KYUNGPOOK Math. J. 63(2023), 1-10
https://doi.org/10.5666/KMJ.2022.63.1.1
pISSN 1225-6951 eISSN 0454-8124

Criteria for Algebraic Operators to be Unitary

Zenon Jan Jabłoński and Jan Stochel
Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland
e-mail : Zenon.Jablonski@im.uj.edu.pl *and* Jan.Stochel@im.uj.edu.pl

Il Bong Jung*
Department of Mathematics, Kyungpook National University, Daegu 41566, Korea
e-mail : ibjung@knu.ac.kr

Abstract. Criteria for an algebraic operator T on a complex Hilbert space $\mathcal{H}$ to be unitary are established. The main one is written in terms of the convergence of sequences of the form $\{\|T^n h\|\}_{n=0}^{\infty}$ with $h \in \mathcal{H}$. Related questions are also discussed.

1. Introduction

By the spectral theorem, a unitary operator with a finite spectrum is algebraic and its spectrum is contained in $\mathbb{T}$, the unit circle centered at 0. The most fundamental example of a unitary algebraic operator is the Fourier transform. According to the famous theorem of Plancherel, the Fourier transform extends uniquely to a unitary operator on $L^2(\mathbb{R})$ (see e.g., [18, Theorem IX.6]). Denote it by $\mathscr{F}$. The Fourier transform $\mathscr{F}$ has the following properties:

$$\mathscr{F}^0 = I,\ \mathscr{F}^1 = \mathscr{F},\ \mathscr{F}^2 = P,\ \mathscr{F}^3 = \mathscr{F}^{-1} \text{ and } \mathscr{F}^4 = I,$$

where I is the identity operator on $L^2(\mathbb{R})$ and $P(f)(x) = f(-x)$ for $f \in L^2(\mathbb{R})$. This implies that $p(x) = x^4$ is the minimal polynomial of $\mathscr{F}$. As a consequence, the Fourier transform is a unitary algebraic operator with (purely point) spectrum $\sigma(\mathscr{F}) = \{1, -1, \mathrm{i}, -\mathrm{i}\}$ (see [18, Theorems IX.1 and IX.6]).

---

* Corresponding Author.
Received July 5, 2022; accepted September 19, 2022.
2020 Mathematics Subject Classification: Primary 47B15; Secondary 47B20.
Key words and phrases: Algebraic operator, unitary operator, normaloid operator, strong stability.
The research of the first and second authors was supported by the National Science Center (NCN) Grant OPUS No. DEC-2021/43/B/ST1/01651. The research of the third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2021R111A1A01043569).

A natural question arises under what additional assumptions an algebraic (bounded linear) operator T on a complex Hilbert space $\mathcal{H}$ with spectrum in $\mathbb{T}$ is unitary. To answer this question let us look at some broader classes $\mathscr{A}$ of operators that can be characterized as follows: an operator T belongs to $\mathscr{A}$ if and only if the sequences of the form $\{\|T^n h\|^2\}_{n=0}^{\infty}$ $(h \in \mathcal{H})$ belong to the corresponding class $\mathscr{S}$ of scalar sequences; in most cases, the class $\mathscr{S}$ appears naturally in harmonic analysis on $*$-semigroups. In particular, the celebrated theorem of Lambert states that the class of subnormal operators corresponds to Stieltjes moment sequences (see [17]). In this line of correspondence, we can list the classes of m-isometric operators [2, 3, 4, 13], completely hypercontractive operators [1], completely hyperexpansive operators [5], alternatingly hyperexpansive operators [20], conditionally positive definite operators [14], and so on. The answer to our question (see Theorem 1.1 below) is written in terms of the convergence of the sequences of the form $\{\|T^n h\|\}_{n=0}^{\infty}$ $(h \in \mathcal{H})$. The condition of their convergence seems to be optimal, since in the light of Remark 3.3 the assumption of their boundedness ceases to be sufficient. It is also worth mentioning that there are contractions (for which the sequences $\{\|T^n h\|\}_{n=0}^{\infty}$, $h \in \mathcal{H}$, automatically converge) with spectrum in $\mathbb{T}$, called *unimodular contractions*, which are not unitary (see [19]). Clearly, unimodular contractions are normaloid. Let us further note that if we replace the class of normaloid operators by a class of more regular operators, it may turn out that members of the latter class with spectrum in $\mathbb{T}$ are unitary. In particular, by Stampfli's theorem (see [21, Corollary, p. 473]), every hyponormal operator with spectrum in $\mathbb{T}$ is unitary.

Before formulating the main result, we establish some notation and terminology. Denote by $\mathbb{C}$ the field of complex numbers. Set $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. Write $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{R}_+$ for the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. Let $\boldsymbol{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $T \in \boldsymbol{B}(\mathcal{H})$, denote by $\mathscr{N}(T)$, $\sigma(T)$ and $r(T)$ the kernel, the spectrum and the spectral radius of T, respectively. An operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be *normaloid* if $r(T) = \|T\|$, or equivalently, by Gelfand's formula for spectral radius, if and only if $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$. Call $T \in \boldsymbol{B}(\mathcal{H})$ *algebraic* if there exists a nonzero polynomial p (in one indeterminate with complex coefficients) such that $p(T) = 0$; such a p is said to be *minimal* if p is the (unique) monic polynomial of least degree among all nonzero polynomials q such that $q(T) = 0$.

The following theorem, which is the main result of the paper, characterizes unitary algebraic operators in terms of the convergence of the sequences of the form $\{\|T^n h\|\}_{n=0}^{\infty}$. Its proof is given in Section 3.

Theorem 1.1. *Suppose that $T \in \boldsymbol{B}(\mathcal{H})$ is algebraic and $\sigma(T) \subseteq \mathbb{T}$. Then the following statements are equivalent*:

(i) *T is unitary,*

(ii) *T is normaloid,*

(iii) $\|T\| \leqslant 1$ (*or equivalently*, $\|T\| = 1$),

(iv) *the sequence* $\{\|T^n h\|\}_{n=0}^{\infty}$ *is convergent in* $\mathbb{R}_+$ *for every* $h \in \mathcal{H}$.

2. Preparatory Facts

In this section we give some basic facts about algebraic operators needed in this paper. We begin with a purely linear algebra result, the proof of which is left to the reader. If $\mathscr{M}$ is a complex vector space, then the identity transformation on $\mathscr{M}$ is denoted by $I_{\mathscr{M}}$ (or simply by I if no ambiguity arises). We write

$$\mathscr{M} = \mathscr{M}_1 \dotplus \ldots \dotplus \mathscr{M}_m \tag{2.1}$$

in the case when $\mathscr{M}$ is a direct sum of (finitely many) vector subspaces $\mathscr{M}_1, \ldots, \mathscr{M}_m$.

Lemma 2.1. *Suppose that* (2.1) *holds. Let* $A\colon \mathscr{M} \to \mathscr{M}$ *be a linear transformation such that* $A(\mathscr{M}_j) \subseteq \mathscr{M}_j$ *for all* $j = 1, \ldots, m$ *and let* $z \in \mathbb{C}$. *Then* $A - zI_{\mathscr{M}}$ *is a bijection if and only if* $A|_{\mathscr{M}_j} - zI_{\mathscr{M}_j}$ *is a bijection for all* $j = 1, \ldots, m$. *Moreover, if* $A - zI_{\mathscr{M}}$ *is a bijection, then* $(A - zI_{\mathscr{M}})^{-1}(\mathscr{M}_j) = \mathscr{M}_j$ *for all* $j = 1, \ldots, m$ *and*

$$(A|_{\mathscr{M}_j} - zI_{\mathscr{M}_j})^{-1} = (A - zI_{\mathscr{M}})^{-1}|_{\mathscr{M}_j}, \quad j = 1, \ldots, m.$$

Corollary 2.2. *Suppose that* $\mathcal{H}$ *is a complex Hilbert space which is a direct sum of finitely many nonzero closed vector subspaces* $\mathcal{H}_1, \ldots, \mathcal{H}_m$. *Let* $T \in \boldsymbol{B}(\mathcal{H})$ *be such that* $T(\mathcal{H}_j) \subseteq \mathcal{H}_j$ *for all* $j = 1, \ldots, m$. *Then* $\sigma(T|_{\mathcal{H}_j}) \subseteq \sigma(T)$ *for all* $j = 1, \ldots, m$.

For the sake of self-containedness, we sketch the proof of the following lemma that collects indispensable facts about algebraic operators.

Lemma 2.3. *Let* $T \in \boldsymbol{B}(\mathcal{H})$. *Then the following conditions are equivalent*:

(i) T *is algebraic*,

(ii) *there exist an integer* $m \geqslant 1$, *integers* $i_1, \ldots, i_m \geqslant 1$, *distinct complex numbers* $z_1, \ldots, z_m$ *and closed nonzero vector subspaces* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *of* $\mathcal{H}$ *such that*

(ii-a) $\mathcal{H} = \mathcal{H}_1 \dotplus \ldots \dotplus \mathcal{H}_m$,

(ii-b) $T(\mathcal{H}_j) \subseteq \mathcal{H}_j$ *for all* $j = 1, \ldots, m$,

(ii-c) $(T_j - z_j I_j)^{i_j} = 0$ *for all* $j = 1, \ldots, m$, *where* $T_j := T|_{\mathcal{H}_j}$ *and* $I_j := I_{\mathcal{H}_j}$,

(ii-d) $\sigma(T) = \{z_1, \ldots, z_m\}$ *and* $\sigma(T_j) = \{z_j\}$ *for all* $j = 1, \ldots, m$,

(ii-e) *there exists a constant* $c \in (0, \infty)$ *such that*

$$\|h_j\| \leqslant c \Big\| \sum_{k=1}^{m} h_k \Big\|, \quad j = 1, \ldots, m,\ h_1 \in \mathcal{H}_1, \ldots, h_m \in \mathcal{H}_m. \tag{2.2}$$

Proof. (i)⇒(ii) Let T be an algebraic operator and p be its minimal polynomial. Clearly, $\deg p \geqslant 1$. It follows from the fundamental theorem of algebra that

$$p(x) = (x - z_1)^{i_1} \cdots (x - z_m)^{i_m}$$

with unique integers $i_1, \ldots, i_m \geqslant 1$ and distinct complex numbers $z_1, \ldots, z_m$. In view of [6, Lemma 6.1], the condition (ii-a) holds with $\mathcal{H}_j := \mathscr{N}((T - z_j I)^{i_j}) \neq \{0\}$ for $j = 1, \ldots, m$. This implies that $\mathcal{H}_1, \ldots, \mathcal{H}_m$ are closed vector subspaces of $\mathcal{H}$ which are invariant for T. As a consequence, (ii-b) and (ii-c) hold. By the spectral mapping theorem, $\sigma(T) \subseteq \{z_1, \ldots, z_m\}$. Since $(T_j - z_j I_j)^{i_j} = 0$ and $\mathcal{H}_j \neq \{0\}$, we infer from the spectral mapping theorem and Corollary 2.2 that

$$\{z_j\} = \sigma(T_j) \subseteq \sigma(T), \quad j = 1, \ldots, m,$$

which implies (ii-d).

Now, we proceed to the proof of (ii-e). Define for $j = 1, \ldots, m$ the linear projection $P_j \colon \mathcal{H} \to \mathcal{H}$ by

$$P_j(h_1 + \ldots + h_m) = h_j, \quad h_1 \in \mathcal{H}_1, \ldots, h_m \in \mathcal{H}_m.$$

By (ii-a), this definition is correct. Using [6, Lemma 6.1(iii)] we see that

$$\mathcal{K}_2 := \mathcal{H}_2 \dotplus \ldots \dotplus \mathcal{H}_m = \mathscr{N}\bigg(\prod_{j=2}^{m} (T - z_j I)^{i_j}\bigg),$$

and so $\mathcal{K}_2$ is a closed vector subspace of $\mathcal{H}$. Since, by (ii-a), $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{K}_2$, we infer from [7, Theorem III.13.2] that $P_1 \in \boldsymbol{B}(\mathcal{H})$. A similar argument shows that $P_j \in \boldsymbol{B}(\mathcal{H})$ for all $j \in \{1, \ldots, m\}$. This implies (2.2).

(ii)⇒(i) It is enough to note that $p(T) = 0$ with $p(x) = (x - z_1)^{i_1} \cdots (x - z_m)^{i_m}$. This completes the proof. □

3. Proof of the Main Result

We begin this section by stating an auxiliary lemma.

Lemma 3.1 ([11, Lemma 2.1]). *Let $b, w \in \mathbb{C}$ be such that $|w| = 1$ and $w \neq \pm 1$. Assume that the sequence $\{\mathrm{Re}\,(w^n b)\}_{n=0}^{\infty}$ is convergent. Then $b = 0$.*

Before proving the main result of this paper, we characterize power bounded algebraic operators with spectrum in the unit circle. Recall that $T \in \boldsymbol{B}(\mathcal{H})$ is said to be *power bounded* if $\sup_{n \in \mathbb{Z}_+} \|T^n\| < \infty$.

Lemma 3.2. *Let $T \in \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent*:

(i) *T is a power bounded algebraic operator such that $\sigma(T) \subseteq \mathbb{T}$,*

(ii) *there exist an integer $m \geqslant 1$, closed nonzero vector subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of $\mathcal{H}$ and distinct complex numbers $z_1, \ldots, z_m$ such that* (ii-a) $\mathcal{H} = \mathcal{H}_1 \dotplus \ldots \dotplus \mathcal{H}_m$,

(ii-b) $T(h_1 + \ldots + h_m) = z_1 h_1 + \ldots + z_m h_m$ *for all* $h_1 \in \mathcal{H}_1, \ldots, h_m \in \mathcal{H}_m$,

(ii-c) $\{z_1, \ldots, z_m\} \subseteq \mathbb{T}$.

Proof. (i)⇒(ii) Assume that T is a power bounded algebraic operator such that $\sigma(T) \subseteq \mathbb{T}$. By Lemma 2.3, there exist an integer $m \geqslant 1$, integers $i_1, \ldots, i_m \geqslant 1$, distinct complex numbers $z_1, \ldots, z_m$ and closed nonzero vector subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of $\mathcal{H}$ that satisfy the conditions (ii-a)-(ii-d) of this lemma. Since $\sigma(T) \subseteq \mathbb{T}$, we have

$$|z_l| = 1, \quad l = 1, \ldots, m. \tag{3.1}$$

Fix $k \in \{1, \ldots, m\}$. Take $h_k \in \mathcal{H}_k \setminus \{0\}$. Using (3.1), Lemma 2.3(ii-c) and [6, Sublemma 6.3], we deduce that

$$\alpha_n(h_k) := \frac{1}{n^{N(h_k)}} \|T_k^n h_k\| \text{ converges to a positive real number as } n \to \infty, \tag{3.2}$$

where $N(h_k)$ is the unique nonnegative integer such that

$$(T_k - z_k I_k)^{N(h_k)} h_k \neq 0 \text{ and } (T_k - z_k I_k)^{N(h_k)+1} h_k = 0. \tag{3.3}$$

In particular, we have

$$\|T_k^n h_k\| = n^{N(h_k)} \alpha_n(h_k), \quad n \geqslant 1. \tag{3.4}$$

Since T_k is power bounded, the sequence $\{\|T_k^n h_k\|\}_{n=0}^{\infty}$ is bounded. Hence, one can infer from (3.2) and (3.4) that $N(h_k) = 0$. This, together with (3.3), implies that $T_k h_k = z_k h_k$. As a consequence, the system T, z_1, $\ldots$, z_m, $\mathcal{H}_1$, $\ldots$, $\mathcal{H}_m$ satisfies the conditions (ii-a), (ii-b) and (ii-c).

(ii)⇒(i) By (ii-a) and (ii-b), $p(T) = 0$ with $p(x) = (x - z_1) \cdots (x - z_m)$, which means that T is an algebraic operator. According to the spectral mapping theorem, $\sigma(T) \subseteq \{z_1, \ldots, z_m\}$. In turn, by (ii-b) and the assumption that each $\mathcal{H}_i$ is nonzero, we deduce that $z_1, \ldots, z_m$ are eigenvalues of T. Therefore, by (ii-c), we have

$$\sigma(T) = \{z_1, \ldots, z_m\} \subseteq \mathbb{T}.$$

Now, using Lemma 2.3(ii-e) (or the uniform boundedness principle), we deduce the power boundedness of T from (ii-a), (ii-b) and (ii-c). This completes the proof. □

Remark 3.3. Regarding Theorem 1.1, we note that in view of Lemma 3.2 there exist algebraic operators T with spectrum in $\mathbb{T}$ that are not unitary but have the property that each sequence $\{\|T^n h\|\}_{n=0}^{\infty}$ is bounded (or equivalently, T is power bounded). ♢

Now, we are ready to prove the main result of this paper.

Proof of Theorem 1.1. The implications (i)⇒(ii)⇒(iii)⇒(iv) are obvious.

(iv)⇒(i) Assume that T is algebraic, $\sigma(T) \subseteq \mathbb{T}$ and the sequence $\{\|T^n h\|\}_{n=0}^{\infty}$ is convergent in $\mathbb{R}_+$ for every $h \in \mathcal{H}$. By the uniform boundedness principle, T is power bounded. In view of the implication (i)⇒(ii) of Lemma 3.2, there exist an integer $m \geqslant 1$, distinct complex numbers $z_1, \ldots, z_m$ and closed nonzero vector subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of $\mathcal{H}$ that satisfy the conditions (ii-a), (ii-b) and (ii-c) of Lemma 3.2. Fix distinct $k, l \in \{1, \ldots, m\}$. Take $h_k \in \mathcal{H}_k$ and $h_l \in \mathcal{H}_l$. Then, by the conditions (ii-a), (ii-b) and (ii-c) of Lemma 3.2, we have

$$\begin{aligned} \|T^n(h_k + h_l)\|^2 &= \|z_k^n h_k + z_l^n h_l\|^2 \\ &= \|h_k\|^2 + 2\mathrm{Re}\left((z_k \bar{z}_l)^n \langle h_k, h_l\rangle\right) + \|h_l\|^2, \quad n \geqslant 0. \end{aligned}$$

Combined with (iv), this implies that the sequence $\left\{\mathrm{Re}\left((z_k\bar{z}_l)^n \langle h_k, h_l\rangle\right)\right\}_{n=0}^{\infty}$ is convergent. If $z_k \bar{z}_l = -1$, then we see that $\mathrm{Re}\,\langle h_k, h_l\rangle = 0$. Substituting $\mathrm{i}h_k$ in place of h_k, we deduce that $\langle h_k, h_l\rangle = 0$. The only possibility left is that $z_k\bar{z}_l \neq \pm 1$. Since, by the condition (ii-c) of Lemma 3.2, $|z_k \bar{z}_l| = 1$, we infer from Lemma 3.1 that $\langle h_k, h_l\rangle = 0$. This shows that $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$. Hence, by the conditions (ii-b) and (ii-c) of Lemma 3.2, we conclude that T is a unitary operator. This completes the proof. □

4. Related Results

For the reader's convenience, we record here some useful facts related to the main topic of this paper concerning certain classes of operators. We begin by discussing the question of the existence of the limits $\lim_{n\to\infty} \|T^n h\|$ in the case of normaloid operators and the issue of strong stability[1)] in the context of subnormal operators. Both are intimately related to Theorem 1.1.

Proposition 4.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a normaloid operator. Then the following conditions are equivalent*:

(i) *the sequence $\{\|T^n h\|\}_{n=0}^{\infty}$ is convergent in $\mathbb{R}_+$ for every $h \in \mathcal{H}$,*

(ii) *T is power bounded,*

(iii) *T is a contraction.*

Proof. By the uniform boundedness principle and Gelfand's formula for $r(T)$, (i) implies (ii) and (ii) implies (iii). That (iii) implies (i) is obvious. □

Before formulating the next result, we give the necessary definitions and facts related to the concept of subnormality. Recall that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator $N \in$

[1)] We refer the reader to [15, 16] for a discussion of the different types of stability of operators.

$\boldsymbol{B}(\mathcal{K})$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Th = Nh$ for all $h \in \mathcal{H}$. Such an N is called a *normal extension* of T; if $\mathcal{K}$ has no proper closed vector subspace containing $\mathcal{H}$ and reducing N, then N is called *minimal.* By a *semispectral measure* of a subnormal operator $T \in \boldsymbol{B}(\mathcal{H})$ we mean the Borel $\boldsymbol{B}(\mathcal{H})$-valued measure F on $\mathbb{C}$ defined by

$$F(\varDelta) = PE(\varDelta)|_{\mathcal{H}}, \quad \varDelta \text{ - Borel subset of } \mathbb{C}, \tag{4.1}$$

where E is the spectral measure of a minimal normal extension $N \in \boldsymbol{B}(\mathcal{K})$ of T and $P \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. In view of [12, Proposition 5], a subnormal operator has exactly one semispectral measure. We also need the following fact (see [12, Proposition 4]).

(4.2) *If N is minimal, then for every Borel subset $\varDelta$ of $\mathbb{C}$, $F(\varDelta) = 0$ if and only if $E(\varDelta) = 0$.*

According to [8, Proposition II.4.6]), the following holds.

(4.3) *Any subnormal operator is normaloid.*

We refer the reader to [8] for the foundations of the theory of subnormal operators.

Proposition 4.2. *Let $S \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator, $N \in \boldsymbol{B}(\mathcal{K})$ be a minimal normal extension of S and F be the semispectral measure of S. Then the following assertions hold*:

- (i) *if S is a contraction, then $\lim_{n\to\infty} \|S^n h\|^2 = \langle F(\mathbb{T})h, h\rangle$ for every $h \in \mathcal{H}$,*
- (ii) *the following conditions are equivalent*:
  - (ii-a) *S is strongly stable, i.e., $\lim_{n\to\infty} S^n h = 0$ for every $h \in \mathcal{H}$,*
  - (ii-b) *S is power bounded and $F(\mathbb{T}) = 0$,*
  - (ii-c) *S is a contraction and $F(\mathbb{T}) = 0$,*
- (iii) *S is strongly stable if and only if N is strongly stable.*

Proof. (i) Suppose $\|S\| \leqslant 1$. Then, by [8, Corollary II.2.17], $\|N\| = \|S\| \leqslant 1$. It follows from the spectral theorem that

$$\|S^n h\|^2 = \|N^n h\|^2 \overset{(4.1)}{=} \int_{\bar{\mathbb{D}}} |z|^{2n} \langle F(\mathrm{d}z)h, h\rangle, \quad n \in \mathbb{Z}_+,\ h \in \mathcal{H},$$

where $\bar{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leqslant 1\}$. Since for $z \in \bar{\mathbb{D}}$, the sequence $\{|z|^{2n}\}_{n=0}^{\infty}$ converges to $\chi_{\mathbb{T}}(z)$ as $n \to \infty$, where $\chi_{\mathbb{T}}$ is the characteristic function of $\mathbb{T}$, we deduce from Lebesgue's dominated convergence theorem that (i) holds.

(ii) Since subnormal operators are normaloid (see (4.3)), the assertion (ii) follows from (i) and Proposition 4.1.

(iii) Recall that $\|S\| = \|N\|$, and also that $F(\mathbb{T}) = 0$ if and only if $E(\mathbb{T}) = 0$, where E is the spectral measure of N (see (4.2)). Combined with (ii) applied to both S and N, this yields (iii). □

We now estimate the growth of norms of powers of an algebraic operator whose spectral radius is less than or equal to 1.

Proposition 4.3. *Suppose that* $T \in \boldsymbol{B}(\mathcal{H})$ *is an algebraic operator such that* $0 < r(T) \leqslant 1$*. Then there exists* $\alpha \in (0, \infty)$ *such that*

$$\|T^n\| \leqslant \alpha\, n^{\kappa} r(T)^n, \quad n \geqslant 1, \tag{4.4}$$

where $\kappa = (\deg p) - 1$ *and* p *is the minimal polynomial of* T*.*

Proof. Since $r(T) > 0$, in light of Lemma 2.3 and its proof it suffices to consider the case when $(T - zI)^i = 0$ for some $1 \leqslant i \leqslant \deg p$ and for some $z \in \mathbb{C}$ such that $0 < |z| \leqslant 1$. Setting $N = T - zI$, it is easily seen that

$$T^n = (zI + N)^n = \sum_{j=0}^{i-1} \binom{n}{j} z^{n-j} N^j, \quad n \geqslant i - 1,$$

which together with $r(T) = |z|$ implies that

$$\|T^n\| \leqslant \sum_{j=0}^{i-1} \binom{n}{j} \|N^j\| |z|^{n-j} \leqslant \bigg(\sum_{j=0}^{i-1} \frac{\|N^j\|}{j!|z|^j}\bigg) n^{i-1} r(T)^n, \quad n \geqslant i - 1.$$

This completes the proof. □

Remark 4.4. a) It is worth mentioning that if $T \in \boldsymbol{B}(\mathcal{H})$ is an algebraic operator such that $r(T) = 0$, then, in view of Lemma 2.3, the estimate (4.4) still holds, but only for $n \geqslant \deg p$.

b) It follows from Gelfand's formula for spectral radius that if $T \in \boldsymbol{B}(\mathcal{H})$ is such that $r(T) < 1$, then T is uniformly stable[2)], i.e., $\lim_{n\to\infty} \|T^n\| = 0$, and hence the sequence $\{T^n h\}_{n=0}^{\infty}$ is convergent for every $h \in \mathcal{H}$. The latter statement ceases to be true if the spectrum of T has a nonempty intersection with $\mathbb{T} \setminus \{1\}$, even if T is algebraic. It could be even worse, namely, if $T = zI$, where $z = \mathrm{e}^{2\pi \mathrm{i}\theta}$ and θ is an irrational number, then by Jacobi's theorem (see [10, Theorem I.3.13]) the closure of the set $\{T^n h \colon n \in \mathbb{Z}_+\}$ is equal to $\{zh \colon z \in \mathbb{T}\}$ for every $h \in \mathcal{H}$.

c) Note that if $T \in \boldsymbol{B}(\mathcal{H})$ is algebraic, then the sequence $\{\|T^n h\|^{1/n}\}_{n=1}^{\infty}$ is convergent in $\mathbb{R}_+$ for all $h \in \mathcal{H}$ (see [6, Proposition 6.2]). Without the assumption that T is algebraic, the sequence $\{\|T^n h\|^{1/n}\}_{n=1}^{\infty}$ may not converge. For more details on this issue, we refer the reader to [9]. ♢

We conclude this section by providing an example illustrating Theorem 1.1 and Proposition 4.1.

---

2) A more detailed discussion of this issue can be found in [16, Proposition 6.22].

Example 4.5. Let $N \in \boldsymbol{B}(\mathcal{H})$ be a nonzero operator such that $N^2 = 0$ and let $\alpha \in \mathbb{T}$. Then the operator $T_\alpha := \alpha I + N$ is algebraic because $p(T_\alpha) = 0$ with $p(x) = (x-\alpha)^2$. Hence, by the spectral mapping theorem $\sigma(T_\alpha) = \{\alpha\}$ and so T_α is invertible in $\boldsymbol{B}(\mathcal{H})$ and $r(T_\alpha) = 1$. However $\|T_\alpha\| > 1$ whenever $\|N\| > 2$ (this can be achieved simply by rescaling N). As a consequence, T_α is not normaloid, hence not subnormal (see (4.3)). Observe that $\mathcal{H} \setminus \mathscr{N}(N) \neq \emptyset$ and

$$\lim_{n\to\infty} \|T_\alpha^n h\| = \infty \text{ if and only if } h \in \mathcal{H} \setminus \mathscr{N}(N). \tag{4.5}$$

Indeed, by Newton's binomial formula (or simply by induction), we have

$$(I + \bar{\alpha}N)^n = I + n\bar{\alpha}N, \quad n \in \mathbb{Z}_+,$$

which implies that

$$\|(\alpha I + N)^n h\|^2 = \|h\|^2 + 2n\mathrm{Re}\,(\alpha\langle h, Nh\rangle) + n^2\|Nh\|^2, \quad n \in \mathbb{Z}_+,\, h \in \mathcal{H}.$$

This yields (4.5). Since $\mathcal{H} \setminus \mathscr{N}(N) \neq \emptyset$, we infer from (4.5) that T_α is not power bounded. A simple example of this kind is $\mathcal{H} = \mathbb{C}^2$, $N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $\alpha = 1$. In this particular case $\|T_1\| > 1$ without rescaling. $\diamondsuit$

References

[1] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory, **13**(1985), 203–217.

[2] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces, I*, Integral Equations Operator Theory, **21**(1995), 383–429.

[3] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces, II*, Integral Equations Operator Theory, **23**(1995), 1–48.

[4] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces, III*, Integral Equations Operator Theory, **24**(1996), 379–421.

[5] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc., **124**(1996), 3745–3752.

[6] D. Cichoń, I. B. Jung and J. Stochel, *Normality via local spectral radii*, J. Operator Theory, **61**(2009), 253–278.

[7] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics **96**, Springer-Verlag, New York(1990).

[8] J. B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, **36**, American Mathematical Society, Providence(1991).

[9] J. Daneš, *On local spectral radius*, Časopis Pěst. Mat., **112**(1987), 177–187.

[10] R. L. Devaney, *An introduction to chaotic dynamical systems*, Reprint of the second (1989) edition, Westview Press, Boulder(2003).

[11] Z. Jabłoński, I. B. Jung and J. Stochel, *On the structure of conditionally positive definite algebraic operators*, Complex Anal. Oper. Theory, **16(6)**(2022), Paper No. 90, 21 pp.

[12] I. Jung and J. Stochel, *Subnormal operators whose adjoints have rich point spectrum*, J. Funct. Anal., **255**(2008), 1797–1816.

[13] Z. J. Jabłoński, I. B. Jung and J. Stochel, *m-Isometric operators and their local properties*, Linear Algebra Appl., **596**(2020), 49–70.

[14] Z. J. Jabłoński, I. B. Jung and J. Stochel, *Conditional positive definiteness in operator theory*, Dissertationes Math., **578**(2022), 64 pp.

[15] C. S. Kubrusly, *An introduction to models and decompositions in operator theory*, Birkhäuser Boston, Boston(1997).

[16] C. S. Kubrusly, *The elements of operator theory*, Birkhäuser/Springer, New York(2011).

[17] A. Lambert, *Subnormality and weighted shifts*, J. London Math. Soc., **14**(1976), 476–480.

[18] M. Reed and B. Simon, *Methods of modern mathematical physics, vol. I: Functional analysis,* Academic Press(1980).

[19] B. Russo, *Unimodular contractions in Hilbert space*, Pacific J. Math., **26**(1968), 163–169.

[20] V. M. Sholapurkar and A. Athavale, *Completely and alternatingly hyperexpansive operators*, J. Operator Theory, **43**(2000), 43–68.

[21] J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc., **117**(1965), 469–476.