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VARIOUS TYPES OF (p,q)-DIFFERENTIAL EQUATIONS RELATED WITH SPECIAL POLYNOMIALS

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Abstract. We introduce several higher-order (p,q)-differential equation of which are related to (p,q)-Bernoulli polynomials. We also find some relations between (p,q)-Bernoulli, (p,q)-Euler, and (p,q)-Genocchi polynomials.

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1. Introduction

For any $n\in\mathbb{C}$, the (p,q)-number is defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}.$$

Wachs and White [9] introduced the (p,q)-numbers in mathematics literature in certain combinatorial problems without any connection to the quantum group related to mathematics and physics literature, see [4], [5] [9].

Definition 1.1. [1], [8] Let z be any complex numbers with $|z|<1$. The two forms of (p,q)-exponential functions are defined by

$$e_{p,q}(z)=\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!},\quad E_{p,q}(z)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}.$$

In [2], Corcino made the theorem of (p,q)-extension of binomials coefficients and found various properties which are related to horizontal function, triangular function, and vertical function.

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Definition 1.2. [2] Let $n \geq k$. (p,q)-Gauss Binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!},$$

where $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$.

Definition 1.3. [1], [8] (p,q)-derivative operator of any function f, also referred to as the Jackson derivative, is defined the as follows:

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \qquad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$.
Let $p = 1$ in Definition 1.3. Then, we can remark

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \qquad x \neq 0,$$

we call D_q is the q-derivative.

Theorem 1.4. [1], [6] *The operator, $D_{p,q}$, has the following basic properties:*

(i) Derivative of a product
$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

(ii) Derivative of a ratio
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In 2016, Araci et al.[1] introduced a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p,q)-numbers and found some properties and identities. After that, several studies have investigated the special functions for various applications, see [3], [6], [7].

Definition 1.5. [3] (p,q)-Euler numbers $\mathcal{E}_{n,p,q}$ and polynomials $\mathcal{E}_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1}, \quad \sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1}e_{p,q}(tx).$$

Consider $p = 1$ in Definition 1.5. Then, we note

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}, \quad \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}e_q(tx),$$

where $\mathcal{E}_{n,q}$ is the q-Euler number and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials.

Definition 1.6. [3] (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty} G_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{2t}{e_{p,q}(t)+1}, \quad \sum_{n=0}^{\infty} G_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx).$$

Consider $p=1$ in Definition 1.6, we note

$$\sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{[n]_{q}!} = \frac{2t}{e_{q}(t)+1}, \quad \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_{q}!} = \frac{2t}{e_{q}(t)+1}e_{q}(tx),$$

where $G_{n,q}$ is the q-Genocchi numbers and $G_{n,q}(x)$ is the q-Genocchi polynomials.

Definition 1.7. [3] (p,q)-Bernoulli numbers $B_{n,p,q}$ and polynomials $B_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{t}{e_{p,q}(t)-1}, \quad \sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{t}{e_{p,q}(t)-1}e_{p,q}(tx).$$

Putting $p=1$ in Definition 1.7, we can note

$$\sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{[n]_{q}!} = \frac{t}{e_{q}(t)-1}, \quad \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_{q}!} = \frac{t}{e_{q}(t)-1}e_{q}(tx),$$

where $B_{n,q}$ is the q-Bernoulli numbers and $B_{n,q}(x)$ is the q-Bernoulli polynomials.

2. Main results

We construct several (p,q)-differential equation for (p,q)-Bernoulli polynomials. We also find some relations between (p,q)-Bernoulli, (p,q)-Euler, and (p,q)-Genocchi polynomials.

Theorem 2.1. *Let $[n]_{p,q} \neq 0$. Then, we have*

$$D_{p,q,x}B_{n,p,q}(x) = [n]_{p,q}B_{n-1,p,q}(px).$$

Proof. From the generating function of (p,q)-Bernoulli polynomials, we have a relation as

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_{p,q} p^{\binom{n-k}{2}}B_{k,p,q}x^{n-k}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{2.1}$$

Comparing the coefficients of the both-sides in Eq. (2.1), we obtain

$$B_{n,p,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_{p,q} p^{\binom{n-k}{2}}B_{k,p,q}x^{n-k} \tag{2.2}$$

By applying (p,q)-derivative in Eq. (2.2), we find

$$\begin{aligned} D^{(1)}_{p,q,x}B_{n,p,q}(x) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}B_{k,p,q}D^{(1)}_{p,q,x}x^{n-k} \\ &= \sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}[n-k]_{p,q}p^{\binom{n-k-1}{2}}B_{k,p,q}(px)^{n-k-1} \end{aligned} \tag{2.3}$$

From Equations (2.2) and (2.3), we find the required result. □

Corollary 2.2. *Considering $p=1$ in Theorem 2.1, one holds*

$$D_{q,x}B_{n,q}(x) = [n]_q B_{n-1,q}(x),$$

where D_q is q-derivative, $[n]_q$ is q-number, and $B_{n,q}(x)$ is the q-Bernoulli polynomials.

Corollary 2.3. *Considering $p=1, q\to 1$ in Theorem 2.1, one holds*

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x),$$

where $B_n(x)$ is the Bernoulli polynomials.

Corollary 2.4. *From Theorem 2.1, we have*

$$D^{(k)}_{p,q,x}B_{n,p,q}(x) = \frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!}B_{n-k,p,q}(p^k x).$$

Theorem 2.5. *The (p,q)-Bernoulli polynomials $B_{n,q}(x)$ satisfies the following higher-order (p,q)-differential equation.*

$$\begin{aligned} &\frac{1}{[n]_{p,q}!}D^{(n)}_{p,q,x}B_{n,p,q}(p^{-n}x) + \frac{1}{[n-1]_{p,q}!}D^{(n-1)}_{p,q,x}B_{n,p,q}(p^{-(n-1)}x) \\ &+\cdots+\frac{1}{[3]_{p,q}!}D^{(3)}_{p,q,x}B_{n,p,q}(p^{-3}x) + \frac{1}{[2]_{p,q}!}D^{(2)}_{p,q,x}B_{n,p,q}(p^{-2}x) \\ &+ D^{(1)}_{p,q,x}B_{n,p,q}(p^{-1}x) - [n]_{p,q}p^{\binom{n-1}{2}}x^{n-1} = 0. \end{aligned}$$

Proof. In order to find higher-order (p,q)-differential equation, we consider $e_{p,q}(t)\neq 1$. Then, (p,q)-Bernoulli polynomials can be transformed as

$$\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!}-1\right) = t\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!}.$$

By using Cauchy product and comparison of coefficients, we find

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{k}{2}}B_{n-k,p,q}(x) - B_{n,p,q}(x) = [n]_{p,q}p^{\binom{n-1}{2}}x^{n-1}. \tag{2.4}$$

From Corollary 2.4, we can note

$$D^{(k)}_{p,q,x}B_{n,p,q}(p^{-k}x) = \frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!}B_{n-k,p,q}(x). \tag{2.5}$$

Replacing Eq. (2.5) in the left-hand side of Eq. (2.4), we find

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}B_{n-k,p,q}(x)-B_{n,p,q}(x)\\&=\sum_{k=0}^{n}\frac{1}{[k]_{p,q}!}D_{p,q,x}^{(k)}B_{n,p,q}(p^{-k}x)-B_{n,p,q}(x).\end{aligned}\tag{2.6}$$

From Equations (2.4) and (2.6), we derive

$$\sum_{k=0}^{n}\frac{1}{[k]_{p,q}!}D_{p,q,x}^{(k)}B_{n,p,q}(p^{-k}x)-B_{n,p,q}(x)-[n]_{p,q}p^{\binom{n-1}{2}}x^{n-1}=0,$$

where the required result is completed at once. □

Corollary 2.6. *Putting $p=1$ in Theorem 2.5, one holds*

$$\begin{aligned}&\frac{1}{[n]_q!}D_{q,x}^{(n)}B_{n,q}(x)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}B_{n,q}(x)+\cdots\\&+\frac{1}{[3]_q!}D_{q,x}^{(3)}B_{n,q}(x)+\frac{1}{[2]_q!}D_{q,x}^{(2)}B_{n,q}(x)+D_{q,x}^{(1)}B_{n,q}(x)-[n]_qx^{n-1}=0,\end{aligned}$$

where $D_q^{(n)}$ is the q-derivative, $B_{n,q}(x)$ is the q-Bernoulli polynomials.

Corollary 2.7. *Setting $p=1, q\to 1$ in Theorem 2.5, one holds*

$$\begin{aligned}&\frac{1}{n!}\frac{d^n}{dx^n}B_n(x)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}B_n(x)+\frac{1}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}B_n(x)+\cdots\\&+\frac{1}{3!}\frac{d^3}{dx^3}B_n(x)+\frac{1}{2!}\frac{d^2}{dx^2}B_n(x)+\frac{d}{dx}^{(1)}B_n(x)-nx^{n-1}=0,\end{aligned}$$

where $B_n(x)$ is the Bernoulli polynomials.

Theorem 2.8. *The (p,q)-Bernoulli polynomials $B_{n,p,q}(x)$ satisfies the following higher-order (p,q)-differential equation which is combined (p,q)-Euler numbers and polynomials.*

$$\begin{aligned}&\frac{\mathcal{E}_{n,p,q}+\mathcal{E}_{n,p,q}(1)}{p^{\binom{n}{2}}[n]_{p,q}!}D_{p,q,x}^{(n)}B_{n,p,q}(x)+\frac{\mathcal{E}_{n-1,p,q}+\mathcal{E}_{n-1,p,q}(1)}{p^{\binom{n-1}{2}}[n-1]_{p,q}!}D_{p,q,x}^{(n-1)}B_{n,p,q}(x)+\cdots\\&+\frac{\mathcal{E}_{2,p,q}+\mathcal{E}_{2,p,q}(1)}{p[2]_{p,q}!}D_{p,q,x}^{(2)}B_{n,p,q}(x)+(\mathcal{E}_{1,p,q}+\mathcal{E}_{1,p,q}(1))D_{p,q,x}^{(1)}B_{n,p,q}(x)\\&+(\mathcal{E}_{0,p,q}+\mathcal{E}_{0,p,q}(1)-2)B_{n,p,q}(x)=0,\end{aligned}$$

where $\mathcal{E}_{n,p,q}$ is (p,q)-Euler numbers and $\mathcal{E}_{n,p,q}(x)$ is the (p,q)-Euler polynomials.

Proof. From the generating function of (p,q)-Bernoulli polynomials, we consider the equation

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= \frac{t}{e_{p,q}(t)-1}e_{p,q}(tx)\\ &= \frac{1}{2}\left(\frac{2}{e_{p,q}(t)+1}+\frac{2}{e_{p,q}(t)+1}e_{p,q}(t)\right)\frac{t}{e_{p,q}(t)-1}e_{p,q}(tx)\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(\mathcal{E}_{k,p,q}+\mathcal{E}_{k,p,q}(1))B_{n-k,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{2.7}$$

In Eq. (2.7), we obtain a relation as

$$2B_{n,p,q}(x) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(\mathcal{E}_{k,p,q}+\mathcal{E}_{k,p,q}(1))B_{n-k,p,q}(x). \tag{2.8}$$

Applying $D_{p,q,x}^{(k)}B_{n,p,q}(x) = \frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!}B_{n-k,p,q}(p^k x)$ in Eq. (2.8), we find

$$\sum_{k=0}^{n}\frac{\mathcal{E}_{k,p,q}+\mathcal{E}_{k,p,q}(1)}{p^{\binom{k}{2}}[k]_{p,q}!}D_{p,q,x}^{(k)}B_{n,p,q}(p^{-k}x)-2B_{n,p,q}(x)=0.$$

From the above equation, we obtain the required result. □

Corollary 2.9. *Setting $p=1$ in Theorem 2.8, one holds*

$$\begin{aligned}&\frac{\mathcal{E}_{n,q}+\mathcal{E}_{n,q}(1)}{[n]_q!}D_{q,x}^{(n)}B_{n,q}(x)+\frac{\mathcal{E}_{n-1,q}+\mathcal{E}_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}B_{n,q}(x)+\cdots\\ &+\frac{\mathcal{E}_{2,q}+\mathcal{E}_{2,q}(1)}{[2]_q!}D_{q,x}^{(2)}B_{n,q}(x)+(\mathcal{E}_{1,q}+\mathcal{E}_{1,q}(1))D_{q,x}^{(1)}B_{n,q}(x)\\ &+(\mathcal{E}_{0,q}+\mathcal{E}_{0,q}(1)-2)B_{n,q}(x)=0,\end{aligned}$$

where $D_q^{(n)}$ is the q-derivative, $\mathcal{E}_{n,q}$ is q-Euler numbers, and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials.

Corollary 2.10. *Setting $p=1, q\to 1$ in Theorem 2.8, one holds*

$$\begin{aligned}&\frac{\mathcal{E}_n+\mathcal{E}_n(1)}{n!}\frac{d^n}{dx^n}B_n(x)+\frac{\mathcal{E}_{n-1}+\mathcal{E}_{n-1}(1)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}B_n(x)+\cdots\\ &+\frac{\mathcal{E}_2+\mathcal{E}_2(1)}{2!}\frac{d^2}{dx^2}B_n(x)+(\mathcal{E}_1+\mathcal{E}_1(1))\frac{d}{dx}B_n(x)+(\mathcal{E}_0+\mathcal{E}_0(1)-2)B_n(x)=0,\end{aligned}$$

where $\mathcal{E}_n$ is the Euler numbers and $\mathcal{E}_n(x)$ is the Euler polynomials.

Theorem 2.11. *The (p,q)-Bernoulli polynomials $B_{n,p,q}(x)$ satisfies the following higher-order (p,q)-differential equation which is combined (p,q)-Genocchi* *numbers and polynomials.*

$$\frac{G_{n,p,q}+G_{n,p,q}(1)}{p^{\binom{n}{2}}[n]_{p,q}!}D_{p,q,x}^{(n)}B_{n,p,q}(p^{-n}x)$$
$$+\frac{G_{n-1,p,q}+G_{n-1,p,q}(1)}{p^{\binom{n-1}{2}}[n-1]_{p,q}!}D_{p,q,x}^{(n-1)}B_{n,p,q}(p^{-(n-1)}x)+\cdots$$
$$+\frac{G_{2,p,q}+G_{2,p,q}(1)}{p[2]_{p,q}!}D_{p,q,x}^{(2)}B_{n,p,q}(p^{-(2)}x)+(G_{1,p,q}+G_{1,p,q}(1))D_{p,q,x}^{(1)}B_{n,p,q}(p^{-1}x)$$
$$+(G_{0,p,q}+G_{0,p,q}(1))B_{n,p,q}(x)-2[n]_{p,q}B_{n-1,p,q}(x)=0,$$

where $G_{n,p,q}$ is the (p,q)-Genocchi numbers and $G_{n,p,q}(x)$ is the (p,q)-Genocchi polynomials.

Proof. From the generating function of $B_{n,p,q}(x)$, we have a relation as

$$\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}$$
$$=\frac{1}{2t}\left(\frac{2t}{e_{p,q}(t)+1}+\frac{2t}{e_{p,q}(t)+1}e_{p,q}(t)\right)\frac{t}{e_{p,q}(t)-1}e_{p,q}(tx)$$
$$=\frac{1}{2t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(G_{k,p,q}+G_{k,p,q}(1))B_{n-k,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}.$$

In a similar way of Theorem 2.5, we derive the following equation.

$$[n]_{p,q}B_{n-1,p,q}(x)=\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(G_{k,p,q}+G_{k,p,q}(1)\right)B_{n-k,p,q}(x).$$

From the equation above, we find the desired result. □

Corollary 2.12. *Setting $p=1$ in Theorem 2.11, the following holds*

$$\frac{G_{n,q}+G_{n,q}(1)}{[n]_q!}D_{q,x}^{(n)}B_{n,q}(x)+\frac{G_{n-1,q}+G_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}B_{n,q}(x)+\cdots$$
$$+\frac{G_{2,q}+G_{2,q}(1)}{[2]_q!}D_{q,x}^{(2)}B_{n,q}(x)+(G_{1,q}+G_{1,q}(1))D_{q,x}^{(1)}B_{n,q}(x)$$
$$+(G_{0,q}+G_{0,q}(1))B_{n,q}(x)-2[n]_qB_{n-1,q}(x)=0,$$

where $D_q^{(n)}$ is the q-derivative, $G_{n,q}$ is the q-Genocchi numbers, and $G_{n,q}(x)$ is the q-Genocchi polynomials.

Corollary 2.13. *Putting $p=1, q\to 1$ in Theorem 2.11, the following holds*

$$\frac{G_n+G_n(1)}{n!}\frac{d^n}{dx^n}B_n(x)+\frac{G_{n-1}+G_{n-1}(1)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}B_n(x)+\cdots+\frac{G_2+G_2(1)}{2!}\frac{d^2}{dx^2}B_n(x)$$
$$+(G_1+G_1(1))\frac{d}{dx}B_n(x)+(G_0+G_0(1))B_{n,q}(x)-2nB_{n-1}(x)=0,$$

where G_n is the Genocchi numbers and $G_n(x)$ is the Genocchi polynomials.

Theorem 2.14. *Let $\alpha \neq 0$, $\beta \neq 0$, and $0 < q < 1$. Then, we find*

$$\begin{aligned}
&\frac{\alpha^n B_{n,p,q}}{p^{\binom{n}{2}}[n]_{p,q}!} D^{(n)}_{p,q,x} B_{n,p,q}(\alpha^{-1}p^{-n}x) + \frac{\alpha^{n-1}\beta B_{n-1,p,q}}{p^{\binom{n-1}{2}}[n-1]_{p,q}!} D^{(n-1)}_{p,q,x} B_{n,p,q}(\alpha^{-1}p^{-(n-1)}x)\\
&+ \frac{\alpha^{n-2}\beta^2 B_{n-2,p,q}}{p^{\binom{n-2}{2}}[n-2]_{p,q}!} D^{(n-2)}_{p,q,x} B_{n,p,q}(\alpha^{-1}p^{-(n-2)}x) + \cdots\\
&+ \alpha\beta^{n-1} B_{1,p,q} D^{(1)}_{p,q,x} B_{n,p,q}(\alpha^{-1}p^{-1}x) + \beta^n B_{0,p,q} B_{n,p,q}(\alpha^{-1}x)\\
&= \frac{\beta^n B_{n,p,q}}{p^{\binom{n}{2}}[n]_{p,q}!} D^{(n)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-n}x) + \frac{\beta^{n-1}\alpha B_{n-1,p,q}}{p^{\binom{n-1}{2}}[n-1]_{p,q}!} D^{(n-1)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-(n-1)}x)\\
&+ \frac{\beta^{n-2}\alpha^2 B_{n-2,p,q}}{p^{\binom{n-2}{2}}[n-2]_{p,q}!} D^{(n-2)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-(n-2)}x) + \cdots\\
&+ \beta\alpha^{n-1} B_{1,p,q} D^{(1)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-1}x) + \alpha^n B_{0,p,q} B_{n,p,q}(\beta^{-1}x).
\end{aligned}$$

Proof. To find a symmetric property of higher-order (p,q)-differential equation for (p,q)-Bernoulli polynomials, we consider a form A as

$$A := \frac{(\alpha\beta t)^2 e_{p,q}(tx)}{(e_{p,q}(\alpha t) - 1)(e_{p,q}(\beta t) - 1)}.$$

From the form A, we obtain

$$\begin{aligned}
A &:= \frac{\alpha\beta t}{e_{p,q}(\alpha t) - 1} \frac{\alpha\beta t}{e_{p,q}(\beta t) - 1} e_{p,q}(tx)\\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha^{k+1}\beta^{n-k+1} B_{k,p,q} B_{n-k,p,q}(\beta^{-1}x) \right) \frac{t^n}{[n]_{p,q}!},
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
A &:= \frac{\alpha\beta t}{e_{p,q}(\beta t) - 1} \frac{\alpha\beta t}{e_{p,q}(\alpha t) - 1} e_{p,q}(tx)\\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \beta^{k+1}\alpha^{n-k+1} B_{k,p,q} B_{n-k,p,q}(\alpha^{-1}x) \right) \frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{2.10}$$

Comparing Eq. (2.9) and (2.10), we have

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha^{k+1}\beta^{n-k+1} B_{k,p,q} B_{n-k,p,q}(\beta^{-1}x)\\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \beta^{k+1}\alpha^{n-k+1} B_{k,p,q} B_{n-k,p,q}(\alpha^{-1}x)
\end{aligned} \tag{2.11}$$

Using a relation of $B_{n-k,p,q}(x)$ and $D^{(K)}_{p,q,x}B_{n,p,q}(x)$ to Eq. (2.11), we derive

$$\sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k} B_{k,p,q}}{p^{\binom{k}{2}}[k]_{p,q}!} D^{(k)}_{p,q,x} B_{n,p,q}(\alpha^{-1}p^{-k}x)$$
$$= \sum_{k=0}^{n} \frac{\beta^k \alpha^{n-k} B_{k,p,q}}{p^{\binom{k}{2}}[k]_{p,q}!} D^{(k)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-k}x).$$

Therefore, we complete the proof of Theorem 2.14. □

Corollary 2.15. *Setting $\alpha = 1$ in Theorem 2.14, one holds*

$$\sum_{k=0}^{n} \frac{\beta^{n-k} B_{k,p,q}}{p^{\binom{k}{2}}[k]_{p,q}!} D^{(k)}_{p,q,x} B_{n,p,q}(p^{-k}x) = \sum_{k=0}^{n} \frac{\beta^{k} B_{k,p,q}}{p^{\binom{k}{2}}[k]_{p,q}!} D^{(k)}_{p,q,x} B_{n,p,q}(\beta^{-1}p^{-k}x).$$

Corollary 2.16. *Considering $p = 1$ in Theorem 2.14, the following holds*

$$\sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k} B_{k,q}}{[k]_q!} D^{(k)}_{q,x} B_{n,q}(\alpha^{-1}x) = \sum_{k=0}^{n} \frac{\beta^k \alpha^{n-k} B_{k,q}}{[k]_q!} D^{(k)}_{q,x} B_{n,q}(\beta^{-1}x),$$

where D_q is q-derivative, $B_{n,q}$ is the q-Bernoulli numbers and $B_{n,q}(x)$ is the q-Bernoulli polynomials.

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References

1. S. Araci, U. Duran, M. Acikgoz, H.M. Srivastava, *A certain (p, q)-derivative operator and associated divided differences*, J. Inequal and Appl., 301 (2016),
2. R.B. Corcino, *On (P, Q)-Binomial coefficients*, Electron. J. Combin. Number Theory, 8 (2008), #A29.
3. U. Duran, M. Acikgoz, S. Araci, *On (p, q)-Bernoulli, (p, q)-Euler and (p, q)-Genocchi Polynomials*, J. Comp. and Theo. Nano., November 2016.
4. R. Jagannathan, K.S. Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, Proceeding of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India, 20-21 December 2005.
5. R. Jagannathan, *(p, q)-Special functions*, Special Functions and Differential Equations, Proceedings of a Workshop held at The Institute of Mathematical Sciences, Matras, India, January (1997), 13-24.
6. C.S. Ryoo, J.Y. Kang, *Various Types of q-Differential Equations of Higher Order for q-Euler and q-Genocchi Polynomials*, Mathematics 10 (2022), 1-16.
7. N. Saba and A. Boussayoud, *New Theorem on Symmetric Functions and Their Applications on Some (p, q)-numbers*, Journal of Applied Mathematics and Informatics, 40 (2022), 243-258.

8. P.N. Sadjang, *On the fundamental theorem of* (p, q)*-calculus and some* (p, q)*-Taylor formulas*, arXiv:1309.3934 [math.QA] (2013).
9. M. Wachs, D. White, (p, q)*-Stirling numbers and set partition statistics*, J. Conbin. Theory, A 56 (1991), 27-46.

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