KYUNGPOOK Math. J. 63(2023), 37-43
https://doi.org/10.5666/KMJ.2022.63.1.37
pISSN 1225-6951 eISSN 0454-8124


# On the Local Cohomology and Formal Local Cohomology Modules

Shahram Rezaei* and Behruz Sadeghi
*Department of Mathematics, Payame Noor University (PNU), Tehran, Iran*
*e-mail* : sha.rezaei@gmail.com *and* b-sadeqi@student.pnu.ac.ir

Abstract. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a commutative Noetherian ring $R$ and $M$ be a finitely generated $R$-module of dimension $d > 0$. We prove some results concerning the top local cohomology and top formal local cohomology modules. Among other things, we determine $\operatorname{Supp}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$ and $\operatorname{Supp}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$. Also, we obtain some relations between $\operatorname{Ann}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, $\operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$ and $\operatorname{Supp}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$ and we get similar results for $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$.

## 1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with identity, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $R$ and $M$ is a finitely generated $R$-module of dimension $d$. Recall that the $i$-th local cohomology module of $M$ with respect to $\mathfrak{a}$ is defined as:

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

The reader can refer to [3], for the basic properties of local cohomology.

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of $M$ with respect to $\mathfrak{a}$. The basic properties of formal local cohomology modules are found in [1], [9] and [2].

In [8], we studied local cohomology module $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ and formal local cohomology module $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$. In this paper, we obtain some new results about them.

Here, we obtain some relations between $\operatorname{Ann}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$, $\operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$ and $\operatorname{Supp}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M))$. Also, we get similar results for $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)$.

We determine the support of the top local cohomology module $\mathfrak{b}\,\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)$. More precisely, we will show that $\operatorname{Supp}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Also,

* Corresponding Author.
Received July 31, 2022; revised November 20, 2023; accepted January 17, 2023.
2020 Mathematics Subject Classification: 13D45, 14B15, 13E99.
Key words and phrases: formal local cohomology, local cohomology.

we prove that for an arbitrary Noetherian local ring $(R, \mathfrak{m})$, $\operatorname{Supp}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$.

## 2. Main Results

A non-zero $R$-module $M$ is called secondary if its multiplication map by any element $a$ of $R$ is either surjective or nilpotent. A secondary representation for an $R$-module $M$ is an expression for $M$ as a finite sum of secondary submodules. If such a representation exists, we will say that $M$ is representable. A prime ideal $\mathfrak{p}$ of $R$ is said to be an attached prime of $M$ if $\mathfrak{p} = (N :_R M)$ for some submodule $N$ of $M$. If $M$ admits a reduced secondary representation, $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\operatorname{Att}_R(M)$ of $M$ is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$.

Recall that for any $R$-module $M$, the cohomological dimension of $M$ with respect to an ideal $\mathfrak{a}$ is defined as $\operatorname{cd}(\mathfrak{a}, M) = \max\{i : \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0\}$ (see [5]). Also, $Assh_R M$ is defined as $\{\mathfrak{p} \in \operatorname{Ass}_R M : \dim R/\mathfrak{p} = \dim M\}$.

We need the following lemmas in the proof of main results.

**Lemma 2.1.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ and $M$ be a non-zero finitely generated $R$-module with finite dimension $d > 0$. Then $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) = 0$ if and only if $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$.*

*Proof.* See [8, Theorem 2.3]. □

**Lemma 2.2.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$. Let $M$ be a non-zero finitely generated $R$-module with finite dimension $d > 0$ such that $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) \neq 0$. Then $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = dim(\mathfrak{b}M) = dim M = d$.*

*Proof.* See [8, Corollary 2.4]. □

The following theorem is a main result of [8] and has a key role in our proofs.

**Theorem 2.3.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$. Let $M$ be a finitely generated $R$-module with dimension $d > 0$. Then*

*i)* $\operatorname{Ann}(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$.

*ii)* $\operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$.

*Proof.* i) See [8, Theorem 2.5].

ii) See [8, Theorem 2.15]. □

In the following, we obtain a generalization of [6, Theorem 2.6].

**Theorem 2.4.** *Let $M$ be a non-zero finitely generated $R$-module of dimension $d$, and let $\mathfrak{b}$ be an ideal of $R$ such that $dim\,\mathfrak{b}M = d > 0$. If $T \subseteq \operatorname{Assh}_R(\mathfrak{b}M)$, then there exists an ideal $\mathfrak{a}$ of $R$ such that $\operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = T$.*

*Proof.* Let $T$ be a subset of $\operatorname{Assh}_R(\mathfrak{b}M)$. By [6, Theorem 2.6], there exists an ideal $\mathfrak{a}$ of $R$ such that $\operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = T$. Therefore Theorem 2.3 (ii) implies that $\operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = T$, as required. □

**Theorem 2.5.** *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be three ideals of a complete local ring $(R, \mathfrak{m})$ and $M$*

*a finitely generated R-module of dimension $d > 0$. Then*

$$\operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M)) \Leftrightarrow \operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M)).$$

*Proof.* Clearly, we can assume that $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) \neq 0$ and $\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M) \neq 0$ and so by Lemma 2.2 we have $\dim \mathfrak{b}M = d$. Assume that $\operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M))$. Theorem 2.3 (ii) implies that $\operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M))$. By [4, Corollary 3.4], we conclude that $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \cong \operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M)$ and so $\operatorname{Ann}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Ann}_R(\operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M))$. Thus by Theorem 2.3 (i) it follows that $\operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M))$. Conversely, let $\operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{c}}(M))$. By Theorem 2.3 (i), we have $\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = \operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M)$. Then by Theorem [7, 2.9] we conclude that $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \cong \operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M)$ and so $\operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \operatorname{Att}_R(\operatorname{H}^d_{\mathfrak{c}}(\mathfrak{b}M))$. Now the result follows from Theorem 2.3,(ii). □

**Theorem 2.6.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ and $M$ a finitely generated R-module of dimension $d > 0$. Then*

$$\operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

*Proof.* By Lemma 2.1 we can assume that $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) \neq 0$ and $\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$. Thus, by Lemma 2.2 we have $\dim_R(\mathfrak{b}M) = d$. By [3, 7.1.7], $\operatorname{H}^d_{\mathfrak{a}}(M)$ is artinian and so $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)$ is artinian. Thus $\operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Max}(R)$. Assume that $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\mathfrak{m} \in \operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M))$. Thus $\mathfrak{b}_{\mathfrak{m}}\operatorname{H}^d_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \neq 0$ and so by [3, Theorem 6.4.1], it follows that $\dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d$. By Lemma 2.1 we conclude that $\operatorname{H}^d_{\mathfrak{a}R_{\mathfrak{m}}}(\mathfrak{b}_{\mathfrak{m}}M_{\mathfrak{m}}) \neq 0$. Thus $\mathfrak{m} \in \operatorname{Supp}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M))$. Therefore

$$\operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)).$$

Similar to the above method, we can show that

$$\operatorname{Supp}_R(\operatorname{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) \subseteq \operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M))$$

and the proof is complete. □

**Corollary 2.7.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ and $M$ a finitely generated R-module of dimension $d > 0$. Let $\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{m} \in \operatorname{Max} R : \exists \mathfrak{p} \in \operatorname{Assh}(\mathfrak{b}M)\ s.t\ \mathfrak{p} \subseteq \mathfrak{m}, \operatorname{cd}_{R_{\mathfrak{m}}}(\mathfrak{a}R_{\mathfrak{m}}, \frac{R_{\mathfrak{m}}}{\mathfrak{p}R_{\mathfrak{m}}}) = d\}.$$

*Proof.* The assertion follows immediately from Theorem 2.6 and [4, 2.7]. □

**Corollary 2.8.** *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be three ideals of $R$ and $M, N$ two finitely generated R-modules of dimension $d > 0$. If $\operatorname{Supp}_R(\mathfrak{b}M) = \operatorname{Supp}_R(\mathfrak{c}N)$ then*

*i)* $\operatorname{Supp}_R(\mathfrak{b}\operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(\mathfrak{c}\operatorname{H}^d_{\mathfrak{a}}(N))$.

*ii)* $\operatorname{Att}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathfrak{c}\,\mathrm{H}^d_{\mathfrak{a}}(N))$.

*Proof.* Since $\operatorname{Supp}_R(\mathfrak{b}M) = \operatorname{Supp}_R(\mathfrak{c}N)$ we conclude that $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{c}N)$ by [5, Theorem 2.2]. If $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = 0$ then $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$ and so $\operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) < d$ by Lemma 2.1. Thus $\operatorname{cd}(\mathfrak{a}, \mathfrak{c}N) < d$ and so $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{c}N) = 0$. By Lemma 2.1, it follows that $\mathfrak{c}\,\mathrm{H}^d_{\mathfrak{a}}(N) = 0$ and the result follows in this case.

Now, assume that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. By Lemma 2.2 we have $\dim_R(\mathfrak{b}M) = \operatorname{cd}(\mathfrak{a}, \mathfrak{b}M) = d$. But $\operatorname{Supp}_R(\mathfrak{b}M) = \operatorname{Supp}_R(\mathfrak{c}N)$ implies that $\dim_R(\mathfrak{c}N) = \operatorname{cd}(\mathfrak{a}, \mathfrak{c}N) = d$ and $\operatorname{Assh}_R \mathfrak{b}M = \operatorname{Assh}_R \mathfrak{c}N$. Now Corollary 2.7 shows that, $\operatorname{Supp}_R \mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = \operatorname{Supp}_R \mathfrak{c}\,\mathrm{H}^d_{\mathfrak{a}}(N)$. On the other hand, since $\operatorname{Assh}_R(\mathfrak{b}M) = \operatorname{Assh}_R(\mathfrak{c}N)$ we have

$$\{\mathfrak{p} \in \operatorname{Assh}_R(\mathfrak{b}M) : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} = \{\mathfrak{p} \in \operatorname{Assh}_R(\mathfrak{c}N) : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

Now, the result (ii) follows immediately from [8, Corollary 2.16], as required. □

In the above results, we saw that some basic properties of two $R$-modules $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ and $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$ are the same. The following result shows that these two $R$-modules are isomorphic under certain conditions.

**Theorem 2.9.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ and $M$ a finitely generated $R$-module of dimension $d > 0$. In each of the following cases, $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)$.*

*i)* $\operatorname{cd}(\mathfrak{a}, M/\mathfrak{b}M) < d - 1$.

*ii)* $\mathfrak{b}M = M$.

*iii)* $\mathfrak{b} = Rx$ *where $x$ is a non-zerodivisor on $M$.*

*Proof.* By Lemma 2.1 we can assume that $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$ and $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$. Thus, by Lemma 2.2, we have $\dim_R(\mathfrak{b}M) = \operatorname{cd}(\mathfrak{a}, M) = d$. Assume that $\mathfrak{c} := \operatorname{Ann}_R M$. Since $\operatorname{Supp}_R(R/\mathfrak{c}) = \operatorname{Supp}_R(M)$, by [5, Theorem 2.2] $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{c}) = \operatorname{cd}(\mathfrak{a}, M) = d$. Thus by the Independence Theorem ([3, Theorem 4.2.1]) $\operatorname{cd}(\mathfrak{a}R/\mathfrak{c}, R/\mathfrak{c}) = d$ and so $\mathrm{H}^i_{\mathfrak{a}R/\mathfrak{c}}(R/\mathfrak{c}) = 0$ for all $i > d$. It follows that $\mathrm{H}^d_{\mathfrak{a}R/\mathfrak{c}}(-)$ is a right exact functor on the category of $R/\mathfrak{c}$-modules and $R/\mathfrak{c}$-homomorphisms. Thus

$$\begin{aligned}\mathrm{H}^d_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) &\simeq (\mathrm{H}^d_{\mathfrak{a}R/\mathfrak{c}}(R/\mathfrak{c}) \otimes_{R/\mathfrak{c}} M) \otimes_R R/\mathfrak{b} \\ &\simeq \mathrm{H}^d_{\mathfrak{a}R/\mathfrak{c}}(R/\mathfrak{c}) \otimes_{R/\mathfrak{c}} M/\mathfrak{b}M \\ &\simeq \mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{b}M).\end{aligned}$$

If $\operatorname{cd}(\mathfrak{a}, M/\mathfrak{b}M) < d-1$ then $\mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{b}M) = 0$ and so by the above isomorphism we conclude that $\mathrm{H}^d_{\mathfrak{a}}(M) = \mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$. On the other hand, the exact sequence $0 \to \mathfrak{b}M \to M \to M/\mathfrak{b}M \to 0$ induces an exact sequence

$$\cdots \to \mathrm{H}^{d-1}_{\mathfrak{a}}(M/\mathfrak{b}M) \to \mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \to \mathrm{H}^d_{\mathfrak{a}}(M) \to \mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{b}M) \to 0.$$

Since $\operatorname{cd}(\mathfrak{a}, M/\mathfrak{b}M) < d - 1$ we have $\mathrm{H}^{d-1}_{\mathfrak{a}}(M/\mathfrak{b}M) = \mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{b}M) = 0$ and so the above exact sequence implies that $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \cong \mathrm{H}^d_{\mathfrak{a}}(M)$. But, we showed that $\mathrm{H}^d_{\mathfrak{a}}(M) = \mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ and so the proof is complete in this case.

If $\mathfrak{b}M = M$ then $\mathrm{H}^d_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) \simeq \mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{b}M) = 0$. Thus $\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M) = \mathrm{H}^d_{\mathfrak{a}}(M)$. Since $\mathfrak{b}M = M$ we have $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = \mathrm{H}^d_{\mathfrak{a}}(M)$ and the result follows in this case.

Now assume that $\mathfrak{b} = Rx$ and $x$ is a non-zerodivisor on $M$. The exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ induces an exact sequence

$$\cdots \to \mathrm{H}^d_{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}^d_{\mathfrak{a}}(M) \to \mathrm{H}^d_{\mathfrak{a}}(M/xM) \to 0.$$

Since $\dim M/xM = d-1$, by [3, 6.1.2] we have $\mathrm{H}^d_{\mathfrak{a}}(M/xM) = 0$. Thus, the above exact sequence implies that $x\,\mathrm{H}^d_{\mathfrak{a}}(M) = \mathrm{H}^d_{\mathfrak{a}}(M)$. On the other hand, it is easy to see that the $R$-module homomorphism $\varphi : M \xrightarrow{x} xM$ is an isomorphism and it follows that $\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{a}}(xM)$. Therefore $x\,\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{a}}(xM)$, as required. □

In the remainder, we prove some results about $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$. For our proofs, we need the following theorems.

**Theorem 2.10.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module. Then*

$$dim_R M/\mathfrak{a}M = \sup\{i \in \mathbb{N}_0 : \mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0\}.$$

*Proof.* See [9, Theorem 4.5]. □

**Theorem 2.11.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$. Then*

$$\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0 \Longleftrightarrow \mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0.$$

*Proof.* See [8, Theorem 3.3]. □

**Corollary 2.12.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$ such that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$ and $dim\,(M/\mathfrak{a}M) = dim\,(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) = dim\,\mathfrak{b}M = d$.*

*Proof.* See [8, Corollary 3.4]. □

**Theorem 2.13.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$. Then*

*i)* $\mathrm{Ann}(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$.

*ii)* $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$.

*Proof.* i) See [8, Theorem 3.6].

ii) See [8, Theorem 3.13]. □

**Theorem 2.14.** *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be three ideals of a local ring $(R, \mathfrak{m})$ and $M, N$ two finitely generated $R$-modules of dimension $d > 0$ such that $\mathrm{Supp}_R(\mathfrak{b}M) = \mathrm{Supp}_R(\mathfrak{c}N)$. Then $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{c}\mathfrak{F}^d_{\mathfrak{a}}(N))$.*

*Proof.* If $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) = 0$ then by Theorem 2.11, $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) = 0$ and so by Theorem 2.10, $\dim\,(\mathfrak{b}M/\mathfrak{a}\mathfrak{b}M) < d$. Since $\mathrm{Supp}_R(\mathfrak{b}M) = \mathrm{Supp}_R(\mathfrak{c}N)$ we conclude that

$\dim(\mathfrak{c}N/\mathfrak{a}\mathfrak{c}N) < d$ and by Theorem 2.10, $\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{c}N) = 0$. Therefore by Theorem 2.11, $\mathfrak{c}\mathfrak{F}_{\mathfrak{a}}^{d}(N) = 0$. Thus $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Att}_R(\mathfrak{c}\mathfrak{F}_{\mathfrak{a}}^{d}(N)) = \emptyset$ and so the result follows in this case.

Now, assume that $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) \neq 0$. If $\mathfrak{c}\mathfrak{F}_{\mathfrak{a}}^{d}(N) = 0$ then similar to the above argument it follows that $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) = 0$. Thus $\mathfrak{c}\mathfrak{F}_{\mathfrak{a}}^{d}(N) \neq 0$ and so by Corollary 2.12 we have $\dim \mathfrak{b}M = \dim \mathfrak{c}N = d$. Since $\mathrm{Supp}_R(\mathfrak{b}M) = \mathrm{Supp}_R(\mathfrak{c}N)$ it is easy to see that $\mathrm{Assh}_R(\mathfrak{b}M) = \mathrm{Assh}_R(\mathfrak{c}N)$ and so $\mathrm{Assh}_R(\mathfrak{b}M) \cap Var(\mathfrak{a}) = \mathrm{Assh}_R(\mathfrak{c}N) \cap Var(\mathfrak{a})$. Therefore, by [8, Corollary 3.14], we have $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Att}_R(\mathfrak{c}\mathfrak{F}_{\mathfrak{a}}^{d}(N))$, as required. □

**Theorem 2.15.** *Let $\mathfrak{b}$ be an ideal of a local ring $(R, \mathfrak{m})$, and let $M$ be a finitely generated $R$-module of dimension $d > 0$ such that $\dim_R(\mathfrak{b}M) = d$. If $T \subseteq \mathrm{Assh}_R(\mathfrak{b}M)$, then $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = T$ where $\mathfrak{a} := \bigcap_{\mathfrak{p}_i \in T} \mathfrak{p}_i$ is an ideal of $R$.*

*Proof.* By [7, Theorem 2.2], $\mathrm{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = T$ where $\mathfrak{a} := \bigcap_{\mathfrak{p}_i \in T} \mathfrak{p}_i$ is an ideal of $R$. Now Theorem 2.13 (ii) implies that $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = T$. □

**Theorem 2.16.** *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be three ideals of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$. Then*

$$\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M)) \Leftrightarrow \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M)).$$

*Proof.* $\Rightarrow)$ : If $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) = 0$ then $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \emptyset$. By assumption we conclude that $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M)) = \emptyset$ and so $\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M) = 0$.

Now we assume that $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) \neq 0$ and $\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M) \neq 0$. Thus, by Corollary 2.12 $\dim \mathfrak{b}M = d$ and by Theorem 2.13 (ii) and our assumption, we have $\mathrm{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Att}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$. Therefore, by [2, 3.4], we have $\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M) \cong \mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M)$. Thus $\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$ and so the assertion follows from Theorem 2.13 (i).

$\Leftarrow)$ : We can (and do) assume that $\mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^{d}(M) \neq 0$ and $\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M) \neq 0$. Then, by Theorem 2.13 (i) and our assumption, we have $\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$. Therefore, by [6, Corollary 3.9 (i)], we conclude that $\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M) \cong \mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M)$. Thus $\mathrm{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Att}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$. Therefore, the assertion follows from Theorem 2.13 (ii). □

**Theorem 2.17.** *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be three ideals of a complete local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$. Then*

$$\mathrm{Att}_R(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M)) \Leftrightarrow \mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M)).$$

*Proof.* $\Rightarrow)$: We can assume that $\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M) \neq 0$ and $\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M) \neq 0$ and so $\dim \mathfrak{b}M = d$. If $\mathrm{Att}_R(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \mathrm{Att}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M))$ then by Theorem 2.3 (ii) and Theorem 2.13 (ii) we conclude that $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Att}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$. By [7, Theorem 2.5], it follows that $\mathrm{H}_{\mathfrak{a}}^{d}(\mathfrak{b}M) \cong \mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M)$ and so $\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{a}}^{d}(\mathfrak{b}M)) = \mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{c}}^{d}(\mathfrak{b}M))$. Thus by Theorem 2.3 (i) and Theorem 2.13 (i) we have $\mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}_{\mathfrak{c}}^{d}(M))$, as required.

$\Leftarrow$): Assume that $\mathrm{Ann}_R(\mathfrak{b}\,\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{c}}(M))$. By Theorem 2.3 (i) and Theorem 2.13 (i), we have $\mathrm{Ann}_R\,\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) = \mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{c}}(\mathfrak{b}M)$. Then by [7, Corollary 2.9] we conclude that $\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M) \cong \mathfrak{F}^d_{\mathfrak{c}}(\mathfrak{b}M)$ and so $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{c}}(\mathfrak{b}M))$. Now, the result follows from Theorem 2.3 (ii) and Theorem 2.13 (ii). □

**Theorem 2.18** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d > 0$. Then*

$$\mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Supp}_R(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M))$$

*Proof.* By Theorem 2.11, we can (and do) assume that $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ and so by Corollary 2.12 we have $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M) \neq 0$ and $\dim \mathfrak{b}M = d$. But, by [2, Lemma 2.2] $\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)$ is an artinian $R$-module and so $\mathrm{Supp}(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \{\mathfrak{m}\}$. On the other hand, $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is artinian and so $\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ is artinian. Thus $\mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M)) = \{\mathfrak{m}\}$. Since $\mathrm{Supp}_R(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \{\mathfrak{m}\}$ we conclude that $\mathrm{Supp}_R(\mathfrak{F}^d_{\mathfrak{a}}(\mathfrak{b}M)) = \mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^d_{\mathfrak{a}}(M))$, as required. □

**Acknowledgments.** The author would like to thank the referee for careful reading and many useful suggestions.

# References

[1] M. Asgharzadeh and K. Divaani-Aazar, *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness*, Comm. Algebra, **39**(2011), 1082–1103.

[2] M. H. Bijan-Zadeh and S. Rezaei, *Artinianness and attached primes of formal local cohomology modules*, Algebra Colloq., **21(2)** (2014), 307–316.

[3] M. P. Brodmann and R. Y. Sharp, *Local cohomology; an algebraic introduction with geometric applications,* Cambridge University Press, Cambridge(1998).

[4] K. Divaani-Aazar, *Vanishing of the top local cohomology modules over Noetherian rings*, Proc. Indian Acad. Sci. Math. Sci., **119(1)**(2009), 23–35.

[5] K. Divaani-Aazar, R. Naghipour and M. Tousi, *Cohomological dimension of certain algebraic varieties,* Proc. Amer. Math. Soc., **130(12)**(2002), 3537–3544.

[6] A. R. Nazari and F. Rastgoo, *Top local cohomology and top formal local cohomology modules with specified attached primes*, J. Algebr. Syst., **8(2)**(2021), 155–164.

[7] S. Rezaei *Some results on top local cohomology and top formal local cohomology modules*, Comm. Algebra, **45(5)** (2017), 1935–1940.

[8] S. Rezaei and B. Sadeghi, *On the top local cohomology and formal local cohomology modules*, Bull. Korean Math. Soc., **60(1)**(2023), 149–160.

[9] P. Schenzel, *On formal local cohomology and connectedness*, J. Algebra, **315(2)**(2007), 894–923.