mathrm{F}^{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$. By Lemma 3.2, we have $\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI of $\mathcal{A}$.

(1) $\Rightarrow$ (5). Assume that $\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI of $\mathcal{A}$. Let $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$ and $n, m \in \mathcal{N}$ such that $n \leq m$. Then, it follows Lemma 3.2 that $\mathrm{F}^{\widetilde{\varepsilon}}$ is a FGBI and $\mathrm{F}_{\widetilde{\kappa}}$ is an AFGBI of $\mathcal{A}$. Thus $\mathrm{F}_{\widetilde{\kappa}}(pzq) \leq \max\{\mathrm{F}_{\widetilde{\kappa}}(p), \mathrm{F}_{\widetilde{\kappa}}(q)\}$ and $\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}^{\widetilde{\varepsilon}}(pzq)$ for all $p, q, z \in \mathcal{A}$. Hence

$$\min\{\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}(p)}{1+2m}, \frac{n+\mathrm{F}^{\widetilde{\varepsilon}}(q)}{1+2m}\} = \frac{n+\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\}}{1+2m} \leq \frac{n+\mathrm{F}^{\widetilde{\varepsilon}}(pzq)}{1+2m},$$

$$\max\{\frac{n+\mathrm{F}_{\widetilde{\kappa}}(p)}{1+2m}, \frac{n+\mathrm{F}_{\widetilde{\kappa}}(q)}{1+2m}\} = \frac{n+\max\{\mathrm{F}_{\widetilde{\kappa}}(p), \mathrm{F}_{\widetilde{\kappa}}(q)\}}{1+2m} \geq \frac{n+\mathrm{F}_{\widetilde{\kappa}}(pzq)}{1+2m}$$

for all $p, q, z \in \mathcal{A}$. Thus $\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}}{1+2m}$ is a FGBI and $\frac{n+\mathrm{F}_{\widetilde{\kappa}}}{1+2m}$ is an AFGBI of $\mathcal{A}$. Therefore, we obtain that the PFS $(\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}}{1+2m}, \frac{n+\mathrm{F}_{\widetilde{\kappa}}}{1+2m})$ is a PFGBI of $\mathcal{A}$.

(4) $\Rightarrow$ (1). Assume that $(\frac{n+\mathrm{F}^{\widetilde{\varepsilon}}}{1+2m}, \frac{n+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{1+2m})$ is a PFGBI of $\mathcal{A}$ for all $n, m \in \mathcal{N}$ such that $n \leq m$. Then the PFS $(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}, \frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3})$ is a PFGBI of $\mathcal{A}$ which implies that $\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}$ is a FGBI and $\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}$ is an AFGBI of $\mathcal{A}$. Since $\mathrm{F}^{\widetilde{\varepsilon}} = 3(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}) - 1$ and $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}} = 3(\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}) - 1$, we have

$$\begin{aligned}
\max\{\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(p), \mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(q)\} &= \max\{3(\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}(p)) - 1, 3(\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}(q)) - 1\} \\
&= 3(\max\{\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}(p), \frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}(q)\}) - 1 \\
&\geq 3(\frac{1+\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}}{3}(pzq)) - 1 \\
&= \mathrm{F}_{\widetilde{\varepsilon}^{\pm}}(pzq), \\
\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} &= \min\{3(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}(p)) - 1, 3(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}(q)) - 1\} \\
&= 3(\min\{\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}(p), \frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}(q)\}) - 1 \\
&\leq 3(\frac{1+\mathrm{F}^{\widetilde{\varepsilon}}}{3}(pzq)) - 1 \\
&= \mathrm{F}^{\widetilde{\varepsilon}}(pzq)
\end{aligned}$$

for all $p, q, z \in \mathcal{A}$. Thus $\mathrm{F}^{\widetilde{\varepsilon}}$ is a FGBI and $\mathrm{F}_{\widetilde{\varepsilon}^{\pm}}$ is an AFGBI of $\mathcal{A}$. By using Lemma 3.10, we obtain that $\mathrm{F}_{\widetilde{\varepsilon}} = 1 - \mathrm{F}_{\widetilde{\varepsilon}^{\pm}}$ is a FGBI of $\mathcal{A}$. Therefore, it follows from Lemma 3.2 that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. □

Lemma 3.12. *Let φ be a NFS of $\mathcal{A}$. Then φ is a NFGBI (resp., NFSS, NFBI) of $\mathcal{A}$ if and only if $-\varphi$ is a FGBI (resp., FSS, FBI) of $\mathcal{A}$.*

Proof. Let φ be a NFGBI of $\mathcal{A}$ and $p, q, z \in \mathcal{A}$. Then $\varphi(pzq) \leq \max\{\varphi(p), \varphi(q)\}$ and thus

$$-\varphi(pzq) \geq -(\max\{\varphi(p), \varphi(q)\}) = \min\{-\varphi(p), -\varphi(q)\}.$$

Therefore, we conclude that $-\varphi$ is a FGBI of $\mathcal{A}$. Conversely, let $-\varphi$ be a FGBI of $\mathcal{A}$ and $p, q, z \in \mathcal{A}$. Then, we have $\max\{\varphi(p), \varphi(q)\} = \max\{-(-\varphi(p)), -(-\varphi(q))\}$ $= -(\min\{-\varphi(p), -\varphi(q)\}) \geq -(-\varphi(pzq)) = \varphi(pzq)$. Thus φ is a NFGBI of $\mathcal{A}$. □

In the following theorem, characterizations of $(\inf, \sup)$-HFGBIs, $(\inf, \sup)$-HFSSs and $(\inf, \sup)$-HFBIs of semigroups are discussed via BFSs.

Theorem 3.13. *Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI (resp., $(\inf, \sup)$-HFSS, $(\inf, \sup)$-HFBI) of $\mathcal{A}$.*
(2) *$\langle -\mathrm{F}_{\widetilde{\varepsilon}}, \mathrm{F}^{\widetilde{\varepsilon}} \rangle$ is a BFGBI (resp., BFSS, BFBI) of $\mathcal{A}$.*

(3) $\langle -\mathrm{F}^{\widetilde{\varepsilon}}, \mathrm{F}_{\widetilde{\varepsilon}}\rangle$ *is a BFGBI (resp., BFSS, BFBI) of* $\mathcal{A}$.
(4) $\langle \mathrm{F}_{\widetilde{\kappa}} - 1, \mathrm{F}^{\widetilde{\varepsilon}}\rangle$ *is a BFGBI (resp., BFSS, BFBI) of* $\mathcal{A}$ *for all* $\widetilde{\kappa} \in \mathrm{IC}(\widetilde{\varepsilon})$.
(5) $\langle \mathrm{F}^{\widetilde{\tau}} - 1, \mathrm{F}_{\widetilde{\varepsilon}}\rangle$ *is a BFGBI (resp., BFSS, BFBI) of* $\mathcal{A}$ *for all* $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\varepsilon})$.

Proof. (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3). They are obvious.

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5). They follow from Lemma 3.2.

(2) $\Rightarrow$ (1). Assume that $\langle -\mathrm{F}_{\widetilde{\varepsilon}}, \mathrm{F}^{\widetilde{\varepsilon}}\rangle$ is a BFGBI of $\mathcal{A}$. Then $-\mathrm{F}_{\widetilde{\varepsilon}}$ is a NFGBI and $\mathrm{F}^{\widetilde{\varepsilon}}$ is a FGBI of $\mathcal{A}$. By using Lemma 3.12, we have $\mathrm{F}_{\widetilde{\varepsilon}} = -(-\mathrm{F}_{\widetilde{\varepsilon}})$ is a FGBI of $\mathcal{A}$. Thus $\mathrm{F}_{\widetilde{\varepsilon}}$ and $\mathrm{F}^{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$ and by Lemma 3.2, we obtain that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$.

(3) $\Rightarrow$ (1). It is similar to prove that (2) $\Rightarrow$ (1) and we omit the details. $\square$

4. Characterizing BIs, FBIs, AFBIs, NFBIs, PFBIs and BFBIs

In this section, we characterize BIs, FBIs, AFBIs, NFBIs, PFBIs and BFBIs of semigroups by $(\inf, \sup)$-type of HFSs.

For any subset $\mathcal{Y}$ of $\mathcal{X}$ and $\Pi, \Psi \in \wp([0,1])$, define a map $\mathcal{C}(\mathcal{Y}, \Pi, \Psi)$ [19, 20] as follows:

$$\mathcal{C}(\mathcal{Y}, \Pi, \Psi) \colon \mathcal{X} \to \wp([0,1]), p \mapsto \begin{cases} \Psi & \text{if } p \in \mathcal{Y}, \\ \Pi & \text{otherwise.} \end{cases}$$

We denote $\mathcal{CI}(\mathcal{Y})$ for $\mathcal{C}(\mathcal{Y}, [0,0], [1,1])$ and is called the *characteristic interval-valued fuzzy set* of $\mathcal{Y}$ on $\mathcal{X}$, and denote $\mathcal{CH}(\mathcal{Y})$ for $\mathcal{C}(\mathcal{Y}, \emptyset, [0,1])$ and is called the *characteristic hesitant fuzzy set* of $\mathcal{Y}$ on $\mathcal{X}$.

Lemma 4.1. *If* $\mathcal{X}$ *is a GBI (resp., SS, BI) of* $\mathcal{A}$*, then* $\mathcal{C}(\mathcal{X}, \Pi, \Psi)$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for each* $\Pi, \Psi \in \wp([0,1])$ *with* $\mathrm{INF}\,\Pi \leq \mathrm{INF}\,\Psi$ *and* $\mathrm{SUP}\,\Pi \leq \mathrm{SUP}\,\Psi$.

Proof. Assume that $\mathcal{X}$ is a GBI of $\mathcal{A}$ and $\Pi, \Psi \in \wp([0,1])$ such that $\mathrm{INF}\,\Pi \leq \mathrm{INF}\,\Psi$ and $\mathrm{SUP}\,\Pi \leq \mathrm{SUP}\,\Psi$. Let $p, q, z \in \mathcal{A}$. In case that $p \notin \mathcal{X}$ or $q \notin \mathcal{X}$, then $\mathcal{C}(\mathcal{X}, \Pi, \Psi)(p) = \Pi$ or $\mathcal{C}(\mathcal{X}, \Pi, \Psi)(q) = \Pi$. Thus

$$\begin{aligned} \mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) \geq \mathrm{INF}\,\Pi &= \min\{\mathrm{INF}\,\mathcal{C}(\mathcal{X}, \Pi, \Psi)(p), \mathrm{INF}\,\mathcal{C}(\mathcal{X}, \Pi, \Psi)(q)\} \\ &= \min\{\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\}, \end{aligned}$$

$$\begin{aligned} \mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) \geq \mathrm{SUP}\,\Pi &= \min\{\mathrm{SUP}\,\mathcal{C}(\mathcal{X}, \Pi, \Psi)(p), \mathrm{SUP}\,\mathcal{C}(\mathcal{X}, \Pi, \Psi)(q)\} \\ &= \min\{\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\}. \end{aligned}$$

On the other hand, let $p, q \in \mathcal{X}$. Since $\mathcal{X}$ is a GBI of $\mathcal{A}$, we have $pzq \in \mathcal{X}$. Then

$$\mathcal{C}(\mathcal{X}, \Pi, \Psi)(pzq) = \mathcal{C}(\mathcal{X}, \Pi, \Psi)(p) = \mathcal{C}(\mathcal{X}, \Pi, \Psi)(q) = \Psi.$$

Thus

$$\begin{gathered} \mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) = \min\{\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\} \text{ and} \\ \mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) = \min\{\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\}. \end{gathered}$$

Hence $\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}$ and $\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}$ are FGBIs of $\mathcal{A}$. Therefore, it follows from Lemma 3.2 that $\mathcal{C}(\mathcal{X}, \Pi, \Psi)$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. $\square$

Theorem 4.2. *Let $\mathcal{X}$ be a nonempty subset of $\mathcal{A}$ and $\Pi, \Psi \in \wp([0,1])$ with one of the following two conditions are true:*

(1) $\mathrm{INF}\,\Pi \le \mathrm{INF}\,\Psi$ *and* $\mathrm{SUP}\,\Pi < \mathrm{SUP}\,\Psi$*,*
(2) $\mathrm{INF}\,\Pi < \mathrm{INF}\,\Psi$ *and* $\mathrm{SUP}\,\Pi \le \mathrm{SUP}\,\Psi$*.*

Then $\mathcal{X}$ is a GBI (resp., SS, BI) of $\mathcal{A}$ if and only if $\mathcal{C}(\mathcal{X}, \Pi, \Psi)$ is an $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$.*

Proof. $(\Rightarrow)$. It follows from Lemma 4.1.

$(\Leftarrow)$. Assume that $\mathcal{C}(\mathcal{X}, \Pi, \Psi)$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. Let $p, q \in \mathcal{X}$ and $z \in \mathcal{A}$. Then $\mathcal{C}(\mathcal{X}, \Pi, \Psi)(p) = \Psi = \mathcal{C}(\mathcal{X}, \Pi, \Psi)(q)$. In the case that $\mathrm{INF}\,\Pi < \mathrm{INF}\,\Psi$ and $\mathrm{SUP}\,\Pi \le \mathrm{SUP}\,\Psi$, then by Lemma 3.2, we have $\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}$ is a GBI of $\mathcal{A}$. Thus

$$\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) \ge \min\{\mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}_{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\} = \mathrm{INF}\,\Psi > \mathrm{INF}\,\Pi.$$

Hence $\mathcal{C}(\mathcal{X}, \Pi, \Psi)(pzq) = \Psi$ and so we get $pzq \in \mathcal{X}$. In the case that $\mathrm{INF}\,\Pi \le \mathrm{INF}\,\Psi$ and $\mathrm{SUP}\,\Pi < \mathrm{SUP}\,\Psi$, then by Lemma 3.2, we obtain that $\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}$ is a GBI of $\mathcal{A}$. Thus

$$\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(pzq) \ge \min\{\mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(p), \mathrm{F}^{\mathcal{C}(\mathcal{X},\Pi,\Psi)}(q)\} = \mathrm{SUP}\,\Psi > \mathrm{SUP}\,\Pi.$$

Hence $\mathcal{C}(\mathcal{X}, \Pi, \Psi)(pzq) = \Psi$ and so we have $pzq \in \mathcal{X}$. Therefore, we conclude that $\mathcal{X}$ is a GBI of $\mathcal{A}$. □

Theorem 4.3. *Let $\mathcal{X}$ be a nonempty subset of $\mathcal{A}$. The following conditions are equivalent.*

(1) *$\mathcal{X}$ is a GBI (resp., SS, BI) of $\mathcal{A}$.*
(2) *$\mathcal{CI}(\mathcal{X})$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$.*
(3) *$\mathcal{CH}(\mathcal{X})$ is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$.*
(4) *$\mathcal{CI}(\mathcal{X})$ is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$.*
(5) *$\mathcal{C}(\mathcal{X}, \breve{a}, \breve{b})$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$ for all $\breve{a}, \breve{b} \in \mathcal{D}[0,1]$ such that $\breve{a} \preceq \breve{b}$.*

Proof. It follows from Lemma 4.1 and Theorems 3.8 and 4.2. □

For each FS φ of $\mathcal{X}$ and $n \in \mathcal{N}$, we define the IvFS $\mathcal{I}(\varphi, n)$ and the HFS $\mathcal{H}(\varphi, n)$ on $\mathcal{X}$ [6, 38] as follows:

$$\mathcal{I}(\varphi, n) : \mathcal{X} \to \mathcal{D}[0,1], p \mapsto [\frac{\varphi}{1+n}(p), \frac{n+\varphi}{1+n}(p)]$$

and

$$\mathcal{H}(\varphi, n) : \mathcal{X} \to \wp([0,1]), p \mapsto \{\frac{\varphi}{1+n}(p), \frac{n+\varphi}{1+n}(p)\}.$$

Then the following are true.

(1) For all $p \in \mathcal{X}$, we have $\mathcal{H}(\varphi, n)(p) \subseteq \mathcal{I}(\varphi, n)(p)$.
(2) $\mathrm{F}_{\mathcal{I}(\varphi,n)} = \frac{\varphi}{1+n} = \mathrm{F}_{\mathcal{H}(\varphi,n)}$ and $\mathrm{F}^{\mathcal{I}(\varphi,n)} = \frac{n+\varphi}{1+n} = \mathrm{F}^{\mathcal{H}(\varphi,n)}$.

(3) If δ is a NFS of $\mathcal{X}$, then $\mathcal{I}(-\delta, n)$ is an IvFS and $\mathcal{H}(-\delta, n)$ is a HFS on $\mathcal{X}$.

Theorem 4.4. *Let φ be a FS of $\mathcal{A}$. The following conditions are equivalent.*

(1) *φ is a FGBI (resp., FSS, FBI) of $\mathcal{A}$.*
(2) *$\mathcal{I}(\varphi, n)$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(3) *$\mathcal{H}(\varphi, n)$ is an* (inf, sup)*-HFGBI (resp.,* (inf, sup)*-HFSS,* (inf, sup)*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(4) *$\mathcal{I}(\varphi, n)$ is an* (inf, sup)*-HFGBI (resp.,* (inf, sup)*-HFSS,* (inf, sup)*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(5) *$\widetilde{\varepsilon}$ is an* (inf, sup)*-HFGBI (resp.,* (inf, sup)*-HFSS,* (inf, sup)*-HFBI) of $\mathcal{A}$ for each HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ and $n \in \mathcal{N}$ such that $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{\varphi}{1+n}$ and $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{n+\varphi}{1+n}$.*

Proof. It follows from Theorems 3.7 and 3.8 that the conditions $(2) - (5)$ are equivalent.

$(1) \Rightarrow (5)$. Assume that φ is a FGBI of $\mathcal{A}$, $\widetilde{\varepsilon}$ is a HFS on $\mathcal{A}$ and $n \in \mathcal{N}$ such that $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{\varphi}{1+n}$ and $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{n+\varphi}{1+n}$. Then, for all $p, q, z \in \mathcal{A}$, we have $\min\{\varphi(p), \varphi(q)\} \leq \varphi(pzq)$. Thus

$$\begin{aligned}\min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} &= \min\{\frac{\varphi(p)}{1+n}, \frac{\varphi(q)}{1+n}\} = \frac{\min\{\varphi(p), \varphi(q)\}}{1+n}\\ &\leq \frac{\varphi(pzq)}{1+n} = \mathrm{F}_{\widetilde{\varepsilon}}(pzq),\\ \min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} &= \min\{\frac{n+\varphi(p)}{1+n}, \frac{n+\varphi(q)}{1+n}\} = \frac{n+\min\{\varphi(p), \varphi(q)\}}{1+n}\\ &\leq \frac{n+\varphi(pzq)}{1+n} = \mathrm{F}^{\widetilde{\varepsilon}}(pzq)\end{aligned}$$

for all $p, q, z \in \mathcal{A}$. Hence $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FGBIs of $\mathcal{A}$ and by Lemma 3.2, we conclude that $\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI of $\mathcal{A}$.

$(5) \Rightarrow (1)$. Assume that $\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI of $\mathcal{A}$ for each HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ and $n \in \mathcal{N}$ such that $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{\varphi}{1+n}$ and $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{n+\varphi}{1+n}$. Let $\widetilde{\varepsilon}$ be a HFS on $\mathcal{A}$ such that $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{\varphi}{2}$ and $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{1+\varphi}{2}$. Then $\widetilde{\varepsilon}$ is an (inf, sup)-HFGBI of $\mathcal{A}$. By using Lemma 3.2, we obtain that $\frac{\varphi}{2} = \mathrm{F}_{\widetilde{\varepsilon}}$ is a FGBI of $\mathcal{A}$. Hence

$$\min\{\varphi(p), \varphi(q)\} = 2(\min\{\frac{\varphi}{2}(p), \frac{\varphi}{2}(q)\}) \leq 2(\frac{\varphi}{2}(pzq)) = \varphi(pzq)$$

for all $p, q, z \in \mathcal{A}$. Consequently, φ is a FGBI of $\mathcal{A}$. $\square$

Theorem 4.5. *Let δ be a NFS of $\mathcal{A}$. The following conditions are equivalent.*

(1) *δ is a NFGBI (resp., NFSS, NFBI) of $\mathcal{A}$.*
(2) *$\mathcal{I}(-\delta, n)$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(3) *$\mathcal{I}(-\delta, n)$ is an* (inf, sup)*-HFGBI (resp.,* (inf, sup)*-HFSS,* (inf, sup)*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(4) *$\mathcal{H}(-\delta, n)$ is an* (inf, sup)*-HFGBI (resp.,* (inf, sup)*-HFSS,* (inf, sup)*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*

(5) $\widetilde{\varepsilon}$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for each HFS* $\widetilde{\varepsilon}$ *on* $\mathcal{A}$ *and* $n \in \mathcal{N}$ *such that* $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{-\delta}{1+n}$ *and* $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{n-\delta}{1+n}$.

Proof. It follows from Lemma 3.12 and Theorem 4.4. □

Theorem 4.6. *Let* φ *be a FS of* $\mathcal{A}$*. The following conditions are equivalent.*

(1) φ *is an AFGBI (resp., AFSS, AFBI) of* $\mathcal{A}$.
(2) $\mathcal{H}(1-\varphi, n)$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(3) $\mathcal{I}(1-\varphi, n)$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(4) $\mathcal{I}(1-\varphi, n)$ *is an IvFGBI (resp., IvFSS, IvFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(5) $\widetilde{\varepsilon}$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for each HFS* $\widetilde{\varepsilon}$ *on* $\mathcal{A}$ *and* $n \in \mathcal{N}$ *such that* $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{1-\varphi}{1+n}$ *and* $\mathrm{F}^{\widetilde{\varepsilon}} = 1 + \frac{-\varphi}{1+n}$.

Proof. It follows from Lemma 3.10 and Theorem 4.4. □

For each PFS $P = (\varphi, \delta)$ in $\mathcal{X}$ and $n \in \mathcal{N}$, define the HFS $\mathcal{H}(P, n)$ and the IvFS $\mathcal{I}(P, n)$ on $\mathcal{X}$ [6, 38] as follows:

$$\mathcal{H}(P, n) : \mathcal{X} \to \wp([0,1]), p \mapsto \{\frac{\varphi}{1+n}(p), 1 - \frac{\delta}{1+n}(p)\}$$

and

$$\mathcal{I}(P, n) : \mathcal{X} \to \mathcal{D}[0,1], p \mapsto [\frac{\varphi}{1+n}(p), 1 - \frac{\delta}{1+n}(p)]$$

Note that $\mathrm{F}_{\mathcal{H}(P,n)} = \frac{\varphi}{1+n} = \mathrm{F}_{\mathcal{I}(P,n)}$ and $\mathrm{F}^{\mathcal{H}(P,n)} = 1 - \frac{\delta}{1+n} = \mathrm{F}^{\mathcal{I}(P,n)}$. In the following theorem, we characterize PFGBIs, PFSSs and PFBIs of semigroups via IvFSs and the $(\inf, \sup)$-type of HFSs.

Theorem 4.7. *Let* $P = (\varphi, \delta)$ *be a PFS in* $\mathcal{A}$*. The following conditions are equivalent.*

(1) P *is a PFGBI (resp., PFSS, PFBI) of* $\mathcal{A}$.
(2) $\mathcal{H}(P, n)$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(3) $\mathcal{I}(P, n)$ *is an IvFGBI (resp., IvFSS, IvFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(4) $\mathcal{I}(P, n)$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for all* $n \in \mathcal{N}$.
(5) $\widetilde{\varepsilon}$ *is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of* $\mathcal{A}$ *for each HFS* $\widetilde{\varepsilon}$ *on* $\mathcal{A}$ *and* $n \in \mathcal{N}$ *such that* $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{\varphi}{1+n}$ *and* $\mathrm{F}^{\widetilde{\varepsilon}} = 1 - \frac{\delta}{1+n}$.

Proof. It follows from Theorems 3.7 and 3.8 that the conditions $(2) - (5)$ are equivalent.

$(1) \Rightarrow (2)$. Assume that P is a PFGBI of $\mathcal{A}$. Then φ is a FGBI and δ is an AFGBI of $\mathcal{A}$. By using Lemma 3.10, we get that $1 - \delta$ is a FGBI of $\mathcal{A}$. Thus, for each $p, q, z \in \mathcal{A}$ and $n \in \mathcal{N}$, we have

$$\min\{\frac{\varphi(p)}{1+n}, \frac{\varphi(q)}{1+n}\} = \frac{\min\{\varphi(p), \varphi(q)\}}{1+n}$$

$$
\begin{aligned}
&\leq \frac{\varphi(pzq)}{1+n}, \\
\min\{1-\frac{\delta(p)}{1+n}, 1-\frac{\delta(q)}{1+n}\} &= \min\{\frac{n+(1-\delta)(p)}{1+n}, \frac{n+(1-\delta)(q)}{1+n}\} \\
&= \frac{n+\min\{(1-\delta)(p),(1-\delta)(q)\}}{1+n} \\
&\leq \frac{n+(1-\delta)(pzq)}{1+n} \\
&= 1-\frac{\delta(pzq)}{1+n}.
\end{aligned}
$$

Hence $\mathrm{F}^{\mathcal{H}(P,n)} = 1-\frac{\delta}{1+n}$ and $\mathrm{F}_{\mathcal{H}(P,n)} = \frac{\varphi}{1+n}$ are FGBIs of $\mathcal{A}$ for all $n \in \mathcal{N}$. By using Lemma 3.2, we conclude that $\mathcal{H}(P,n)$ is an (inf, sup)-HFGBI of $\mathcal{A}$ for all $n \in \mathcal{N}$.

(2) $\Rightarrow$ (1). Assume that $\mathcal{H}(P,n)$ is an (inf, sup)-HFGBI of $\mathcal{A}$ for all $n \in \mathcal{N}$. Then the HFS $\mathcal{H}(P,1)$ is an (inf, sup)-HFGBI of $\mathcal{A}$. By Lemma 3.2, we obtain that $\frac{2-\delta}{2} = \mathrm{F}^{\mathcal{H}(P,1)}$ and $\frac{\varphi}{2} = \mathrm{F}_{\mathcal{H}(P,1)}$ are FGBIs of $\mathcal{A}$. Note that $\delta = 2-2(\frac{2-\delta}{2})$ and $\varphi = 2(\frac{\varphi}{2})$. Thus, for each $p,q,z \in \mathcal{A}$, we get

$$
\begin{aligned}
\min\{\varphi(p),\varphi(q)\} &= \min\{2(\frac{\varphi(p)}{2}), 2(\frac{\varphi(q)}{2})\} \\
&\leq 2(\frac{\varphi(pzq)}{2}) \\
&= \varphi(pzq), \\
\max\{\delta(p),\delta(q)\} &= \max\{2-2(\frac{2-\delta(p)}{2}), 2-2(\frac{2-\delta(q)}{2})\} \\
&= 2-2(\min\{\frac{2-\delta(p)}{2}, \frac{2-\delta(q)}{2}\}) \\
&\geq 2-2(\frac{2-\delta(pzq)}{2}) \\
&= \delta(pzq).
\end{aligned}
$$

Hence φ is a FGBI and δ is an AFGBI of $\mathcal{A}$. Therefore P is a PFGBI of $\mathcal{A}$. $\square$

For each BFS $B = \langle\varphi,\delta\rangle$ on $\mathcal{X}$ and $n \in \mathcal{N}$, define the IvFS $\mathcal{I}\langle B,n\rangle$ and the HFS $\mathcal{H}\langle B,n\rangle$ on $\mathcal{X}$ [6, 38] as follows:

$$\mathcal{I}\langle B,n\rangle(p) = [\tfrac{-\varphi(p)}{1+n}, \tfrac{n+\delta(p)}{1+n}] \text{ and } \mathcal{H}\langle B,n\rangle(p) = \{\tfrac{-\varphi(p)}{1+n}, \tfrac{n+\delta(p)}{1+n}\} \text{ for all } p \in \mathcal{X}.$$

Note that $\mathrm{F}_{\mathcal{H}\langle B,n\rangle} = \frac{-\varphi}{1+n} = \mathrm{F}_{\mathcal{I}\langle B,n\rangle}$ and $\mathrm{F}^{\mathcal{H}\langle B,n\rangle} = \frac{n+\delta}{1+n} = \mathrm{F}^{\mathcal{I}\langle B,n\rangle}$. In Theorem 4.8, we characterize BFGBIs, BFSSs and BFBIs of semigroups via IvFSs and the (inf, sup)-type of HFSs .

Theorem 4.8. *Let $B = \langle\varphi,\delta\rangle$ be a BFS in $\mathcal{A}$. The following conditions are equivalent.*

(1) *B is a BFGBI (resp., BFSS, BFBI) of $\mathcal{A}$.*

(2) *$\mathcal{H}\langle B, n\rangle$ is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(3) *$\mathcal{I}\langle B, n\rangle$ is an IvFGBI (resp., IvFSS, IvFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(4) *$\mathcal{I}\langle B, n\rangle$is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$ for all $n \in \mathcal{N}$.*
(5) *$\widetilde{\varepsilon}$ is an* $(\inf, \sup)$*-HFGBI (resp.,* $(\inf, \sup)$*-HFSS,* $(\inf, \sup)$*-HFBI) of $\mathcal{A}$ for each HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ and $n \in \mathcal{N}$ such that $\mathrm{F}_{\widetilde{\varepsilon}} = \frac{-\varphi}{1+n}$ and $\mathrm{F}^{\widetilde{\varepsilon}} = \frac{n+\delta}{1+n}$.*

Proof. By Theorems 3.7 and 3.8, the conditions $(2) - (5)$ are equivalent.

$(1) \Rightarrow (2)$. Assume that B is a BFGBI of $\mathcal{A}$. Then φ is a NFGBI and δ is a FGBI of $\mathcal{A}$. Thus $-\varphi$ is a FGBI of $\mathcal{A}$ because using Lemma 3.12. Hence, for all $p, q, z \in \mathcal{A}$ and $n \in \mathcal{N}$, we have

$$\begin{aligned}
\mathrm{F}_{\mathcal{H}\langle B,n\rangle}(pzq) = \frac{-\varphi(pzq)}{1+n} &\geq \min\{\frac{-\varphi(p)}{1+n}, \frac{-\varphi(q)}{1+n}\} \\
&= \min\{\mathrm{F}_{\mathcal{H}\langle B,n\rangle}(p), \mathrm{F}_{\mathcal{H}\langle B,n\rangle}(q)\}, \\
\mathrm{F}^{\mathcal{H}\langle B,n\rangle}(pzq) = \frac{n+\delta(pzq)}{1+n} &\geq \min\{\frac{n+\delta(p)}{1+n}, \frac{n+\delta(q)}{1+n}\} \\
&= \min\{\mathrm{F}^{\mathcal{H}\langle B,n\rangle}(p), \mathrm{F}^{\mathcal{H}\langle B,n\rangle}(q)\}.
\end{aligned}$$

Hence $\mathrm{F}^{\mathcal{H}\langle B,n\rangle}$ and $\mathrm{F}_{\mathcal{H}\langle B,n\rangle}$ are FGBIs of $\mathcal{A}$ for all $n \in \mathcal{N}$. Therefore, it follows from Lemma 3.2 that $\mathcal{H}\langle B, n\rangle$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ for all $n \in \mathcal{N}$.

$(2) \Rightarrow (1)$. Assume that $\mathcal{H}\langle B, n\rangle$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ for all $n \in \mathcal{N}$. Then the BFS $\mathcal{H}\langle B, 1\rangle$ is an $(\inf, \sup)$-HFGBI of $\mathcal{A}$. By using Lemma 3.2, we obtain that $\frac{1+\delta}{2} = \mathrm{F}^{\mathcal{H}\langle B,1\rangle}$ is a FGBI and $\frac{\varphi}{2} = -\mathrm{F}_{\mathcal{H}\langle B,1\rangle}$ is a NFGBI of $\mathcal{A}$. Thus, for each $p, q, z \in \mathcal{A}$, we get

$$\begin{aligned}
\varphi(pzq) &= 2(\frac{\varphi(pzq)}{2}) \leq 2(\max\{\frac{\varphi(p)}{2}, \frac{\varphi(q)}{2}\}) = \max\{\varphi(p), \varphi(q)\}, \\
\delta(pzq) &= 2(\frac{1+\delta(pzq)}{2}) - 1 \geq \min\{2(\frac{1+\delta(p)}{2}) - 1, 2(\frac{1+\delta(q)}{2}) - 1\} \\
&= \min\{\delta(p), \delta(q)\}.
\end{aligned}$$

Hence φ is a NFGBI and δ is a FGBI of $\mathcal{A}$. Therefore, we conclude that B is a BFGBI of $\mathcal{A}$. □

5. Characterizing semigroups in terms of $(\inf, \sup)$-HFBIs

In this section, we characterize a completely regular semigroup, a group and a semigroup which is a semilattice of groups via $(\inf, \sup)$-HFBIs.

We recall that a semigroup $\mathcal{A}$ is said to be

- *regular* [33] if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = pqp$,
- *intra-regular* [33] if for each $z \in \mathcal{A}$, there exist $p, q \in \mathcal{A}$ such that $z = pz^2q$,
- *completely regular* [33] if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = pqp$ and $pq = qp$,

- *right (left) regular* [33] if for each $p \in \mathcal{A}$ there exists $q \in \mathcal{A}$ such that $p = p^2 q$ ($p = qp^2$),
- *normal* [33] if $p\mathcal{A} = \mathcal{A}p$ for all $p \in \mathcal{A}$,
- a *band* [7] if $p = p^2$ for all $p \in \mathcal{A}$,
- *commutative* [7] if $pq = qp$ for all $p, q \in \mathcal{A}$,
- a *semilattice of groups* [33] if it is the set-theoretical union of a family of mutually disjoint subgroups $G_i (i \in I)$ such that for all $i, j \in I$ the products G_iG_j and G_jG_i are both contained in the same subgroup $G_k (k \in I)$,
- a *group* [33] if an identity element exists and every element has an inverse.

Theorem 5.1. *In a regular semigroup $\mathcal{A}$, every* $(\inf, \sup)$*-HFGBI of $\mathcal{A}$ is an* $(\inf, \sup)$*-HFSS of $\mathcal{A}$, and so an* $(\inf, \sup)$*-HFBI of $\mathcal{A}$.*

Proof. Let $\widetilde{\varepsilon}$ be an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. There exists $z \in \mathcal{A}$ such that $p = pzp$. Thus $\min\{\mathrm{F}_{\widetilde{\varepsilon}}(p), \mathrm{F}_{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}_{\widetilde{\varepsilon}}(p(zp)q) = \mathrm{F}_{\widetilde{\varepsilon}}(pq)$ and $\min\{\mathrm{F}^{\widetilde{\varepsilon}}(p), \mathrm{F}^{\widetilde{\varepsilon}}(q)\} \leq \mathrm{F}^{\widetilde{\varepsilon}}(p(zp)q) = \mathrm{F}^{\widetilde{\varepsilon}}(pq)$. Hence $\mathrm{F}^{\widetilde{\varepsilon}}$ and $\mathrm{F}_{\widetilde{\varepsilon}}$ are FSSs of $\mathcal{A}$. Therefore, it follows from Lemma 3.2 that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFSS of $\mathcal{A}$. □

Lemma 5.2. [8] *A semigroup $\mathcal{A}$ is a group if and only if it contains no proper bi-ideal.*

For elements $\Pi, \Psi \in \wp([0, 1])$, we define $\Pi \approxeq \Psi$ if and only if $\mathrm{SUP}\,\Pi = \mathrm{SUP}\,\Psi$ and $\mathrm{INF}\,\Pi = \mathrm{INF}\,\Psi$. A HFS $\widetilde{\varepsilon}$ on $\mathcal{X}$ is called *constant* if $\widetilde{\varepsilon}(p) = \widetilde{\varepsilon}(q)$ for all $p, q \in \mathcal{X}$, and called $(\inf, \sup)$-*constant* if $\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(q)$ for all $p, q \in \mathcal{X}$. Then the following conditions are true.

(1) If $\Pi = \Psi$, then $\Pi \approxeq \Psi$.
(2) If $\mathcal{Y}$ is a subset of $\mathcal{X}$ and $p, q \in \mathcal{X}$, then $\mathcal{CH}(\mathcal{Y})(p) \approxeq \mathcal{CH}(\mathcal{Y})(q)$ if and only if $\mathcal{CH}(\mathcal{Y})(p) = \mathcal{CH}(\mathcal{Y})(q)$.
(3) If a HFS $\widetilde{\varepsilon}$ on $\mathcal{X}$ is constant, then $\widetilde{\varepsilon}$ is $(\inf, \sup)$-constant.
(4) If a HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ is $(\inf, \sup)$-constant, then $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFBI of $\mathcal{A}$.

Theorem 5.3. *Let $\mathcal{A}$ be a semigroup. The following conditions are equivalent.*

(1) *$\mathcal{A}$ is a group.*
(2) *Every* $(\inf, \sup)$*-HFGBI of $\mathcal{A}$ is* $(\inf, \sup)$*-constant.*
(3) *Every* $(\inf, \sup)$*-HFBI of $\mathcal{A}$ is* $(\inf, \sup)$*-constant.*

Proof. (1) ⇒ (2). Assume that $\mathcal{A}$ is a group with the identity e. Let $\widetilde{\varepsilon}$ be an $(\inf, \sup)$-HFGBI of $\mathcal{A}$ and $p \in \mathcal{A}$. Then

$$\begin{aligned}\mathrm{INF}\,\widetilde{\varepsilon}(p) &= \mathrm{INF}\,\widetilde{\varepsilon}(epe) \geq \mathrm{INF}\,\widetilde{\varepsilon}(e) = \mathrm{INF}\,\widetilde{\varepsilon}(ee)\\ &= \mathrm{INF}\,((pp^{-1})(p^{-1}p)) = \mathrm{INF}\,(p(p^{-1}p^{-1})p) \geq \mathrm{INF}\,\widetilde{\varepsilon}(p).\end{aligned}$$

Thus $\mathrm{INF}\,(e) = \mathrm{INF}\,(p)$. By the similar arguments, we obtain that $\mathrm{SUP}\,\widetilde{\varepsilon}(e) = \mathrm{SUP}\,\widetilde{\varepsilon}(p)$. Hence $\widetilde{\varepsilon}(e) \approxeq \widetilde{\varepsilon}(p)$ for all $p \in \mathcal{A}$, which implies that $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-constant.

(2) ⇒ (3). It is obvious.

(3) $\Rightarrow$ (1). Let $\mathcal{X}$ be a BI of $\mathcal{A}$ and $p \in \mathcal{X}$. By Theorem 4.3, we have $\mathcal{CH}(\mathcal{X})$ is an $(\inf, \sup)$-HFBI of $\mathcal{A}$. By the assumption (3), we obtain that $\mathcal{CH}(\mathcal{X})$ is $(\inf, \sup)$-constant. Hence $\mathcal{CH}(\mathcal{X})(q) \approxeq \mathcal{CH}(\mathcal{X})(p) = [0, 1]$ for all $q \in \mathcal{A}$, which implies that $\mathcal{X} = \mathcal{A}$. It is follows from Lemma 5.2 that $\mathcal{A}$ is a group. $\square$

Lemma 5.4. [36] *Let $\mathcal{A}$ be a semigroup. Then $\mathcal{A}$ is completely regular if and only if $p \in p^2\mathcal{A}p^2$ for all $p \in \mathcal{A}$.*

Theorem 5.5. *Let $\mathcal{A}$ be a semigroup. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is completely regular.*
(2) *$\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(p^2)$ for each $(\inf, \sup)$-HFGBI $\widetilde{\varepsilon}$ of $\mathcal{A}$ and $p \in \mathcal{A}$.*
(3) *$\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(p^2)$ for each $(\inf, \sup)$-HFBI $\widetilde{\varepsilon}$ of $\mathcal{A}$ and $p \in \mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2). Let $\widetilde{\varepsilon}$ be an $(\inf, \sup)$-HGBI of $\mathcal{A}$ and $p \in \mathcal{A}$. By the assumption (1) and Lemma 5.4, there exists $q \in \mathcal{A}$ such that $p = p^2qp^2$. Since $\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HGBI of $\mathcal{A}$ and Lemma 3.2, we have

$$\mathrm{F}_{\widetilde{\varepsilon}}(p) = \mathrm{F}_{\widetilde{\varepsilon}}(p^2qp^2) \geq \mathrm{F}_{\widetilde{\varepsilon}}(p^2) = \mathrm{F}_{\widetilde{\varepsilon}}(p(pqp^2)p) \geq \mathrm{F}_{\widetilde{\varepsilon}}(p).$$

Thus $\mathrm{INF}\,\widetilde{\varepsilon}(p) = \mathrm{F}_{\widetilde{\varepsilon}}(p) = \mathrm{F}_{\widetilde{\varepsilon}}(p^2) = \mathrm{INF}\,\widetilde{\varepsilon}(p^2)$. By the similar arguments, we can prove that $\mathrm{SUP}\,\widetilde{\varepsilon}(p) = \mathrm{SUP}\,\widetilde{\varepsilon}(p^2)$. Hence $\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(p^2)$.
(2) $\Rightarrow$ (3). It is clear.
(3) $\Rightarrow$ (1). Let $p \in \mathcal{A}$. Then $B[p^2] := \{p^2\} \cup \{p^4\} \cup p^2\mathcal{A}p^2$ is a BI of $\mathcal{A}$. By Theorem 4.3, we have that $\mathcal{CH}(B[p^2])$ is an $(\inf, \sup)$-HFBI of $\mathcal{A}$. By the assumption (3) and $p^2 \in B[p^2]$, we get

$$\mathcal{CH}(B[p^2])(p) \approxeq \mathcal{CH}(B[p^2])(p^2) = [0, 1].$$

Thus $p \in B[p^2]$ which implies that $p \in p^2\mathcal{A}p^2$. It follows from Lemma 5.4 that $\mathcal{A}$ is completely regular. $\square$

Definition 5.6. An $(\inf, \sup)$-HFBI $\widetilde{\varepsilon}$ of $\mathcal{A}$ is called *B-normal* if $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$ for all $p, q \in \mathcal{A}$.

Note that if a HFS $\widetilde{\varepsilon}$ on $\mathcal{A}$ is $(\inf, \sup)$-constant, then $\widetilde{\varepsilon}$ is B-normal.

Definition 5.7. A semigroup $\mathcal{A}$ is called $(\inf, \sup)$-*hesitant fuzzy B^*-normal* ($(\inf, \sup)$-HFB^*-normal) if every $(\inf, \sup)$-HFBI of $\mathcal{A}$ is B-normal.

Theorem 5.8. *Every $(\inf, \sup)$-HFB^*-normal semigroup is normal.*

Proof. Assume that $\mathcal{A}$ is an $(\inf, \sup)$-HFB^*-normal semigroup. Let $p \in \mathcal{A}$ and $q \in p\mathcal{A}$. Then, there exists $z \in \mathcal{A}$ such that $q = pz$. Thus $B[zp] := \{zp\} \cup \{(zp)^2\} \cup zp\mathcal{A}zp$ is a BI of $\mathcal{A}$ and by using Theorem 4.3, we get that $\mathcal{CH}(B[zp])$ is an $(\inf, \sup)$-HFBI of $\mathcal{A}$. By the assumption, we obtain that $\mathcal{CH}(B[zp])$ is B-normal and so

$$\mathcal{CH}(B[zp])(q) = \mathcal{CH}(B[zp])(pz) \approxeq \mathcal{CH}(B[zp])(zp) = [0, 1].$$

Then $q \in B[zp] \subseteq \mathcal{A}p$. Hence $p\mathcal{A} \subseteq \mathcal{A}p$. By the similar arguments, we can show that $\mathcal{A}p \subseteq p\mathcal{A}$. Therefore $p\mathcal{A} = \mathcal{A}p$. Consequently, we have $\mathcal{A}$ is normal. $\square$

Theorem 5.9. *If a semigroup $\mathcal{A}$ is regular (resp., completely regular, left regular, right regular, intra-regular), then $\mathcal{A}$ is normal if and only if $\mathcal{A}$ is* $(\inf, \sup)$-*HFB***-normal.*

Proof. $(\Rightarrow)$. Assume that $\mathcal{A}$ is regular and normal. Let $\widetilde{\varepsilon}$ be an $(\inf, \sup)$-HFBI of $\mathcal{A}$ and $p, q \in \mathcal{A}$. There exists $w \in \mathcal{A}$ such that $pq = pqwpq$. Then $pq = pqwpqwpq \in pq(\mathcal{A}p)(q\mathcal{A})pq \subseteq (\mathcal{A}q)p(\mathcal{A}\mathcal{A})q(p\mathcal{A}) \subseteq q(\mathcal{A}p)\mathcal{A}(q\mathcal{A})p = qp(\mathcal{A}\mathcal{A}\mathcal{A})qp$ $\subseteq qp\mathcal{A}qp$. There exists $z \in \mathcal{A}$ such that $pq = qpzqp$.
Thus $\mathrm{INF}\,\widetilde{\varepsilon}(pq) = \mathrm{INF}\,\widetilde{\varepsilon}((qp)z(qp)) \geq \mathrm{INF}\,\widetilde{\varepsilon}(qp)$ and
$\mathrm{SUP}\,\widetilde{\varepsilon}(pq) = \mathrm{SUP}\,\widetilde{\varepsilon}((qp)z(qp)) \geq \mathrm{SUP}\,\widetilde{\varepsilon}(qp)$. Hence $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$. Therefore $\mathcal{A}$ is $(\inf, \sup)$-HFB^*-normal.
$(\Leftarrow)$. It follows from Theorem 5.8. □

Theorem 5.10. *Let $\mathcal{A}$ be a band. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is commutative.*
(2) *$\mathcal{A}$ is normal.*
(3) *$\mathcal{A}$ is* $(\inf, \sup)$-*HFB***-normal.*

Proof. Since $\mathcal{A}$ is a band, we have $\mathcal{A}$ is regular. It follows from Theorem 5.9 that the conditions (2) and (3) are equivalent.

(1) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Assume that $\mathcal{A}$ is $(\inf, \sup)$-HFB^*-normal and $p, q \in \mathcal{A}$. Then, by Theorem 5.8, we have $pq \in \mathcal{A}q = q\mathcal{A}$ and $qp \in q\mathcal{A} = \mathcal{A}q$. There exist $w, z \in \mathcal{A}$ such that $pq = qz$ and $qp = wq$. Since $\mathcal{A}$ is a band, we get

$$pq = qz = q(qz) = (qp)q = w(qq) = wq = qp.$$

Thus $\mathcal{A}$ is commutative. □

Theorem 5.11. *For an* $(\inf, \sup)$-*HFB***-normal semigroup $\mathcal{A}$, the following conditions are equivalent.*

(1) *$\mathcal{A}$ is regular.*
(2) *$\mathcal{A}$ is right regular.*
(3) *$\mathcal{A}$ is left regular.*
(4) *$\mathcal{A}$ is intra-regular.*
(5) *$\mathcal{A}$ is completely regular.*

Proof. It follows from Lemma 5.4 and Theorem 5.8. □

Theorem 5.12. [33] *Let $\mathcal{A}$ be a semigroup. Then $\mathcal{A}$ is a semilattice of groups if and only if $\mathcal{A}$ is regular and normal.*

Theorem 5.13. *For a semigroup $\mathcal{A}$, the following conditions are equivalent.*

(1) *$\mathcal{A}$ is a semilattice of groups.*
(2) *$\mathcal{A}$ is right regular and* $(\inf, \sup)$-*HFB***-normal.*
(3) *$\mathcal{A}$ is left regular and and* $(\inf, \sup)$-*HFB***-normal.*
(4) *$\mathcal{A}$ is intra-regular and* $(\inf, \sup)$-*HFB***-normal.*
(5) *$\mathcal{A}$ is regular and* $(\inf, \sup)$-*HFB***-normal.*

(6) $\mathcal{A}$ *is completely regular and* $(\inf, \sup)$*-HFB***-normal.*
(7) $\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(p^2)$ *and* $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$ *for each* $(\inf, \sup)$*-HFBI* $\widetilde{\varepsilon}$ *of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$*.*
(8) $\widetilde{\varepsilon}(p) \approxeq \widetilde{\varepsilon}(p^2)$ *and* $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$ *for each* $(\inf, \sup)$*-HFGBI* $\widetilde{\varepsilon}$ *of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$*.*

Proof. It follows from Theorems 5.1, 5.5, 5.11 and 5.12. □

Theorem 5.14. *Let* $\mathcal{A}$ *be a regular (resp., intra-regular, completely regular, left regular, right regular) semigroup. The following conditions are equivalent.*

(1) $\mathcal{A}$ *is a semilattice of groups.*
(2) $\mathcal{A}$ *is normal.*
(3) $\mathcal{A}$ *is* $(\inf, \sup)$*-HFB***-normal.*
(4) $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$ *for each* $(\inf, \sup)$*-HFGBI* $\widetilde{\varepsilon}$ *of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$*.*

Proof. It follows from Theorems 5.9, 5.11 and 5.13. □

Theorem 5.15. *Let* $\mathcal{A}$ *be a band. The following conditions are equivalent.*

(1) $\mathcal{A}$ *is a semilattice of groups.*
(2) $\mathcal{A}$ *is commutative.*
(3) $\mathcal{A}$ *is normal.*
(4) $\mathcal{A}$ *is* $(\inf, \sup)$*-HFB***-normal.*
(5) $\widetilde{\varepsilon}(pq) \approxeq \widetilde{\varepsilon}(qp)$ *for each* $(\inf, \sup)$*-HFGBI* $\widetilde{\varepsilon}$ *of* $\mathcal{A}$ *and* $p, q \in \mathcal{A}$*.*

Proof. It follows from Theorems 5.10 and 5.14. □

6. Conclusions and future work

In present paper, we have introduced the concept of $(\inf, \sup)$-HFBIs (resp., $(\inf, \sup)$-HFSSs, $(\inf, \sup)$-HFGBIs), which is a generalization of the concept of IvFBIs (resp., IvFSSs, IvFGBIs), of semigroups and investigated its related properties. $(\inf, \sup)$-HFSSs, $(\inf, \sup)$-HFGBIs and $(\inf, \sup)$-HFBIs of semigroups have been characterized in terms of sets, FSs, NFSs, IvFSs, PFSs, HFSs and BFSs. Also, characterizations of BIs, FBIs, AFBIs, NFBIs, PFBIs and BFBIs have been investigated in terms of the $(\inf, \sup)$-type of HFSs. As important study results, we have provided conditions under which a semigroup can be completely regular, a group and a semilattice of groups in terms of $(\inf, \sup)$-HFBIs.

In the future, the concepts and findings from this study will be applied to ternary semigroups, Γ-semigroups, LA-semigroups, BCK/BCI/BCC-algebras, and other important algebraic systems to further examine their applicability as a mathematical tool for automation systems, decision theory, etc.

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References

1. M.Y. Abbasi, A.F. Talee, S.A. Khan and K. Hila, *A hesitant fuzzy set approach to ideal theory in G-semigroups*, Adv. Fuzzy Syst. **2018** (2018), Article ID 5738024, 6 pages.
2. S. Anis, M. Khan and Y.B. Jun, *Hybrid ideals in semigroups*, Cogent Math. **4** (2017), Article ID 1352117.
3. K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. **20** (1986), 87-96.
4. M.S. Cheong and K. Hur, *Interval-valued fuzzy ideals and bi-ideals of a semigroup*, Int. J. Fuzzy Log. Intell. Syst. **11** (2011), 259-266.
5. V. Chinnadurai and A. Arulselvam, *On Pythagorean Fuzzy Ideals in Semigroups*, J. Xi'an Univ. Arch. Tech. **12** (2020), 1005-1012.
6. N. Chunsee, R. Prasertpong, P. Khamrot, T. Gaketem, A. Iampan and P. Julatha, (inf, sup)-*Hesitant fuzzy subalgebras of BCK/BCI-algebras*, J. Math. Comput. Sci. **29** (2023), 142-155.
7. A.H. Clifford, *Bands of semigroups*, Proc. Am. Math. Soc. **5** (1954), 499-504.
8. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, American Mathematical Society Providence, Rhode Island, 1977.
9. H. Harizavi and Y.B. Jun, *Sup-hesitant fuzzy quasi-associative ideals of BCI-algebras*, Filomat **34** (2020), 4189-4197.
10. C. Jana, M. Pal and A.B. Saied, $(\in, \in \vee q)$*-bipolar fuzzy BCK/BCI-algebras*, Missouri J. Math. Sci. **29** (2017), 139-160.
11. C. Jana, T. Senapati, M. Bhowmik and M. Pal, *On intuitionistic fuzzy G-subalgebras of G-algebras*, Fuzzy Inf. Eng. **7** (2015), 195-209.
12. C. Jana, T. Senapati and M. Pal, $(\in, \in \vee q)$*-Intuitionistic fuzzy BCI-subalgebras of a BCI-algebra*, J. Intell. Fuzzy Syst. **31** (2016), 613-621.
13. C. Jana, T. Senapati, K.P. Shum and M. Pal, *Bipolar fuzzy soft subalgebras and ideals of BCK/BCI-algebras based on bipolar fuzzy points*, J. Intell. Fuzzy Syst. **37** (2019), 2785-2795.
14. U. Jittburus, P. Julatha and A. Iampan, *INF-Hesitant fuzzy ideals of semigroups and their INF-hesitant fuzzy translations*, J. Discrete Math. Sci. Cryptography **25** (2022), 1487-1507.
15. U. Jittburus and P. Julatha, inf*-hesitant fuzzy interior ideals of semigroups*, Int. J. Math. Comput. Sci. **17** (2022), 775-783.
16. U. Jittburus and P. Julatha, *New generalizations of hesitant and interval-valued fuzzy ideals of semigroups*, Adv. Math. Sci. J. **10** (2021), 2199-2212.
17. P. Julatha, T. Gaketem, P. Khamrot, R. Chinram and A. Iampan, sup*-Hesitant fuzzy ideals and bi-ideals of semigroups*, Submitted.
18. P. Julatha and A. Iampan, *A new generalization of hesitant and interval-valued fuzzy ideals of ternary semigroups*, Int. J. Fuzzy Log. Intell. Syst. **21** (2021), 169-175.
19. P. Julatha and A. Iampan, inf*-Hesitant and* (sup, inf)*-hesitant fuzzy ideals of ternary semigroups*, Missouri J. Math. Sci. Accepted.
20. P. Julatha and A. Iampan, *On* inf*-hesitant fuzzy* Γ*-ideals of* Γ*-semigroups*, Adv. Fuzzy Syst. **2022** (2022), Article ID 9755894.
21. P. Julatha and A. Iampan, SUP *-Hesitant fuzzy ideals of* Γ*-semigroups*, J. Math. Comput. Sci. **26** (2022), 148-161.
22. P. Julatha and M. Siripitukdet, *Some characterizations of anti-fuzzy (generalized) bi-ideals of semigroups*, Thai J. Math. **16** (2018), 335-346.
23. Y.B. Jun, M.S. Kang and C.H. Park, *N-Subalgebras in BCK/BCI-algebras based on point N-structures*, Int. J. Math. Math. Sci. **2010** (2010), Article ID 303412.

24. Y.B. Jun, C.S. Kim and M.S. Kang, *Cubic subalgebras and ideals of BCK/BCI-algebras*, Far East J. Math. Sci. **44** (2010), 239-250.
25. Y.B. Jun and A. Khan, *Cubic ideals in semigroups*, Honam Math. J. **35** (2013), 607-623.
26. Y.B. Jun, K.J. Lee and S.Z. Song, *Hesitant fuzzy bi-ideals in semigroups*, Commun. Korean Math. Soc. **30** (2015), 143-154.
27. Y.B. Jun, K.J. Lee and S.Z. Song, *N-Ideals of BCK/BCI-algebras*, J. Chungcheong Math. Soc. **22** (2009), 417-437.
28. Y.B. Jun, S.Z. Song and G. Muhiuddin, *Hybrid structures and applications*, Ann. Commun. Math. **1** (2018), 11-25.
29. M. Khan and T. Asif, *Characterizations of semigroups by their anti fuzzy ideals*, J. Math. Res. **2** (2010), 134-143.
30. C.S. Kim, J.G. Kang and J.M. Kang, *Ideal theory of semigroups based on the bipolar valued fuzzy set theory*, Ann. Fuzzy Math. Inform. **2** (2011), 193-206.
31. K.J. Lee, *Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras*, Bull. Malays. Math. Sci. Soc. **32** (2009), 361-373.
32. K.C. Lee, H.W. Kang and K. Hur, *Interval-valued fuzzy generalized bi-ideals of a semigroup*, Honam Math. J. **33** (2011), 603-616.
33. J.N. Mordeson, D.S. Malik and N. Kuroki, *Fuzzy semigroups*, Springer, 2003.
34. G. Muhiuddin and Y.B. Jun, *Sup-hesitant fuzzy subalgebras and its translations and extensions*, Ann. Commun. Math. **2** (2019), 48-56.
35. A.L. Narayanan and T. Manikantan, *Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semigroups*, J. Appl. Math. Comput. **20** (2006), 455-464.
36. M. Petrich, *Introduction to semigroups*, Columbus, Ohio, 1973.
37. P. Phummee, S. Papan, C. Noyoampaeng, U. Jittburus, P. Julatha and A. Iampan, sup-*Hesitant fuzzy interior ideals of semigroups and their* sup-*hesitant fuzzy translations*, Int. J. Innov. Comput. Inf. Control **18** (2022), 121-132.
38. N. Ratchakhwan, P. Julatha, T. Gaketem, P. Khamrot, R. Prasertpong and A. Iampan, $(\inf, \sup)$-*Hesitant fuzzy ideals of BCK/BCI-algebras*, Int. J. Anal. Appl. **20** (2022), 34.
39. M. Shabir and Y. Nawaz, *Semigroups characterized by the properties of their anti fuzzy ideals*, J. Adv. Res. Pure Math. **1** (2009), 42-59.
40. M.M. Takallo, R.A. Borzooei and Y.B. Jun, *Sup-Hesitant fuzzy p-ideals of BCI-algebras*, Fuzzy Inf. Eng. **13** (2021), 460-469.
41. A.F. Talee, M.Y. Abassi and S.A. Khan, *Hesitant fuzzy ideals in semigroups with a frontier*, Arya Bhatta J. Math. Inf. **9** (2017), 163-170.
42. N. Thillaigovindan and V. Chinnadurai, *On interval valued fuzzy quasi-ideals of semigroups*, East Asian Math. J. **25** (2009), 449-467.
43. V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. **25** (2010), 529-539.
44. V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, In: 2009 IEEE International Conference on Fuzzy Systems, IEEE, Jeju Island, South Korea, 2009, 1378-1382.
45. R.R. Yager and A.M. Abbasov, *Pythagorean membership grades, complex numbers, and decision making*, Int. J. Intell. Syst. **28** (2013), 436-452.
46. R.R. Yager, *Pythagorean fuzzy subsets*, In: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE, Edmonton, AB, Canada, 2013, 57-61.
47. P. Yiarayong, *An* (α, β)*-hesitant fuzzy set approach to ideal theory in semigroups*, Bull. Sect. Log., Univ. Łódź, Dep. Log. 2022, 28 pages.
48. L.A. Zadeh, *Fuzzy sets*, Inform. Control. **8** (1965), 338-353.
49. L.A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning I*, Inform. Sci. **8** (1975), 199-249.
50. W.R. Zhang, *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis*, In: NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, IEEE, San Antonio, TX, USA, 1994, 305-309.

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