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# SIMPLE RANKED SAMPLING SCHEME: MODIFICATION AND APPLICATION IN THE THEORY OF ESTIMATION OF ERLANG DISTRIBUTION[†]

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Abstract. This paper deals in the study of the estimation of the parameters of Erlang distribution based on rank set sampling and some of its modifications. Here we considered Maximum Likelihood (ML) and the Bayesian technique to estimate the shape and scale parameter of Erlang distribution based on RSS and its some modifications such as ERSS, MRSS, and MRSSu. The derivation for unknown parameters of Erlang distribution is well presented using normal approximation to the asymptotic distribution of ML estimators. But due to the complexity involves in the integral, the Bayes estimator of unknown parameters is obtained using MCMC method. Further, we compared the MSE of estimation in different sampling schemes with different set sizes and cycle size. A real-life data application is also given to illustrate the efficiency of the proposed scheme.

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## 1. Introduction

Ranked set sampling (RSS) is designed for situations where measurements are difficult to obtain but judgment ranking of a set of sampling units is fairly easy and reliable. It utilizes information gained without formal measurement to provide more structure to the final measured data as compared with the usual simple random sampling (SRS). Such an approach is applicable in environmental and agricultural issues, where it is clear that pre-sampling judgment can be quite cheap relative to the cost of detailed measurement of many quantities of interest. Ranked set sampling (RSS), introduced by [23], is a technique designed for situations where the sampling units are difficult or expensive to measure, but

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can be easily ordered by some means without actual quantification. [28] study the Keplan-Meier estimator of survival probability based on the RSS scheme under random censoring time setup while [21] used the estimator for estimating an extension of the well-known stress strength reliability in a non-parametric setup and compare it with the parametric analogous in the case of the exponential distribution. A logistic regression model was proposed by [10] to aid in the ranking of a binary variable of interest. [20] discussed in detail the goodness of fit tests of Laplace distribution based on sample entropy when the data are collected according to some ranked set sampling-based scheme.

A continuous probability distribution that has a wide range of applications and a strong relationship with exponential and Gamma distribution is known as Erlang distribution. It was developed by [11] to determine the number of telephone calls at the time of switching station operators. It can be expressed by the length of messages and waiting time in telephone traffic. If the exponential distribution is expressed as the individual call's duration, then the Erlang distribution is expressed as the succession of the call's duration. The Erlang distribution becomes Gamma variate when a shape parameter is an integer form [16]. [13] used two prior densities to obtain its Bayes estimator. The problem raised at the time of Bayesian parameter estimation of Erlang distribution was addressed by [16]. A simulator design was constructed of the project management process for time estimation [29]. The expected time between failure measures was obtained by [12]. In addition, they showed that the estimated failure time and actual failure time are approximately the same. Further, [17] investigated the procedure of computing the median asymptotic expansion of the Erlang distribution. The pdf and cdf of Erlang distribution are given by

$$g(x;\theta,k)=\frac{\theta^k x^{k-1}e^{-\theta x}}{(k-1)!} \tag{1}$$

$$G(x;\theta,k)=1-\sum_{n}\frac{1}{n!}e^{-\theta x}(\theta x)^n \tag{2}$$

Where $\theta$ and k are rate and shape parameters respectively, such that x is an integer number. If the shape parameter is 1, then the Erlang distribution becomes an exponential distribution. If the scale parameter is 2, then the Erlang distribution becomes the Chi-square distribution. Basically, the sub-models of the Erlang distribution are exponential and chi-square distribution. Moreover, the gamma distribution is a generalized form of Erlang distribution. The pdf curve of the Erlang distribution is shown in Figure 1. In this paper, we presented well-known sampling schemes like Simple random sampling, Rank set sampling and some of its modifications including Median ranked set sampling, Extreme rank set sampling, and Maximum rank set sampling scheme with unequal samples size for Erlang distributions. The main purpose of this paper is to compare the estimators that are based on different sampling schemes, Simple random sampling, Rank set sampling, Median ranked set sampling, Extreme rank set sampling, and Maximum rank set sampling scheme with unequal samples size for Erlang distributions. In section 2, our focus is to derive Erlang distribution parameters based on the above-mentioned schemes by using Maximum Likelihood Methods (ML). Further in the subsection, the Bayesian technique is used to obtain the point estimator of Erlang distribution parameters. For this, we

used three different loss functions that are PLF, SELF, and LELF. The explicit form of an integral using Bayes point estimate cannot handle easily. So instead of derivations, we use the Metro polio Hasting within Gibba (MHG) algorithm to solve this problem.
The rest of the article is organized as follows. The parameter estimation using maximum likelihood estimation and Bayesian techniques are presented in section 2. In section 3, a simulation study is carried out to check the performance of different sampling schemes for maximum likelihood and Bayesian methods. It also considers real-life data set to further check the performance of the proposed estimator in section 4. Finally, our conclusion and remarks are presented in Section 5.

## 2. Parameter Estimation under different sampling scheme

In this section, we use two different methods for parameter estimation of Erlang distribution. The first method comes from a classical approach which was established by Fisher around about 1930. Alternatively, the Bayesian approach is used which was first discovered by Reverend Thomas Bayes. Moreover, we have used one prior and three loss functions, which are elaborated on in their respective sections.

**2.1. Maximum Likelihood Estimation Method.** The most commonly used method is maximum likelihood estimation method, which was first discovered by Gaurs. In early 1920, Fisher later used in a series of paper to estimate parameters. It is the only method that gives sufficient estimators, which are known to be asymptotically MVUES. Therefore, the most important feature of this method is to take a random sample and then pick a maximum probability of an observed data. In this way, we can easily find the value of unknown parameter.

**2.1.1. Parameter Estimation under RSS.** RSS was the first time suggested by [23] for estimating the population means. He stated that RSS gives more precise estimators than SRS for estimating a population parameter. It is used in conditions, where accurate measurements of sample units are hard to observe due to high cost and more time utilization. For this reason, RSS may accurately rank a set of sample units without considering cost and time. For more application see [3, 9, 22] and references therein. Based on the same number of selected observations, RSS is a more representative technique than any other probability sampling technique. The following steps are used for the appliance of RSS:
1. Select 'm' units each of 'm' size.
2. Rank the units according to ascending order. The ranking is to be done with respect to variables of interest without actual measurements.
3. Actual measurements are grabbed only on the largest ith units in the ith sample. where i=1,2,......m.
4. Repeat the above steps 'r' times to get $n^* = r.m^*$
The developed sample is represented by X(i:m)j where ith unit is the largest unit in the jth cycle in a set of size 'm'. The resulted function of RSS is given by the

following equation (see e.g. [1])

$$g_{x_{(i:m)}i}(x_{(i:m)j}) = \prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}G^{i-1}(x_{(i:m)j})[1-G(x_{(i:m)j})]^{m-i}g(x_{(i:m)j}) \tag{3}$$

$$LRSS(k,\theta;x) = \prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}[1-\sum_{n}\frac{1}{n!}e^{-\theta x}(\theta x)^{n}]^{i-1}[1-\sum_{n}\frac{1}{n!}e^{-\theta x}(\theta x)^{n}]^{m-1}$$
$$* \frac{\theta^{n}k}{(k-1)!}[\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}]^{k-1}exp[-\theta\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}] \tag{4}$$

$$LRSS(k,\theta;x) = \frac{\theta^{n}k}{(k-1)!}\left[\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}\right]^{k-1} exp[-\theta\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}]\prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}$$
$$*[1-\sum_{n}\frac{1}{n!}e^{-\theta x}(\theta x)^{n}]^{i-1}[1-\sum_{n}\frac{1}{n!}e^{-\theta x}(\theta x)^{n}]^{m-1} \tag{5}$$

$$LRSS(k,\theta;x) = \frac{\theta^{n}k}{(k-1)!}\left[\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}\right]^{k-1} exp[-\theta \quad *[1-\zeta_{ij}]^{i-1}\zeta_{ij}^{m-1} \tag{6}$$

where $\zeta = \sum_{n}\frac{1}{n!}e^{-\theta}(\theta x)^{n}$

$$l_{RSS} = nklog\theta - \theta\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j} + (k-1)\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j} + log\prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}$$
$$+(i-1)log[1-\zeta_{ij}] + (m-1)log\zeta_{ij} + (k-1)! \tag{7}$$

Taking derivatives w.r.t $\theta$

$$\frac{dl_{RSS}}{d\theta} = \frac{nk}{\theta} - \sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j} - \frac{(i-1)}{1-\zeta_{ij}}\left(\frac{n\zeta_{ij}}{\theta x} - x\zeta_{ij}\right) + \frac{(m-i)}{\zeta_{ij}}\left(\frac{n\zeta_{ij}}{\theta x} - x\zeta_{ij}\right) \tag{8}$$

After solving the above equation, we get

$$\frac{dl_{RSS}}{d\theta} = \frac{nk}{\theta} - \sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j} + \left(\frac{(m-i)}{\zeta_{ij}} - \frac{(i-1)}{1-\zeta_{ij}}\right)\left(\frac{n\zeta_{ij}}{\theta x} - x\zeta_{ij}\right) \tag{9}$$

Taking derivative w.r.t $'k'$

$$\frac{dl_{RSS}}{d\theta} = nlog\theta - log\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j} + \left(-\gamma + \sum_{c=1}^{\infty}\frac{k-1}{c(c+k-1)}\right) \tag{10}$$

The MLEs for the parameter $\theta$ and k are obtained by maximizing the log likelihood in (7). This can be seen that on solving (9) and (10) equations simultaneously (and setting them is equals to zero), which do not have closed form solution. Therefore, Newton Raphson method is used to obtain the estimates. The algorithm consists of the following steps:
1. Set an initial guess $(\theta^{(0)}, k^{(0)})$
2. For iterations $t \geq 1$, do the following
i. Estimate the slope $S(\theta, k) = \left(\frac{dl_{RSS}}{d\theta}, \frac{dl_{RSS}}{dk}\right)$ at $(\theta^{(t-1)}, k^{(t-1)})$
ii. Estimate the observed Fisher Information matrix as $I(\theta, k)$ at $(\theta^{(t-1)}, k^{(t-1)})$

iii. Upgrade parameter vector as $(\theta^{(t)}, k^{(t)}) = (\theta^{(t-1)}, k^{(t-1)}) + S(\theta^{(t-1)}, k^{(t-1)}) \times I^{-1}(\theta^{(t-1)}, k^{(t-1)})$
where $I^{-1}$ is the Inverse of Fisher information.
3. Repeat step 2 if $|(\theta^{(t)}, k^{(t)}) - (\theta^{(t-1)}, k^{(t-1)}| < 10^{-6}$ (threshold value).
4. The last iteration of MLEs of parameters $(\theta, k)$ denoted by $\hat{\theta_{ML}} = (\hat{\theta_{ML}}, \hat{k_{ML}})$
The asymptotic properties of MLE were used to get the confidence intervals of parameters that is: $\hat{\theta_{ML}} \approx N(\theta, I^{-1}(\theta, k))$.
Here the MLE is asymptotically normal with the mean and variance-covariance matrix are true parameter values and inverse of the observed fisher information matrix respectively see e.g. [17] and $I(\theta, k)$ can be defined as:

$$I(\theta, k) = -\begin{pmatrix} \frac{d^2 l_{RSS}}{d\theta^2} & \frac{dl_{RSS}}{d\theta dk} \\ \frac{dl_{RSS}}{dkd\theta} & \frac{d^2 l_{RSS}}{dk^2} \end{pmatrix} \tag{11}$$

Hence, $a(1-\alpha)100\%$ confidence interval for Erlang distribution parameters are $\left(\hat{\theta_{ML}} \pm Z_{\frac{\alpha}{2}}\sqrt{v(\hat{\hat{\theta_{ML}}})}\right)$, $\left(\hat{k_{ML}} \pm Z_{\frac{\alpha}{2}}\sqrt{v(\hat{\hat{k_{ML}}})}\right)$ where $Z_{\frac{\alpha}{2}}$ is the standard normal distribution of $\frac{\alpha}{2}$
upper quantile and $v(\hat{\hat{\theta_{ML}}})$ and $v(\hat{\hat{k_{ML}}})$ are diagonal elements of the inverse of the observed Fisher Information matrix. The elements of (12) are as follows:

$$\frac{d^2 l_{RSS}}{d\theta^2} = -\frac{nk}{\theta^2} + \zeta_{ij}\left(\frac{n}{\theta x} - x\right)^2 \left(\frac{(m-i)}{\zeta_{ij}} - \frac{(i-1)}{1-\zeta_{ij}}\right) + \zeta_{ij}^2\left(\frac{n}{\theta x} - x\right)^2 \left(\frac{(n-i)}{\zeta_{ij}} - \frac{(i-1)}{1-\zeta_{ij}}^2\right) \tag{12}$$

$$\frac{d^2 l_{RSS}}{dk^2} = -n\left[-\gamma + \sum_{c=1}^{\infty}\left(\frac{k-1}{c(c+k-1)}\right)\right] \tag{13}$$

$$\frac{dl_{RSS}}{d\theta dk} = \frac{nk}{\theta} \tag{14}$$

$$\frac{dl_{RSS}}{dkd\theta} = \frac{nk}{\theta} \tag{15}$$

**2.1.2. Parameter Estimation under SRS.** Let $x_1, .., x_n$ be a random sample from Erlang Distribution with parameters k and $\theta$ . The likelihood and log-likelihood are as followed:

$$LRSS(k, \theta, x) = \left(\frac{\theta^n k}{(k-1)!}\right)^n \prod_{i=1}^{m} x_i^{k-1} e^{-\theta \sum_{i=1}^{m} x_i} \tag{16}$$

$$l_{RSS}(k, \theta, x) = nklog\theta + (k-1)\sum_{i=1}^{m} logx_i - \theta\sum_{i=1}^{m} x_i - nlog(k-1)! \tag{17}$$

The parameters of $\theta$ and k can be estimated by equating the following equations

$$\frac{dl_{RSS}}{d\theta dk} = \frac{nk}{\theta} - \sum_{i=1}^{m} x_i \tag{18}$$

$$\frac{dl_{RSS}}{dkdk} = nlog\theta + \sum_{i=1}^{m} logx_i - n\left(-\gamma + \sum_{c=1}^{\infty}\frac{k-1}{c(c+k-1)}\right) \tag{19}$$

As the above equations do not have a closed-form solution, for the Newton-Raphson method is used. The elements of $I(\theta, k)$ are as follows:

$$\frac{d^2 l_{RSS}}{d\theta^2} = -\frac{nk}{\theta^2} \tag{20}$$

$$\frac{d^2 l_{RSS}}{d\theta dk} = \frac{n}{\theta} \tag{21}$$

$$\frac{d^2 l_{RSS}}{dk d\theta} = \frac{n}{\theta} \tag{22}$$

$$\frac{d^2 l_{RSS}}{dk^2} = -n\left(-\gamma + \sum_{c=1}^{\infty} \frac{k-1}{c(c+k-1)}\right) \tag{23}$$

**2.1.3. Boot-p Confidence Interval.** In the case of a small sample size of n, the Confidence Interval followed by Normal approximation may not work well. An alternative way of the confidence interval is the Resampling method which may give an accurate approximation. The most widely used resampling method is known to be the bootstrap (Boot-p) confidence interval.

i. A random sample is selected from the population (RSS or SRS) and $\hat{\theta_{ML}}$ is obtained as discussed in eq (11).

ii. Using specific sampling scheme (RSS or SRS), a bootstrap random sample is generated of Erlang distribution with parameter $\hat{\theta_{ML}}$

iii. The generated bootstrap parameter is denoted by $\hat{\theta^*}$ .

iv. The above(ii) and (iii) step is repeated N times to get $\hat{\theta_1^*}, \hat{\theta_2^*}, \hat{\theta_3^*}, ......., \hat{\theta_N^*}$

v. For order estimate, the above estimates are arranged in ascending orders $\hat{\theta_{(1)}^*}, \hat{\theta_{(2)}^*}, \hat{\theta_{(3)}^*}, ......., \hat{\theta_{(N)}^*}$

vi. A $(1-\alpha)100\%$ confidence interval is thus obtained by $100\frac{\alpha}{2}$ and $100(1-\frac{\alpha}{2})$ of bootstrap estimates.

**2.1.4. Parameter Estimation under ERSS.** In this section, parameter estimation of Erlang distribution of ML equation under ERSS will be obtained. [27] was the first who proposed the modification of RSS, to estimate the population mean by considering only the maximum or minimum ranked unit from each set. The following procedure is used for estimation under ERSS. The "m" random set of each of size "m" units is selected from the population, then ranked with respect to an interesting variable by inspection or any cost-free method. For even or odd set sizes, the selection method may be changed. For the first $m = 2$ sets in the case of even set size "m", the smallest ranked units are selected whereas from the other $m = 2$ sets, the largest ranked units are selected. In the case of odd set size "m", for the first $(m-1)/2$ sets, the smallest units are selected whereas from the other $(m-1)/2$ sets the largest units are selected and for the rest of the sets "median" is selected. The size of ERSS can be increased by cycling the procedure "r" times. Thus, we have $n = mr$ sample size.

$$g_1(u_{ij}, \theta, k) = mf(u_{ij}, \theta, k)\left[1 - F(u_{ij}, \theta, k)\right]^{m-1} \tag{24}$$

For an even set of size "m", the ERSS procedure and densities rolled into one, the likelihood function of the parameter given U = u is as follows:

$$LERSS_\theta(k,\theta;u) = \prod_{j=1}^{r}\prod_{i=1}^{m/2} g_1(u_{ij},\theta,k)\prod_{j=1}^{r}\prod_{i=m/2+1}^{m/2} g_1(u_{ij},\theta,k) \tag{25}$$

$$\begin{aligned} &LERSS_\theta(k,\theta;u)\\ &= \frac{\theta^{nk}}{(k-1)!}\prod_{j=1}^{r}\prod_{i=1}^{m}(u_{ij})^{k-1}e^{-\theta\sum_{i=1}^{n}u_i}\prod_{j=1}^{r}\prod_{i=1}^{m/2}\left(\sum_{mr}\frac{1}{mr!}e^{-\theta u_{ij}}(\theta u_{ij})^{mr}\right)^{m-1}\\ &*\prod_{j=1}^{r}\prod_{i=m/2+1}^{m}\left(1-\sum_{mr}\frac{1}{mr!}e^{-\theta u_{ij}}(\theta u_{ij})^{mr}\right)^{m-1} \end{aligned} \tag{26}$$

For odd set size, the likelihood function of the parameter given V = v can be written as

$$\begin{aligned} &LERSS_\theta(k,\theta;u)\\ &= \prod_{j=1}^{r}\prod_{i=1}^{(m-1)/2} g_1(v_{ij},\theta,k)\prod_{j=1}^{r}\prod_{i=(m+1)/2}^{m-1} g_m(v_{ij},\theta,k)\prod_{j=1}^{r}\prod_{i=1}^{m} g_{(m+1)/2}(v_{ij},\theta,k) \end{aligned} \tag{27}$$

$$\begin{aligned} &LERSS_\theta(k,\theta;u)\\ &= E^{mr}\frac{\theta^{mrk}}{(k-1)!}\prod_{j=1}^{r}\prod_{i=1}^{(m-1)/2}(v_{ij})^{k-1}e^{-\theta\sum_{i=1}^{n}v_i}\prod_{j=1}^{r}\prod_{i=(m+1)/2}^{m-1}\\ &*\left(\sum_{mr}\frac{1}{mr!}e^{-\theta v_{ij}}(\theta v_{ij})^{mr}\right)^{m-1}\prod_{j=1}^{r}\prod_{i=1}^{m}\left(1-\sum_{mr}\frac{1}{mr!}e^{-\theta v_{ij}}(\theta v_{ij})^{mr}\right)^{m-1} \end{aligned} \tag{28}$$

where $E = \frac{m!}{((m-1)/2)!^2}$

**2.1.5. Parameter Estimation under MRSS.** [22] proposed modification of RSS, for estimating a population mean known as MRSS. He investigated that MRSS is more efficient than SRS and provides an unbiased estimator in the case of systematic distribution. The procedure for the selection of units is the same as RSS in terms of random selection and order of each "m" sets with respect to interested variables. Median element is selected, if the set size "m" is odd. In the case of even "m" sets, $(m/2)th$ ranked units are selected from the first $m = 2$ sets and $(m+2)/2$ ranked units are selected from the remaining m=2 sets. A procedure may be repeated "r" a number of times. For odd set size, let $O = X_i((m+1) = 2 : m)j, i = 1, \ldots\ldots, m = 2; j = 1, \ldots\ldots., r.$
Then the likelihood function for the parameter

$$LERSS_o(k,\theta;o) = \prod_{j=1}^{r}\prod_{i=1}^{m} g_{(m+1)/2}(o_{ij},\theta,k) \tag{29}$$

$$\begin{aligned} &LERSS_o(k,\theta;o)\\ &= M^{mr}\frac{\theta^{mrk}}{(k-1)!^{mr}}\prod_{j=1}^{r}\prod_{i=1}^{m}e^{-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}o_{ij}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}o_{ij}\right)^{k-1}\\ &*\left(1-\sum_{mr}\frac{1}{mr!}e^{-\theta o_{ij}}(\theta o_{ij})^{mr}\right)^{(m-1)/2}\left(\sum_{mr}\frac{1}{mr!}e^{-\theta o_{ij}}(\theta o_{ij})^{mr}\right)^{(m-1)/2} \end{aligned} \tag{30}$$

For even set size ”m”, the likelihood function for parameter E = e is given as

$$LERSS_e(k,\theta;e)=\prod_{j=1}^{r}\prod_{i=1}^{m/2}g_{(m+2)/2}(e_{ij},\theta,k) \tag{31}$$

$$\begin{aligned} &LERSS_o(k,\theta;o)\\ &=M^{mr}\frac{\theta^{mrk}}{(k-1)!^{mr}}\prod_{j=1}^{r}\prod_{i=1}^{m}e^{-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e_{ij}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}e_{ij}\right)^{k-1}\\ &*\left(1-\sum_{mr}\frac{1}{mr!}e^{-\theta e_{ij}}(\theta e_{ij})^{mr}\right)\\ &*\prod_{j=1}^{r}\prod_{i=1}^{m/2}\left(1-\sum_{mr}\frac{1}{mr!}e^{-\theta e_{ij}}(\theta e_{ij})^{mr}\right)^{m/2-1}\left(\sum_{mr}\frac{1}{mr!}e^{-\theta e_{ij}}(\theta e_{ij})^{mr}\right)^{m/2} \end{aligned} \tag{32}$$

where $M=\frac{m!}{\frac{m-2}{2}!\frac{m}{2}!}$

**2.1.6. Parameter Estimation under MRSSUs.** [7] were the first who proposed maximum ranked set sampling procedure with unequal samples. The process of estimation based on MRSSUs is as follow:
1) ’n’ random sample is selected, where the jth sample contains ’m’ observations (where$1\leq j\leq r$)
2) then each sample observation is arranged in order of magnitude as
$1:V_{(1:1)1}$........................................... $\rightarrow U_1=V_{(1:1)1}$
$2:V_{(1:2)1}...V_{(2:2)2}$........................... $\rightarrow U_2=V_{(2:2)2}$
.
.
$r:V_{(1:r)r}...V_{(2:r)r}$........................... $\rightarrow U_r=V_{(r:r)r}$

$$LMRSS(k,\theta;u)=\prod_{i=1}^{m}\prod_{j=1}^{r}jf(u_{ij})\left[F(u_{ij})\right]^{j-i} \tag{33}$$

$$\begin{aligned} &LMRSS(k,\theta;u)\\ &=\frac{\theta^{mr}}{(k-1)!}e^{-\theta\sum_{j=1}^{r}\sum_{i=1}^{m}u_{ij}}\prod_{i=1}^{m}\prod_{j=1}^{r}j(u_{ij})^{k-1}\left[1-\sum_{mr}\frac{1}{mr!}e^{-\theta u_{ij}}(\theta u_{ij})^{mr}\right]^{j-i} \end{aligned} \tag{34}$$

**2.2. Bayesian Estimation.** In Bayesian inference, there are different approaches based on different interpretations. Among those, objective and subjective Bayesian inferences are widely used [30] which used Bayesian methods for data analysis. Popular approaches presented by [4] and [19]. Nowadays, an updated version of Bayesian data analysis Markov Chain Monte Carlo techniques is conferred by [5, 14] and [26]. Further, [2, 6] made a strong contribution to this subject. In this approach, the probability is not a defined frequency of occurrences, rather it is based on a person’s belief given the available information. To make inferences about the parameters, the rules of probability are used, and parameters are considered random variables. The interpretation of parameters must be based on the” Degree of Belief”. Bayes Theorem is then used by revising these beliefs after getting the data about parameters. After analysis of the data, the posterior distribution is then estimated which gives relative weights to each parameter. There are two sources of the posterior distribution, the prior distribution, and the observed data. For inferences, not all but only the occurring data set is used.

**2.2.1. Prior Distribution.** In Bayesian analysis, the prior distribution represents all the assumed or known information about the parameter. It is basically based on a person's judgment, belief, or a statement of degree of belief of parameters. The choice of prior distribution plays a vital role in Bayesian analysis. The impact of prior can be estimated, by the reliability of posterior distribution based on different choices of prior. A best prior still does not pay an impact on the posterior, even if the data have sufficient information. If the posterior distribution is highly dependent on prior, in this case, the data may not contain sufficient information. Whereas the data may cover sufficient information if the posterior is relatively constant over a choice of priors. Prior distribution may be categorized as Proper Prior and Improper Prior. The prior distribution is considered "proper" if it does not depend on data and integral or summation equals to 1. Conversantly, it may be considered as improper, if it does not depend on data and the distribution does not integrate or sum to 1. The gamma prior distribution is given as

$$\pi(\theta) \propto \theta^{a_1-1} e^{b_1\theta} k^{a_2-1} e^{b_2 k} \tag{35}$$

where $a_1, b_1, a_2, b_2$ are hyperparameters assumed to be 0.001. Bayesian inference is then obtained based on the posterior distribution of parameter k and $\theta$ , given the data 's'. that is:

$$\pi(\theta|S) \propto l(\theta|S)\pi(\theta) \tag{36}$$

where $l(\theta|S)\pi(\theta)$ is the likelihood function.

**2.2.2. Loss Function.** In this paper, we used the following two loss function. Laplace was the first who gave the concept of the Loss function, then [18] introduced it. It is used to measure the 'error' or 'loss'. Our aim is to choose an estimator which makes this error or loss small or ideally choose an estimator that has a small loss or risk. In this work, we have used two different loss function which is widely used in Bayesian inferences. The well-known loss function is the squared error loss function, defined as

$$L_1(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2 \tag{37}$$

It is symmetric in nature and distributes equal weights to both overestimation and underestimation. Hence, it is not best in every situation. [18] founded Linear-exponential Loss Function (LELF) reintroduced by [30] which is asymmetric in nature, and is defined as

$$L_2(\theta, \hat{\theta}) = a\left[e^{b(\hat{\theta}-\theta)} - b(\hat{\theta} - \theta)^2 - 1\right] \tag{38}$$

where 'a' and 'b' are parameters of the linear exponential loss function. We take$a = 1$, without loss of generality. In the case of a positive value of 'a', overestimation becomes more serious than underestimation. Another loss function introduced by [24] is the alternative asymmetric loss function known as the precautionary loss function. To avoid underestimation, this loss function is close to infinity near the origin. Hence, it gives a close estimation, in circumstances of low failure rates are being estimated. When underestimation becomes a serious problem, these estimators than very useful. The formula is given by:

$$L_3(\theta, \hat{\theta}) = \frac{(\hat{\theta} - \theta)^2}{\theta} \tag{39}$$

where $\theta$ and $\hat{\theta}$ are actual and estimated values of the parameter.

**2.2.3. Bayesian Estimation under SRS.** Under SRS, the joint posterior distribution of the parameter$\theta$ and k can be obtained by joining the likelihood in eq(17) and prior in eq (32) via Bayes theorem. It can be written as

$$\pi(\theta, k|x) \propto \frac{\pi(\theta|k) L_{SRS}(\theta, k, x)}{\int\int \pi(\theta|k) L_{SRS}(\theta, k, x) d\theta dk} \tag{40}$$

by finding the posterior, one can obtain the Bayes estimator using the above posterior, that is

$$\hat{g} \propto \int\int \pi(\theta, k|x) d\theta dk \tag{41}$$

The above explicit form of integral is complicated to solve. Therefore, we may use the 'Markov Chain Monte Carlo(MCMC) method to obtain Bayes estimate of the parameter. To execute MCMC methodology from the joint posterior density $\pi(\theta, k|x)$, we consider the MHG algorithm to generate large samples. The algorithm is intimate as follows:
Steps
1. Start with an initial guess $(k^0, \theta^0) = (k_{MLE.SRS}^{\hat{}}, \theta_{MLE.SRS}^{\hat{}})$
2. For $t = 1$, choose a proposal Kernel from positive truncated Normal Distribution $N(k, \theta, \sigma^2) I_{x>0}$
3. For $t \geq 1$, calculate the acceptance ratio as:

$$a^* = \frac{\pi(\theta^*, k^*|x) q(\theta^{t-1}, k^{t-1})}{\pi(\theta^{t-1}, k^{t-1}|x) q(\theta^*, k^*)} \tag{42}$$

Set $(\theta^t, k^t) = (\theta^*, k^*)$ with probability $p = min(1, \alpha^*$ and otherwise set $(\theta^t, k^t) = (\theta^{t-1}, k^{t-1})$
4. For a large number of iterations say 'T' set t = t + 1 and repeat step 3.
In our simulation, the proposal kernel is independent of the normal kernel. Its mean and standard deviation is equal to the previously sampled value and $\sqrt{(}I^(- 1))$ respectively, with a factor of $2.38/\sqrt{d}$. Where $I^( - 1)$ is observed Fisher Information space dimension. According to [15], we took $d = 3$. Further, we noticed that the MHG algorithm does not produce a fast convergence rate due to its association among parameters. Another algorithm developed by [8] is known as differential Evolution M-H (DE-M-H). This algorithm overcomes the problem of non-convergence and collinear parameters. It has the ability to run multiple chains say N, which can be intimated through over-dispersed states. In each chain, the proposed value takes information from the remaining two randomly selected chains. In this process, chains get full information from each other. This algorithm is executed as follows:
Steps
1. N chains are initiated as $(\theta_1^{(0)}, ........, \theta_1^{(n)})$

2. For $t \geq 1$, calculate the acceptance ratio as:

$$a^* = \frac{\pi(\theta^*, k^*|x)q(\theta^{t-1}, k^{t-1})}{\pi(\theta^{t-1}, k^{t-1}|x)q(\theta^*, k^*)} \tag{43}$$

The proposed model for the ith chain is defined as:

$$\theta_t^{(T)} = \theta_t^{(t-1)} + \beta(\theta_j^{(t-1)} + \theta_k^{(t-1)}) + \epsilon \tag{44}$$

With probability, $min(1, \alpha^*)$ where $\theta_t^{(t-1)}$ is the preceding stages of the ith chain, $\theta_j^{(t-1)}$ and $\theta_k^{(t-1)}$ are remaining two randomly selected chains without replacement. This gives a guarantee that the chains are getting full information from each other. Whereas $\epsilon$ belongs to $N(0, \sigma)$ and $\sigma^2$ is variance of observed fisher information. The following posterior samples may be used to obtain the posterior estimates and credible intervals as follow:

$$\hat{\alpha_\beta} = (\hat{\theta_\beta}, \hat{k_\beta}) \tag{45}$$

where $\hat{\theta_\beta} = \frac{\sum\sum \theta_i^t}{T}$ and $\hat{\theta_\beta} = \frac{\sum\sum k_i^t}{T}$
Moreover, the point estimator of $\hat{\theta_\beta}$ and $\hat{k_\beta}$ can be calculated from posterior samples.

**2.2.4. Bayesian Estimation under RSS.** Just like SRS, the joint posterior distribution of parameter $\theta$ and k under RSS can be obtained by combining the likelihood in eq (6) and prior in eq(32) using the Bayes theorem. Up to normalizing constant, it can be written as:

$$\pi(\theta, k|x) \propto \frac{\pi(\theta|k)L_{RSS}(\theta, k, x)}{\int\int \pi(\theta|k)L_{RSS}(\theta, k, x)d\theta dk} \tag{46}$$

$$\pi(\theta, k|x) \propto \theta^{a_1-1}e^{b_1\theta}k^{a_2-1}e^{b_2 k}\frac{\theta^{mk}}{(k-1)!}^{mr} exp\left[-\theta\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}\right] exp\left[\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i:m)j}\right]^{k-1}$$
$$* \prod_{j=1}^{r}\prod_{i=1}^{m}\left[1-\sum\frac{1}{n!}e^{\theta x}(\theta x)^n\right]^{i-1}\left[1-\sum\frac{1}{n!}e^{\theta x}(\theta x)^n\right]^{m-1} \tag{47}$$

The posterior mean can be obtained by using the above posterior as:

$$\hat{g} \propto \int\int \pi(\theta, k|x)d\theta dk \tag{48}$$

by using Bayes theorem,

$$\hat{g} \propto \frac{\int\int g\pi(\theta|k)L_{RSS}(\theta, k, x)d\theta dk}{\int\int \pi(\theta|k)L_{RSS}(\theta, k, x)d\theta dk} \tag{49}$$

The above explicit form of integral is complicated to solve. Therefore, we may use the 'Markov Chain Monte Carlo(MCMC) method as defined in the previous section, to obtain Bayes's estimate of the parameter.

**2.2.5. Bayesian Estimation under MRSSu.** The Bayesian approach under MRSSu can be obtained by combining the prior in eq (32) and the likelihood function in eq(29), is given by

$$\pi(\theta, k|u) \propto \frac{\pi(\theta|k) L_{MRSSu}(\theta, k, x) d\theta dk}{\int \int \pi(\theta|k) L_{MRSSu}(\theta, k, x) d\theta dk} \tag{50}$$

$$\pi(\theta, k|u) \propto \frac{\theta^{mr}}{(k-1)!} exp\left[-\theta \sum_{j=1}^{r} \sum_{i=1}^{m}\right] \prod_{j=1}^{r} \prod_{i=1}^{m} j u_{ij}^{k-1} \left[1 - \sum_{mr} \frac{1}{mr!} e^{-\theta u_{ij}} (\theta u_{ij})^{mr}\right]^{i-1} \tag{51}$$

We use the DH-EM algorithm to generate the large samples form of complicated joint posterior density $\pi(\theta, k|u)$ we do not state it here, as it is similar to the Algorithm defined in SRS and RSS sections.

**2.2.6. Bayesian Estimation ERSS under MRSS.** The Bayesian approach under ERSS and MRSS is similar to the estimation under SRS and RSS. The only difference is the likelihood that LERSS and LMRSS is used in place of LSRS for extreme rank set sampling and maximum rank set sampling separately, respectively. The bayesian estimator of $\theta_{MRSS}, \theta_{ERSS}, k_{MRSS}, k_{ERSS}$ are solved by numerical method because of the complexity of integral.

## 3. Simulation study

For the support of this research, the Monte Carlo simulation study is conducted on a perfect ranking setup. In order to illustrate the performance of ML and Bayesian estimation methods based on different sampling schemes. Perfect ranking setup is one of the fundamental assumptions of RSS, under which the units in each sample are ranked without error. Many authors have tried to show that their proposed techniques based on RSS remain valid or at least as effective as those based on SRS when the assumption of perfect ranking is moderately relaxed. As such, it is an ideal setup for practical situations. Comparison is also made using different sets and sample sizes based on Bias and MSE. The data were simulated from Erlang Distribution for each pair i-e.($\theta = 0.5; 1$) and ($k = 1; 2$). Here we considered three sample sizes n = 25, 50, 100 for each set of parameter values with set sizes (m=3, 4). Here n=25, 50, and 100 are considered to be small, moderate, and large respectively. The finite number of iterations say T = 2000 are generated using five kinds of schemes (SRS, RSS, MRSS, ERSS, and MRSSu) for Erlang distribution. In this process of iteration, negative ML estimates are excluded. Those iterations are also excluded which are unable to find the roots of ML equations adequately. For Bayesian estimation, parameters are estimated by using the MHG algorithm with different chains each having T = 2000 iterations. For LELF, the values of 'b' are taken to be 1 and -1, and 'a' is fixed to be 1. In the case of a small sample size, it is observed that the Bayesian approach shows better results than MLE in terms of MSE. As the sample size increases, the ML method acquires a smaller MSE than the Bayesian approach. It is also observed under different sampling scheme for large sample size both methods Bayesian and MLE tends to behave almost the same. Therefore, for small sample sizes Bayesian approach is recommended due to its better performance than MLE. For moderation to large sample sizes, the ML approach is recommended because of its simplicity and better convergence.

## 4. Real Data Example

This section aims to implement the method of statistical inference discussed in the previous section through real data examples.

**4.1. Data Set-I.** We considered the data related to failure times for each of 23 ball bearings from the number of million revolutions, reported by [4]. The individual bearings were examined periodically to check whether "Failure" had occurred or not. The data in Table 8 are assumed to be our population and number of times "Failure" are considered continuous. Lawless stated that Erlang distribution is suitable for modeling the mentioned data. To implement the proposed estimation methods, we execute 1000 bootstrap samples that were chosen to be with replacement from the data. We determined sample size and set sizes are n = 10 and m = 3 and 4 respectively, with different sampling schemes. Moreover, we had no prior information about parameters. Bayesian estimates were obtained through MHG algorithm. Therefore, the ML and Bayesian estimators were obtained by these procedures. The resulting MSE of estimators is presented in Table 9.

**4.2. Data Set-II.** The second data set was previously used by [14]. It shows the waiting time (min) of 100 bank clients before the service is being executed. The summary of the data is shown in Table 1. The results of $-2logL$, Alkaike Information Criteria (AIC), and Bayesian (BIC) are presented in Table 10.

**Table 1. Summary of data on waiting time of bank clients**

| Min | Max | Mean | $Q_1$ | Median | $Q_3$ | Variance | Kurtosis | skewness |
|---|---|---|---|---|---|---|---|---|
| 10.8 | 38.5 | 9.877 | 4.675 | 8.1 | 13.02 | 52.374 | 5.54 | 1.473 |

**4.3. Data Set-III.** The third data set formerly used by [14]. It illustrates the lifetime of 20 electronic components. The summary of the data is given in Table 2. The results of -2logL, Alkaike Information Criteria (AIC) and Bayesian (BIC) are presented in Table 11.

**Table 2. Summary of data on electronic components**

| Min | Max | Mean | $Q_1$ | Median | $Q_3$ | Variance | Kurtosis | skewness |
|---|---|---|---|---|---|---|---|---|
| 0.03 | 5.09 | 1.936 | 0.775 | 1.795 | 2.9 | 2.063 | 2.72 | 0.603 |

**4.4. Data Set-IV.** The fourth data set was previously used by [19]. It illustrates the remission time (Months) of 128 bladder cancer patients. The summary of the data is presented in Table 3. The resulting comparison using different sampling schemes based on log-likelihood, AIC, and BIC is shown in Table 11.

**Table 3: Summary of data on remission time of blood cancer patients**

| Min | Max | Mean | $Q_1$ | Median | $Q_3$ | Variance | Kurtosis | skewness |
|---|---|---|---|---|---|---|---|---|
| 0.08 | 75.366 | 3.348 | 6.395 | 2.9 | 2.187 | 2.87 | 0.887 | 0.213 |

**4.5. Data Set-V.** The fifth data were previously used by [25]. It represents an accelerated life test of 59 conductors, the summary is shown in Table 4. The resulting comparison of sampling schemes is shown in Table 11.

**Table 4: Summary of data on conductors**

| Min | Max | Mean | $Q_1$ | Median | $Q_3$ | Variance | Kurtosis | skewness |
|---|---|---|---|---|---|---|---|---|
| 2.997 | 11.04 | 6.929 | 6.052 | 6.869 | 7.810 | 2.48 | 3.280 | 0.22 |

**Table 5: A Confidence interval of MLE parameters estimates using Normal and Bootstrapping** method under SRS and RSS ($\theta = 0.5, k = 1$)

| n | Samp. | M | Normal ($\theta$) | Normal (k) | Bootstrap ($\theta$) | Bootstrap (k) |
|---|---|---|---|---|---|---|
| 25 | SRS | | 0.7316 | 0.6741 | 0.9891 | 1.1984 |
| | RSS | 3 | 0.7265 | 0.6890 | 0.9910 | 1.1899 |
| | | 4 | 0.7198 | 0.6913 | 0.9900 | 1.1872 |
| 50 | SRS | | 0.7180 | 0.7344 | 0.9812 | 1.1743 |
| | RSS | 3 | 0.7216 | 0.7659 | 0.9806 | 1.1724 |
| | | 4 | 0.7209 | 0.7715 | 0.9815 | 1.1703 |
| 100 | SRS | | 0.7114 | 0.9143 | 0.9716 | 1.1587 |
| | RSS | 3 | 0.7123 | 0.9265 | 0.9734 | 1.1590 |
| | | 4 | 0.7016 | 0.9289 | 0.9701 | 1.1592 |

**Table 6: A Coverage probability (CP) and of the MLE and Bayes estimates under RSS and SRS for different sample sizes and different set sizes when** ($\theta = 0.5, k = 1$)

| n | Samp. | M | MLE ($\theta$) | Bayes ($\theta$) | MLE (k) | Bayes (k) |
|---|---|---|---|---|---|---|
| 25 | SRS | | 0.628 | 0.782 | 0.841 | 0.898 |
| | RSS | 3 | 0.632 | 0.784 | 0.843 | 0.899 |
| | | 4 | 0.636 | 0.787 | 0.849 | 0.896 |
| 50 | SRS | | 0.711 | 0.812 | 0.868 | 0.910 |
| | RSS | 3 | 0.720 | 0.826 | 0.866 | 0.907 |
| | | 4 | 0.723 | 0.829 | 0.869 | 0.911 |
| 100 | SRS | | 0.745 | 0.844 | 0.872 | 0.922 |
| | RSS | 3 | 0.755 | 0.834 | 0.877 | 0.923 |
| | | 4 | 0.765 | 0.841 | 0.873 | 0.929 |

**Table 7: MSE of Erlang distribution through MLE and Bayesian estimation under SRS, RSS, ERSS, MRSS and MRSSu ($\theta = 0.5, k = 1$) and $n = 25$**

| Samp. | m | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| SRS | | 0.1634 | 0.1359 | 0.1063 | 0.1742 | 0.1617 | 0.2192 | 0.2004 | 0.1942 | 0.1990 | 0.2015 |
| MRSS | 3 | 0.0859 | 0.0728 | 0.0524 | 0.0714 | 0.0706 | 0.2087 | 0.1895 | 0.1624 | 0.1789 | 0.1774 |
| | 4 | 0.0819 | 0.0703 | 0.0600 | 0.0646 | 0.0621 | 0.2034 | 0.1745 | 0.1539 | 0.1734 | 0.1688 |
| RSS | 3 | 0.0903 | 0.0729 | 0.0628 | 0.0741 | 0.0725 | 0.2072 | 0.1834 | 0.1638 | 0.1767 | 0.1734 |
| | 4 | 0.0911 | 0.0647 | 0.0575 | 0.0628 | 0.0751 | 0.2041 | 0.1801 | 0.1516 | 0.1762 | 0.1657 |
| ERSS | 3 | 0.1183 | 0.1174 | 0.0997 | 0.1129 | 0.1193 | 0.2153 | 0.1997 | 0.1774 | 0.1862 | 0.1909 |
| | 4 | 0.1129 | 0.1194 | 0.1002 | 0.1136 | 0.1131 | 0.2149 | 0.1927 | 0.1729 | 0.1818 | 0.1876 |
| MRSSu | 3 | 0.1228 | 0.1209 | 0.1097 | 0.1149 | 0.1185 | 0.2184 | 0.1934 | 0.1839 | 0.1927 | 0.1837 |
| | 4 | 0.1216 | 0.1190 | 0.1109 | 0.1173 | 0.1105 | 0.2177 | 0.1928 | 0.1769 | 0.1899 | 0.1821 |
| SRS | | 0.1168 | 0.1112 | 0.1093 | 0.1139 | 0.1115 | 0.2127 | 0.2019 | 0.1837 | 0.2006 | 0.1899 |
| MRSS | 3 | 0.0627 | 0.0599 | 0.0318 | 0.0464 | 0.0465 | 0.1949 | 0.1734 | 0.1416 | 0.1678 | 0.1684 |
| | 4 | 0.0534 | 0.0518 | 0.0268 | 0.0451 | 0.0448 | 0.1928 | 0.1705 | 0.1329 | 0.1663 | 0.1639 |
| RSS | 3 | 0.0692 | 0.0583 | 0.0413 | 0.0486 | 0.0516 | 0.1995 | 0.1762 | 0.1403 | 0.1671 | 0.1688 |
| | 4 | 0.0576 | 0.0479 | 0.0328 | 0.0471 | 0.0483 | 0.1945 | 0.1725 | 0.1389 | 0.1658 | 0.1626 |
| ERSS | 3 | 0.0928 | 0.0937 | 0.0829 | 0.0719 | 0.0639 | 0.2064 | 0.1918 | 0.1609 | 0.1784 | 0.1708 |
| | 4 | 0.0874 | 0.0863 | 0.0714 | 0.0713 | 0.0537 | 0.2073 | 0.1911 | 0.1587 | 0.1723 | 0.1693 |
| MRSSu | 3 | 0.0963 | 0.0847 | 0.0739 | 0.0839 | 0.0729 | 0.2038 | 0.1912 | 0.1619 | 0.1829 | 0.1710 |
| | 4 | 0.0914 | 0.0729 | 0.0657 | 0.0795 | 0.0593 | 0.2018 | 0.1904 | 0.1600 | 0.1803 | 0.1700 |
| SRS | | 0.0972 | 0.0925 | 0.0764 | 0.0817 | 0.0899 | 0.1132 | 0.1091 | 0.0936 | 0.1002 | 0.1103 |
| MRSS | 3 | 0.0105 | 0.0102 | 0.0083 | 0.0093 | 0.0086 | 0.1019 | 0.0096 | 0.0070 | 0.0099 | 0.0098 |
| | 4 | 0.0095 | 0.0093 | 0.0074 | 0.0079 | 0.0081 | 0.0093 | 0.0086 | 0.0068 | 0.0087 | 0.0078 |
| RSS | 3 | 0.0116 | 0.0324 | 0.0091 | 0.0096 | 0.0074 | 0.1020 | 0.0099 | 0.0071 | 0.0096 | 0.0090 |
| | 4 | 0.0100 | 0.0293 | 0.0078 | 0.0082 | 0.0069 | 0.0089 | 0.0082 | 0.0061 | 0.0078 | 0.0083 |
| ERSS | 3 | 0.0394 | 0.0289 | 0.0119 | 0.0254 | 0.0199 | 0.1939 | 0.0982 | 0.0318 | 0.0846 | 0.0748 |
| | 4 | 0.0308 | 0.0264 | 0.0120 | 0.0219 | 0.0148 | 0.1912 | 0.0920 | 0.0310 | 0.0839 | 0.0652 |
| MRSSu | 3 | 0.0510 | 0.0305 | 0.0229 | 0.0329 | 0.0284 | 0.1898 | 0.0964 | 0.0327 | 0.0819 | 0.0829 |
| | 4 | 0.0505 | 0.0300 | 0.0218 | 0.0316 | 0.0248 | 0.1823 | 0.0924 | 0.0313 | 0.0794 | 0.0739 |

**Table 8: MSE of Erlang distribution through MLE and Bayesian estimation under SRS, RSS, ERSS, MRSS and MRSSu** $(\theta = 1, k = 2, a = 1)$**and** $n = 25$

| Samp. | m | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| SRS | | 2.2183 | 2.1619 | 2.0948 | 2.1346 | 2.1298 | 3.3091 | 2.9827 | 2.1919 | 2.8494 | 2.8818 |
| MRSS | 3 | 2.0034 | 1.9790 | 1.6537 | 1.8864 | 1.9004 | 3.2878 | 2.6093 | 1.9964 | 2.5295 | 2.6018 |
| | 4 | 2.0018 | 1.9346 | 1.5935 | 1.8413 | 1.8572 | 3.2739 | 2.4175 | 1.9875 | 2.4241 | 2.5005 |
| RSS | 3 | 2.0056 | 1.9728 | 1.6642 | 1.8341 | 1.8703 | 3.2826 | 2.6784 | 1.9942 | 2.6493 | 2.6506 |
| | 4 | 2.0037 | 1.9501 | 1.5914 | 1.8701 | 1.8694 | 3.2655 | 2.5003 | 1.9979 | 2.5101 | 2.4976 |
| ERSS | 3 | 2.1098 | 2.0100 | 1.8093 | 1.8990 | 1.9819 | 3.2991 | 2.8265 | 2.0109 | 2.8109 | 2.7784 |
| | 4 | 2.1001 | 2.0067 | 1.8000 | 1.8962 | 1.9398 | 3.2856 | 2.7904 | 2.0045 | 2.6485 | 2.7019 |
| MRSSu | 3 | 2.1075 | 2.1005 | 1.9513 | 1.9742 | 1.9002 | 3.2901 | 2.9150 | 2.0708 | 2.7859 | 2.8000 |
| | 4 | 2.1006 | 2.0909 | 1.9348 | 1.9720 | 1.8909 | 3.2853 | 2.9109 | 2.0665 | 2.6316 | 2.7629 |
| SRS | | 2.1993 | 2.1123 | 1.9792 | 2.1081 | 2.1067 | 2.1096 | 2.1036 | 2.1001 | 2.0930 | 2.0916 |
| MRSS | 3 | 1.9734 | 1.7450 | 1.3819 | 1.8100 | 1.8300 | 2.0832 | 1.8903 | 1.7170 | 1.8192 | 1.7340 |
| | 4 | 1.9664 | 1.4811 | 1.1104 | 1.8035 | 1.8071 | 2.0701 | 1.8745 | 1.6923 | 1.6356 | 1.7112 |
| RSS | 3 | 1.9722 | 1.6995 | 1.4027 | 1.8069 | 1.9101 | 2.0816 | 1.8976 | 1.7209 | 1.7999 | 1.7363 |
| | 4 | 1.9599 | 1.5540 | 1.2219 | 1.8014 | 1.8032 | 2.0792 | 1.8743 | 1.7175 | 1.6804 | 1.7172 |
| ERSS | 3 | 2.0937 | 1.8832 | 1.8184 | 1.8809 | 1.9832 | 2.1012 | 1.9804 | 1.9653 | 2.0014 | 1.9093 |
| | 4 | 2.0911 | 1.7399 | 1.7966 | 1.7963 | 1.9709 | 2.0952 | 1.9713 | 1.9564 | 1.9328 | 1.8832 |
| MRSSu | 3 | 2.1090 | 1.9039 | 1.9003 | 1.8749 | 1.9856 | 2.0910 | 1.9852 | 1.9609 | 1.9908 | 1.9990 |
| | 4 | 2.1023 | 1.8205 | 1.8044 | 1.8504 | 1.9605 | 2.0901 | 1.9801 | 1.9500 | 1.9704 | 1.8650 |
| SRS | | 1.7681 | 1.6991 | 1.1029 | 1.5928 | 1.5929 | 2.1016 | 1.9934 | 1.7285 | 1.9543 | 1.9384 |
| MRSS | 3 | 1.0087 | 1.4843 | 0.9529 | 1.2983 | 1.3027 | 1.8735 | 1.7430 | 1.4643 | 1.6983 | 1.7121 |
| | 4 | 0.9160 | 1.4277 | 0.8132 | 1.1164 | 1.1187 | 1.8700 | 1.6582 | 1.1485 | 1.4090 | 1.6230 |
| RSS | 3 | 0.9998 | 1.5003 | 0.9910 | 1.2872 | 1.3013 | 1.8790 | 1.7560 | 1.5003 | 1.6972 | 1.7190 |
| | 4 | 0.9091 | 1.4325 | 0.8515 | 1.1109 | 1.1125 | 1.7935 | 1.6449 | 1.1253 | 1.5103 | 1.6304 |
| ERSS | 3 | 1.6631 | 1.5884 | 1.1011 | 1.4995 | 1.5135 | 2.0546 | 1.8840 | 1.6859 | 1.8376 | 1.8950 |
| | 4 | 1.5639 | 1.5029 | 1.0984 | 1.4832 | 1.4364 | 2.0316 | 1.8794 | 1.6774 | 1.7965 | 1.8504 |
| MRSSu | 3 | 1.7490 | 1.5578 | 1.0943 | 1.5293 | 1.5024 | 2.0915 | 1.8929 | 1.7015 | 1.9203 | 1.9034 |
| | 4 | 1.7279 | 1.5009 | 1.0853 | 1.5016 | 1.4925 | 2.0813 | 1.8648 | 1.6905 | 1.8370 | 1.9005 |

**Table.9: Failure time of 23 ball bearings**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17.88 | 28.92 | 33.00 | 41.52 | 42.12 | 45.60 | 48.40 | 51.84 |
| 54.02 | 55.56 | 68.64 | 68.64 | 68.88 | 84.12 | 93.12 | 98.64 |
| 105.12 | 105.84 | 127.92 | 128.04 | 173.40 | 67.80 | 51.96 | 0 |

**Table 10: MSE of Erlang distribution for data set-I** $(n = 10, a = 1)$

| Samp. | m | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ | MLE | $L_1$ | $L_2$ | $L_2$ | $L_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| SRS | | 0.0221 | 0.0210 | 0.0032 | 0.0219 | 0.0199 | 0.3132 | 0.1991 | 0.1053 | 0.1892 | 0.1793 |
| M RSS | 3 | 0.0193 | 0.0203 | 0.0028 | 0.0184 | 0.0176 | 0.1953 | 0.1862 | 0.1003 | 0.1527 | 0.1409 |
| | 4 | 0.0186 | 0.0184 | 0.0019 | 0.0154 | 0.0132 | 0.1618 | 0.1764 | 0.0989 | 0.1339 | 0.1381 |
| RSS | 3 | 0.0192 | 0.0176 | 0.0028 | 0.0174 | 0.0172 | 0.2016 | 0.1790 | 0.1018 | 0.1530 | 0.1510 |
| | 4 | 0.0173 | 0.0164 | 0.0014 | 0.0162 | 0.0151 | 0.1763 | 0.1668 | 0.0921 | 0.1401 | 0.1435 |
| ERSS | 3 | 0.0208 | 0.0200 | 0.0037 | 0.0199 | 0.0193 | 0.2947 | 0.2915 | 0.1027 | 0.1762 | 0.1669 |
| | 4 | 0.0203 | 0.0198 | 0.0025 | 0.0183 | 0.0191 | 0.2189 | 0.2111 | 0.1013 | 0.1711 | 0.1590 |
| MRSSu | 3 | 0.0210 | 0.0203 | 0.0097 | 0.0200 | 0.0185 | 0.2190 | 0.2004 | 0.1050 | 0.1818 | 0.1769 |
| | 4 | 0.0219 | 0.0211 | 0.0081 | 0.0201 | 0.0190 | 0.2278 | 0.2010 | 0.1021 | 0.1782 | 0.1723 |

**Table 11: MSE of Erlang distribution for data set-II**$(n = 10, m = 3)$

| | Sampling Scheme | -2log | AIC | BIC |
|---|---|---|---|---|
| 54em Data-I | SRS | $-193.377$ | 398.478 | 402.484 |
| | MRSS | -204.765 | 386.496 | 400.382 |
| | RSS | -199.478 | 392.471 | 401.920 |
| | ERSS | -200.070 | 382.289 | 402.206 |
| | MRSSu | -189.023 | 395.381 | 410.478 |
| 54em Data-II | SRS | -33.198 | 81.293 | 110.247 |
| | MRSS | -30.122 | 72.395 | 103.842 |
| | RSS | -36.782 | 79.940 | 99.379 |
| | ERSS | -34.441 | 73.202 | 103.480 |
| | MRSSu | -39.128 | 80.190 | 108.476 |
| 54em Data-III | SRS | -683.294 | 639.604 | 1003.63 |
| | MRSS | -672.485 | 623.189 | 1002.48 |
| | RSS | -680.008 | 633.754 | 999.37 |
| | ERSS | -671.479 | 620.489 | 999.590 |
| | MRSSu | -684.093 | 636.464 | 1009.45 |
| 54em Data-IV | SRS | -44.92 | 143.68 | 203.58 |
| | MRSS | -40.27 | 139.57 | 196.34 |
| | RSS | -49.75 | 141.19 | 200.38 |
| | ERSS | -39.83 | 138.20 | 194.05 |
| | MRSSu | -40.38 | 149.58 | 202.74 |

## 5. Discussion and Conclusion

This paper studied estimation problems for Erlang distribution based on five different sampling schemes namely Simple, Ranked Set Sampling, Extreme Rank Set Sampling, Median Rank, and Maximum Rank Set Sampling of unequal sizes. The estimation of unknown parameters was investigated using ML and Bayesian approaches. Explicit forms are obtained by the ML method for both shape and scale parameters. The Bayesian method is obtained under three different loss functions namely Squared Error Loss Function (SELF), Linear Exponential Loss Function (LELF), Precautionary Loss Function (PLF), and weakly information prior that is Gamma Prior. MCMC method is used to get Bayes estimation of unknown parameters due to the implicit form of integral. In Table 5, scale and shape parameters are obtained through both ML and Bayesian estimation using Gamma prior and Squared Error Loss Function based on all above- mentioned sampling schemes. Comparisons are made based on the bias, mean square error (MSE)(see tables 7 and 8), and coverage probability (CP) of confidence intervals (see table 6). Different combinations of the parameter values are considered to cover different shapes of the probability density function of the Erlang distribution. Therefore, the conclusions were almost the same for all combinations of the parameter values, the results of the two combinations are considered as $(\theta = 0.5; 1)$ and (k = 1; 2).

Tables 7 and 8 illustrate the MSE of Erlang distribution based on MLE and Bayesian methods using SRS, RSS, ERSS, MRSS, and MRSSu. In Table 11, different sampling schemes are compared using log-likelihood, AIC, and BIC criteria. The lowest AIC is considered to be the best. In our comparison, the

RSS and ERSS are shown the lowest values of AIC and BIC and log-likelihood as compared to other sampling schemes. Moreover, confidence interval construction using bootstrapping plays better than one obtained using the normal approach. Both techniques perform almost the same as the sample size increases. We concluded from the study that the Bayesian estimation technique performs better than ML techniques if the sample size is small. From moderate to large, the ML estimation method performs better. Further, it is observed that the MSE of MRSS is smaller than others like SRS and other modifications of RSS. This showed that MRSS is more efficient than all other estimation schemes.

**Conflicts of interest** : The authors declare no conflict of interest.

**Data availability** : Not applicable

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