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On ϑ-quasi-Geraghty Contractive Mappings and Application to Perturbed Volterra and Hypergeometric Operators

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Abstract. In this paper we suggest an enhanced Geraghty-type contractive mapping for examining the existence properties of classical nonlinear operators with or without prior degenerates. The nonlinear operators are proved to exist with the imposition of the Geraghty-type condition in a non-empty closed subset of complete metric spaces. To showcase some efficacies of the Geraghty-type condition, convergent rate and stability are deduced. The results are used to study some asymptotic properties of perturbed integral and hypergeometric operators. The results also extend and generalize some existing Geraghty-type conditions.

1. Introduction

Application of Banach's contraction map [3] in the area of applied and social sciences has birthed many general concepts by abstracting some common properties of Banach's condition. Two of these general concepts appear in [11, 20]. The Banach-type map is reformulated by:

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha(d(x,y))d(x,y),\ \forall x, y \in X, \tag{1.1}$$

where X is a complete metric space and T is a self-map of X. In [11], if $\mathcal{F} = \{\alpha | \alpha : \mathbb{R}^+ \to [0,1)\}$ is a class of functions for which $\alpha(s_n) \to 1$ (not continuous) implies $s_n \to 0$, then T has a unique fixed point. Then again in [20], it is proved that if $\alpha : \mathbb{R}^+ \to [0,1)$ is a monotone decreasing function, then T satisfies (1.1). The former and latter results have prompted several generalizations in the last few decades. Presently, there exists a vast amount of literature on the results concerning the Geraghty map. In [12], a generalized Geraghty-type for the class of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is proved under the condition

$$\psi(d(\mathrm{T}x, \mathrm{T}y)) \leq \alpha(\psi(d(x,y)))\psi(d(x,y)),\ \forall x, y \in X, \tag{1.2}$$

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in partially ordered set $(X, \leq)$ with $\alpha \in \mathcal{F}$. Also see [6] for an improvement on (1.2) in modular metric spaces. Martnez-Moreno et al. [14] studied the common fixed point theorems of Geraghty-type for the two mappings $\mathsf{S}, \mathrm{T} : X \to X$ such that T has the S-monotone property and satisfies

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha(d(Sx, Sy))d(Sx, Sy), \ \forall x, y \in X. \tag{1.3}$$

In [5], a generalized $(\psi, \alpha, \beta)-$Geraghty type condition for the three maps $\mathcal{R}, \mathsf{S}, \mathrm{T}$ satisfying

$$\psi(d(Tx, \mathcal{R}y)) \leq \alpha(d(Sx, Sy))\beta(d(Sx, Sy)), \ \forall x \geq y. \tag{1.4}$$

was introduced and proved in the framework of partially ordered metric spaces. Another recent extension was proved in [10] for the set of all functions $\alpha : [0, \infty) \to [0, \frac{1}{s})$ satisfying

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha(\mathrm{M}(x, y))d(\mathrm{M}(x, y)), \ \forall x, y \in X. \tag{1.5}$$

where $\mathrm{M}(x, y) = max\{d(x, y), d(x, \mathrm{T}x), d(y, \mathrm{T}y), \frac{1}{2s}\left[d(x, \mathrm{T}y) + d(y, \mathrm{T}x)\right]\}$.
Few results regarding the admissibility of the Geraghty-type operators can be seen in [2, 7, 13, 19]. Some related results to the contractive condition in [20] shall be treated as consequences in the main results. Though, the results above are suitable for studying the existence properties of nonlinear self-maps satisfying Geraghty-type conditions with $\alpha \in \mathcal{F}$. However, some results regarding the estimates such as convergent rate and stability have received low attention concerning perturbations of the operator T. This is because as $\alpha \to 1$, the estimates become costly, and worse if eventually α attains 1 (nonexpansive case, where no refinement is not allowed). In this case, the fixed point of T is degenerate and a problem of its finding is a prior unstable (see page 8 [1]). Motivated by the above reasons, the present paper suggests a quasi-Geraghty contractive condition to study both the existence properties and the effectiveness of some nondegenerate nonlinear operators with applications to perturbed integral operators and hypergeometric-type operators.

2. The ϑ-Quasi-Geraghty Mappings

Let $\mathrm{T} : X \to X$ be a self-map which has, but not limited to, the following properties:

I. $d(\mathrm{T}x, \mathrm{T}y) \leq \lambda d(x, y)$;

II. $d(\mathrm{T}x, \mathrm{T}y) \leq \lambda d(x, \mathrm{T}x)$;

III. $d(\mathrm{T}x, \mathrm{T}y) \leq \lambda d(y, \mathrm{T}y)$.

at some distinct points $x, y \in X$ and for $\lambda \in [0, 1)$. The map T is called λ-pseudocontraction if it satisfies at least one of the conditions I - III and λ-contraction if condition I holds, see [24]. In what follow, λ-pseudocontraction map is likened to the class of $\mathcal{F}$.

Lemma 2.1. *Suppose that* T : $X \to X$ *is* λ*-pseudocontraction mapping. Let* $u_0 \in X$ *and set* $u_{n+1} = \mathrm{T}u_n$ *for* $n = 0, 1, 2, \ldots$. *Then* u_n *converges to a unique fixed point if and only if there exists* $\lambda \in \mathcal{F}$ *such that for all* $n, m \in \mathbb{N}$, *at least one of the conditions I - III hold.*

Proof. Obviously, condition I is an analogous of (1.1) if $\lambda \in \mathcal{F}$. So let $u_0 \in X$ and assume that u_n converges to a unique fixed point p. It suffices to prove that the constant function $\lambda \in \mathcal{F}$ exists. Define $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\lambda(t_n) = \sup\left\{\frac{d(Tu_n, Tu_m)}{d(u_n, u_{n+1})} : t_n \leq d(u_n, u_{n+1})\right\}$$

Since T is a λ-pseudocontraction map (II), the above quotient does not exceed 1. Let $t_n = d(u_{n-1}, u_n)$, then $\alpha(t_n) \to 1$, and indeed, $t_n \to 0$ as $n \to \infty$ since u_n converges.
Next, assume that $\lambda \in \mathcal{F}$ exists, that is, $\lambda(t_n) \to 1$ for each t_n in $\mathbb{R}^+$. We show that T is asymptotically regular and $t_n \to 0$. Since $u_n = T^n u_0$ for $n = 0, 1, 2, \ldots$, by hypothesis:

$$\begin{aligned} t_{n+1} &= d(u_n, u_{n+1}) = d(\mathrm{T}^n u_0, \mathrm{T}^{n+1} u_0) \\ &\leq \lambda(d(T^{n-1}u_0, \mathrm{T}^n u_0)) d(\mathrm{T}^{n-1}u_0, T^n u_0) \\ &= \lambda(t_n) t_n \end{aligned}$$

Since $\lambda \in \mathcal{F}$, the last inequality implies that

$$t_{n+1} \leq t_n.$$

Hence, t_n is a nonincreasing nonnegative terms, and thus converges to a nonnegative real number ϵ for which

$$\epsilon = \liminf_{n\to\infty} t_n.$$

By replacing $n = n_k$, it follows that $\epsilon \to 0$.
Now assume that u_n is not Cauchy. For given $\varepsilon > 0$, there are positive integers m_k and n_k with $n_k > m_k > k$ such that $d(u_{m_k}, u_{n_k}) \geq \varepsilon$.
But, by hypothesis of the lemma, it follows that

$$\varepsilon \leq d(u_{m_k}, u_{n_k}) \leq \lambda(t_{m_k-1}) t_{m_k-1} \to 0$$

This violates the latter condition. Hence, u_n is Cauchy and by completeness, u_n converges to a unique fixed point in X. Condition III also follows. □

Now, let $\mathcal{F}^*$ be a class of finite functions $(\alpha_1, \alpha_2, \ldots, \alpha_{q_0})$ for $q_0 \in \mathbb{N}$ such that for each $i \in \{1, 2, \ldots, q_0\}$ there corresponds a set of independent nonnegative terms $\{t_{n,1}; t_{n,2}; \ldots; t_{n,q_0}\}$, for each $t_{n,i} \in \mathbb{R}^+$, with the property that each function $\alpha_i(t_{n,i}) \to \frac{1}{q_0}$ implies $t_{n,i} \to 0$ as $n \to \infty$ and that

$$\alpha_1(t_{n,1}) + \alpha_2(t_{n,2}) + \cdots + \alpha_{q_0}(t_{n,q_0}) \to 1 \tag{2.1}$$

Observe that for $q_0 = 1$, the class $\mathcal{F}^*$ is a refined form of $\mathcal{F}$ or simply a subclass of $\mathcal{F}$. Also, if $t_{n,i} = t_n$, then the characterisation (2.1) is similar to the results in [6, 16], see also [21] for earlier work.
Motivated by Lemma 2.1, let $T : X \to X$ be a self-map satisfying the λ_i-pseudocontraction like conditions with independent inputs
$\{t_1; t_2; t_{p_0}\} = \{d(x, y); d(x, Tx); d(y, Ty)\}$ such that for $p_0 < q_0$,

$$d(\mathrm{T}x, \mathrm{T}y) \leq (\lambda_1(t_{n,1})t_1 + \lambda_2(t_{n,2})t_2 + \lambda_{p_0}(t_{n,p_0})t_{p_0})\, p_0^{-1} \tag{2.2}$$

By resolving in terms of t_1 and t_2 while t_{p_0} is restrained, this gives

$$d(\mathrm{T}x, \mathrm{T}y) \leq \lambda_1(t_{n,1})\frac{(1 + \frac{\lambda_{p_0}(t_{n,p_0})}{\lambda_1(t_{n,1})})}{p_0 - \lambda_{p_0}(t_{n,p_0})}t_1 + \lambda_2(t_{n,2})\frac{(1 + \frac{\lambda_{p_0}(t_{n,p_0})}{\lambda_2(t_{n,2})})}{p_0 - \lambda_{p_0}(t_{n,p_0})}t_2 \tag{2.3}$$

If $\lambda_1(t_{n,1})$ and $\lambda_2(t_{n,2})$ grow when $\lambda_{p_0}(t_{n,p_0})$ diminishes. Then, inequality (2.3) is equivalent to

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha_1(t_{n,1})\frac{2}{p_0}t_1 + \alpha_2(t_{n,2})\frac{2}{p_0}t_2 \tag{2.4}$$

where $p_0 > 0$; $\alpha_1(t_{n,1})$ and $\alpha_2(t_{n,1})$ are in $\mathcal{F}^*$ with the property that $\alpha_1(t_{n,1}) + \alpha_2(t_{n,2}) \to 1$. If $p_0 = 2$, then

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha_1(t_{n,1})t_1 + \alpha_2(t_{n,2})t_2 \tag{2.5}$$

The inequalities (2.4) and (2.5) shall be formalised in the sequel. Before then, the following class of test functions are defined.

Definition 2.2. For $q_0 = 2$, the set $\mathcal{F}^*$ is the class of functions $\alpha_i : \mathbb{R}^+ \to [0, \frac{1}{2})$ with the property that $\alpha_i(t_{n,i}) \to \frac{1}{2}$ implies $t_{n,i} \to 0$ for $i = 1, 2$.

Definition 2.3. Let Φ be the class of functions $\vartheta : [0, \infty) \to [0, \infty)$ with the property that

- ϑ_1: ϑ is lower semi-continuous and non-decreasing function;
- ϑ_2: $\vartheta(t) < t$;
- ϑ_3: $\vartheta(t) = 0$ if and only if $t = 0$; and
- ϑ_4: ϑ is subadditive.

Motivated by the above classes of functions, Lemma 2.1, conditions (2.4) and (2.5), the $\vartheta-$Geraghty-type map is presented as follow:

Definition 2.4. Let X be a complete metric space and $T : X \to X$ be a self map. The map T is called a ϑ-quasi-Geraghty contractive map if it satisfies

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha(d(x, y))\vartheta(d(x, y)) + \beta(d(x, \mathrm{T}x))\vartheta(d(x, \mathrm{T}x)), \tag{2.6}$$

for $x, y \in X$, where $\vartheta \in \Phi$ and $\alpha, \beta \in \mathcal{F}^*$.

By comparing condition (2.4) and (2.6), the role of function ϑ is obvious.

Definition 2.5. Let X be a complete metric space and $T : X \to X$ be a self map. The map T is called quasi-Geraghty contractive map if it satisfies

$$d(\mathrm{T}x, \mathrm{T}y) \leq \alpha(d(x, y))d(x, y) + \beta(d(x, \mathrm{T}x))d(x, \mathrm{T}x), \tag{2.7}$$

for $x, y \in X$, where $\alpha, \beta \in \mathcal{F}^*$.

Condition (2.7) is a special case of condition (2.6) if ϑ is an identity and it is also similar to (2.4) for $p_0 = 2$. Both (2.6) and (2.7) are class of general nonlinear contractive maps of second kind of the form

$$d(\mathrm{T}x, \mathrm{T}y) \leq \psi_1(s) + \psi_2(t), \text{ for } x, y \in X, \tag{2.8}$$

where ψ_1, ψ_2 are upper semi-continuous functions and $s, t \in [0, \infty)$, see [22]. So, these conditions are independent and have advantages over other like conditions in the literature.

3. Main Results

In this section, the existence properties and stability of the nonlinear self-operator $T : \mathcal{K} \to \mathcal{K}$ are established and proved with the imposition of conditions (2.6) and (2.7). The convergent rate of the quasi-Geraghty conditions is deduced and compared using practical examples.

Theorem 3.1. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and $T : \mathcal{K} \to \mathcal{K}$ be a Picard map satisfying (2.6) for which $\vartheta \in \Phi$ and $\alpha, \beta \in \mathcal{F}^*$. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.*

Proof. Let $x_0 \in K$ and let x_n be defined by the Picard sequence

$$x_{n+1} = \mathrm{T}x_n, \; n = 0, 1, 2, \ldots$$

By the condition (2.6), there gives

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(\mathrm{T}x_{n-1}, \mathrm{T}x_n\right) \\ &\leq \alpha\left(d(x_{n-1}, x_n)\right) \vartheta\left(d(x_{n-1}, x_n)\right) \\ &\quad + \beta\left(d(x_{n-1}, \mathrm{T}x_{n-1})\right) \vartheta\left(d(x_{n-1}, \mathrm{T}x_{n-1})\right) \end{aligned}$$

This further implies that

$$d(x_n, x_{n+1}) \leq \left[\alpha\left(d(x_{n-1}, x_n)\right) + \beta\left(d(x_{n-1}, x_n)\right)\right] \vartheta\left(d(x_{n-1}, x_n)\right) \tag{3.1}$$

Since $\alpha, \beta \in \mathcal{F}^*$, the last inequality reduces to

$$d(x_n, x_{n+1}) \leq \vartheta\left(d(x_{n-1}, x_n)\right)$$

More so, since $\vartheta \in \Phi$, thus,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

This implies that $d(x_n, x_{n+1})$ is nonincreasing nonnegative term, and thus converges to a nonnegative real number ϵ such that

$$\epsilon = \liminf_{n\to\infty} d(x_n, x_{n+1}) = \liminf_{n\to\infty} d(x_{n-1}, x_n)$$

By taking limit of inequality (3.1) as $n \to \infty$ and using the properties on ϑ, α and β, we obtain

$$\epsilon \leq [\alpha(\epsilon) + \beta(\epsilon)]\, \vartheta\,(\epsilon) < \epsilon$$

This contradicts the hypothesis. Hence,

$$\epsilon = \liminf_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3.2}$$

Next is to prove that $\{x_n\}$ is a Cauchy sequence. On contrary, suppose $\{x_n\}$ is not Cauchy, Then for given $\varepsilon > 0$, there exist positive integers m_k and n_k such that $n_k > m_k > k$ for all positive integer $\mathcal{K}$,

$$d(x_{n_k}, x_{m_k}) > \varepsilon \text{ and } d(x_{n_k}, x_{m_k-1}) \leq \varepsilon \tag{3.3}$$

Using (3.2) and triangle inequality in (3.3), we have the following:

$$\varepsilon < d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \leq \varepsilon$$

Taking limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon \tag{3.4}$$

Thus, by using (2.6), (3.2), (3.4) and triangle inequality, there results

$$\begin{aligned}
\varepsilon < d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k}) \\
&= d(\mathrm{T}x_{n_k}, \mathrm{T}x_{m_k-1}) \\
&\leq \alpha\,(d(x_{n_k}, x_{m_k-1}))\, \vartheta\,(d(x_{n_k}, x_{m_k-1})) \\
&+ \beta\,(d(x_{n_k}, \mathrm{T}x_{n_k}))\, \vartheta\,(d(x_{n_k}, \mathrm{T}x_{n_k})) \\
&= \alpha\,(d(x_{n_k}, x_{m_k-1}))\, \vartheta\,(d(x_{n_k}, x_{m_k-1})) \\
&+ \beta\,(d(x_{n_k}, x_{n_k+1}))\, \vartheta\,(d(x_{n_k}, x_{n_k+1}))
\end{aligned}$$

Applying limit as $n_k \to \infty$ to the last inequality, we get

$$\varepsilon < \alpha\,(\varepsilon)\, \vartheta\,(\varepsilon)$$

Since $\alpha \in \mathcal{F}^*$ and $\vartheta \in \Phi$, then

$$\varepsilon < \frac{1}{2}\varepsilon$$

This contradicts ε being a positive real number. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, x_n converges to $x^* \in X$.
Next is to prove that $\mathrm{T}x^* \in X$, that is, $x^* = \mathrm{T}x^*$. By (2.6) and triangle inequality, we have

$$d(\mathrm{T}x^*, x^*) \leq d(\mathrm{T}x^*, \mathrm{T}x_n) + d(x_{n+1}, x^*)$$

But then,

$$d(\mathrm{T}x^*, \mathrm{T}x_n) \leq \alpha\left(d(x^*, x_n)\right)\vartheta\left(d(x^*, x_n)\right) + \beta\left(d(x_n, x_{n+1})\right)\vartheta\left(d(x_n, x_{n+1})\right)$$

By (3.2) and convergence of x_n, $\vartheta\left(d(x^*, x_n)\right) \to 0$ and $\vartheta\left(d(x_n, x_{n+1})\right) \to 0$ as $n \to \infty$.
Thus, T is continuous on X. Therefore, $d(\mathrm{T}x^*, x^*) = 0$ if and only if $x^* = \mathrm{T}x^*$.
Next, assume that x^* and y^* are two fixed points of T with $d(x^*, y^*) \neq 0$. Then, by hypothesis

$$d(x^*, y^*) = d(\mathrm{T}x^*, \mathrm{T}y^*) \leq \alpha\left(d(x^*, y^*)\right)\vartheta\left(d(x^*, y^*)\right) + \beta\left(d(x^*, \mathrm{T}x^*)\right)\vartheta\left(d(x^*, \mathrm{T}x^*)\right)$$

By the properties on α, β and ϑ, this reduces to

$$d(x^*, y^*) \leq \frac{1}{2}d(x^*, y^*)$$

Hence a contradiction. Therefore, $x^* = y^*$. □

Remark 3.2. If $\vartheta(s) \equiv s$, for all $s \geq 0$ in the condition (2.6), then the proof is analogue of Theorem 3.1.

Example 3.3. Let $X = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3)\}$ be a 2D diamond-pentagon set endowed with the taxicab metric
$d(y, z) = |y_1 - z_1| + |y_2 - z_2|$ for all $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in X. Also, let T on X be a self-map defined as follows:

$$T(y_1, y_2) = \begin{cases} (y_1, 1); & y_1 \leq y_2 \\ (2, y_2); & y_1 > y_2 \end{cases}$$

with $\vartheta(t) = t$, for $t \in \mathbb{R}^+$.

It is easily verified that the fixed nodes are $(1,1)$ and $(2,1)$. Then, the map T satisfies all hypotheses of Theorem 3.1. It is worthy to note that the condition (2.6) in Theorem 3.1 is weaker than those in the previous studies.

Corollary 3.4. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and $T : \mathcal{K} \to \mathcal{K}$ be a Picard map satisfying* (2.6) *for which $\vartheta \in \Phi$ and*

$\alpha, \beta : [0, \infty) \to [0, \frac{1}{2})$ are monotone decreasing. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.

Proof. Obviously, such α, β are in the subclass $\mathcal{F}^*$. □

Corollary 3.5. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and $T : \mathcal{K} \to \mathcal{K}$ be a Picard map satisfying* (2.6) *for which $\vartheta \in \Phi$ and $\alpha, \beta : [0, \infty) \to [0, \frac{1}{2})$ are monotone increasing. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.*
This follows from Corollary 3.4.

Corollary 3.6. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and $T : \mathcal{K} \to \mathcal{K}$ be a Picard map satisfying* (2.6) *for which $\vartheta \in \Phi$ and $\alpha, \beta : [0, \infty) \to [0, \frac{1}{2})$ are continuous test functions. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.*
Immediate from Corollary 3.4.

Corollary 3.7. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and $T : \mathcal{K} \to \mathcal{K}$ be a Picard map satisfying* (2.6) *for which $\vartheta \in \Phi$ and $\alpha, \beta : [0, \infty) \to [0, \frac{1}{2})$ are constant functions such that $\alpha + \beta < 1$. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.* This also follows from Theorem 3.1. See also [18] for related result.
The estimate of an operator satisfying (2.6) is presented as follow:

Theorem 3.8. *Let $\mathcal{K}$ be a non-empty closed subset of X and let $T : \mathcal{K} \to \mathcal{K}$ be a self-map satisfying* (2.6) *for which $\vartheta \in \Phi$ and $\alpha, \beta \in \mathcal{F}^*$. Let $F(T)$ be a non-empty set of all fixed points in $\mathcal{K}$. Then, the sequence $\{x_n\}$ defined by the Picard iterative process converges to the fixed point $x^* \in F(T)$ with the following estimate:*

$$d(x_n, x^*) \leq \frac{1}{p_0^n} d(x_0, x^*), \ p_0 > 2 \tag{3.5}$$

Proof. Suppose T satisfies condition (2.6), by Theorem 3.1, T has a fixed point $x^* \in F(\mathrm{T})$. Now, select $x_0 \in \mathcal{K}$ and let x_n be a Picard sequence, then by the property on ϑ, we have

$$\begin{aligned}
d(x_n, x^*) &= d(\mathrm{T}x_{n-1}, \mathrm{T}x^*) \\
&\leq \alpha(d(x_{n-1}, x^*))\vartheta(d(x_{n-1}, x^*)) + \beta(d(x_{n-1}, \mathrm{T}x_{n-1}))\vartheta(d(x_{n-1}, \mathrm{T}x_{n-1})) \\
&= \alpha(d(x_{n-1}, x^*))\vartheta(d(x_{n-1}, x^*)) + \beta(d(x_{n-1}, x_n))\vartheta(d(x_{n-1}, x_n)) \\
&\leq \alpha(d(x_{n-1}, x^*))\vartheta(d(x_{n-1}, x^*)) + \beta(d(x_{n-1}, x_n))\vartheta(d(x_{n-1}, x^*)) \\
&\quad + \beta(d(x_{n-1}, x_n))d(x_n, x^*)
\end{aligned}$$

This further implies

$$d(x_n, x^*) \leq \frac{\alpha(d(x_{n-1}, x^*)) + \beta(d(x_{n-1}, x_n))}{1 - \beta(d(x_{n-1}, x_n))}\vartheta(d(x_{n-1}, x^*))$$

Since $\alpha, \beta \in \mathcal{F}^*$, then

$$\frac{\alpha(d(x_{n-1}, x^*)) + \beta(d(x_{n-1}, x_n))}{1 - \beta(d(x_{n-1}, x_n))} \to 2$$

implies that

$$d(x_n, x^*) \leq 2\vartheta(d(x_{n-1}, x^*))$$

By induction,

$$d(x_n, x^*) \leq 2^n \vartheta^n(d(x_0, x^*)) = \frac{1}{p_0^n} d(x_0, x^*) \equiv \xi_{n,1} d(x_0, x^*)$$

Observe that $d(x_n, x^*) \to 0$ as $n \to \infty$. □

Remark 3.9. If $\alpha \in \mathcal{F}^*$ and $\vartheta(t) = t$, then the estimate $d(x_n, x^*) \leq \frac{1}{2^n} d(x_0, x^*) \equiv \xi_{n,2}(d(x_0, x^*)$ is obtained. If $\alpha \in \mathcal{F}$, then $d(x_n, x^*) \leq d(x_0, x^*) \equiv \xi_{n,3} d(x_0, x^*)$. Both estimates $\xi_{n,2}$ and $\xi_{n,3}$ are costly compare to estimate $\xi_{n,1}$.

Suppose, by Theorem 3.1, that $\{x_n\} \subset \mathcal{K}$ converges to a fixed point x^* of T and denote $F(T) = \{x^* \in \mathcal{K} : x^* = \mathrm{T}x^*\}$ as the set of all fixed points of T. Let $\{y_n\}$ be an arbitrary sequence in $\mathcal{K}$ and set $\tau_n = d(y_n, \mathrm{T}y_n)$, for $n = 0, 1, 2, \ldots$. The stability of an operator satisfying (2.6) is stated in the next theorem. See [17, 4, 23] for few results on stability.

Theorem 3.10. *Let $(\mathcal{K}, d)$ be an arbitrary closed subset of X and T is an operator satisfying contractive condition* (2.6) *with the property that $\alpha, \beta \in \mathcal{F}^*$ and $F(T)$ is nonempty. Then, for $x_0 \in \mathcal{K}$, the sequence $\{x_n\}$ defined by Picard operator is stable.*

Proof. Let $\{y_n\} \subset \mathcal{K}$ be an arbitrary sequence and let $\tau_n = d(y_n, \mathrm{T}y_n)$. Let $x^* \in F(T)$ and assume that $\tau_n \to 0$ as $n \to \infty$. Then, by hypothesis

$$d(y_n, x^*) \leq d(y_n, \mathrm{T}y_n) + d(\mathrm{T}y_n, \mathrm{T}x^*)$$

But,

$$d(\mathrm{T}y_n, \mathrm{T}x^*) \leq \alpha(d(y_n, x^*))\vartheta(d(y_n, x^*)) + \beta(d(y_n, \mathrm{T}y_n))\vartheta(d(y_n, \mathrm{T}y_n))$$

Thus,

$$d(y_n, x^*) \leq \tau_n + \alpha(d(y_n, x^*))\vartheta(d(y_n, x^*)) + \beta(\tau_n)\vartheta(\tau_n) \tag{3.6}$$

Since $\tau_n \to 0$ and by the properties on α, β and ϑ, then (3.6) becomes

$$\frac{1}{2} d(y_n, x^*) \leq \tau_n + \frac{1}{2}\vartheta(\tau_n) \to 0, \text{ as } n \to \infty \tag{3.7}$$

Hence, y_n converges to $x^* \in F(T)$.
On the other hand, suppose $d(y_n, x^*) \to 0$ for large n, where $x^* \in F(T)$, then by hypothesis

$$\begin{aligned}
\tau_n &= d(y_n, \mathrm{T}y_n) \\
&\le d(y_n, x^*) + d(\mathrm{T}x^*, \mathrm{T}y_n) \\
&\le d(y_n, x^*) + \alpha(d(x^*, y_n))\vartheta(d(x^*, y_n)) + \beta(d(x^*, \mathrm{T}x^*))\vartheta(d(x^*, \mathrm{T}x^*)) \\
&= d(y_n, x^*) + \alpha(d(x^*, y_n))\vartheta(d(x^*, y_n))
\end{aligned}$$

This further implies

$$\tau_n \le \frac{1}{2} d(x^*, y_n) \to 0 \tag{3.8}$$

Both (3.7) and (3.8) give that $\tau_n \to 0 \iff y_n \to x^*$.
Therefore, the operator satisfying (2.6) is stable. □

4. Applications

The existence properties of some non-degenerate nonlinear operators given by the solutions of differential equations (DEs), namely, perturbed Volterra and hypergeometric operators are studied in this section with the imposition of the ϑ-quasi-Geragthy condition (2.6) in complete metric spaces.

4.1. Application I

Here, Theorem 3.1. is employed to study the existence theorem of solutions for the perturbed integral equations of Volterra-like in complete metric spaces. This is facilitated by the collection of results in [5, 10, 8, 9, 15].
Let $X = C(I, \mathbb{R})$ be the set of all real-valued continuous functions defined on $I = [0, L]$ and $d : X \times X \to \mathbb{R}^+$ be defined by:

$$d(\upsilon, \omega) = \sup_{t \in [0,L]} \{|\upsilon(t) - \omega(t)|\}, \ \upsilon, \omega \in C(I, \mathbb{R}). \tag{4.1}$$

Clearly, the pair $(C(I, \mathbb{R}), d)$ is a complete metric space.
Now, consider the Volterra equation

$$\upsilon(t) = f(t) + \int_I \rho(t, s)\upsilon(s) ds \tag{4.2}$$

where $f(t) \in X$, $\rho : I \times I \to \mathbb{R}$ is a positive function with $(t, s) \in I \times I$ and $\upsilon(t) \in X$.
Also, consider the problem related to the perturbed integral equation

$$\upsilon(t) = f(t) + \int_I \varrho(t, s, \upsilon(s)) ds + \int_I \rho(t, s)\upsilon(s) ds \equiv f(t; y) + \int_I \rho(t, s)\upsilon(s) ds \tag{4.3}$$

where

$$f(t;y) = f(t) + \int_I \varrho(t,s,\upsilon(s))ds \tag{4.4}$$

is a perturbed operator, $\varrho : I \times I \times \mathbb{R} \to CB(\mathbb{R})$ is a continuous function such that $\varrho(t,s,0) = 0$.
The purpose of equation (4.3) is to study some asymptotic properties of solutions of the perturbed equation. To ensure that the solutions of (4.2) and (4.3) are the same, the condition imposed on (4.2) is also on (4.3).
Let C_h be a class of function spaces which is stronger than $(C(I,\mathbb{R}),d)$ and has the property that

$$\upsilon(t) \in C_h \iff \int_I \rho(t,s)\upsilon(s)ds \in C_h \quad \text{whenever} \quad f(t) \in C_h.$$

This condition leads to the admissibility of the pair (C_h, C_h) with respect to the Volterra equation (4.2). In order that the perturbed operator (4.4) acts from C_h to C_h, it suffices to impose the following hypotheses:

I. The pair (C_h, C_h) is admissible with respect to (4.2);

II. For each $\upsilon(t) \in C_h$, there corresponds $\omega(t) \in \mathrm{T}\upsilon(t)$ such that $\mathrm{T}\upsilon(t) \in C_h$ for $t \in I$.

III. There exist positive function $\rho_0 : I \times I \to \mathbb{R}$ and $\rho : I \times I \to \mathbb{R}$ such that

$$|\varrho(t,s,\upsilon(s)) - \varrho(t,s,\omega(s))| \leq \rho_0(t,s)(1 - e^{-|\upsilon(s)-\omega(s)|/2})$$

and

$$|\upsilon(s) - \nu(s)| \leq \rho(t,s)\ln(1 + |\upsilon(s) - \nu(s)|/2), \ \forall\, \nu \in \mathrm{T}\upsilon,$$

respectively.

IV. For all $s,t \in I$ and $L > 0$, there give $h_1, h_2 \in C_h$ such that

$$\int_I \rho_0(t,s)h_1(s)ds \leq \frac{h_1(t)}{L} \quad \text{and} \quad \int_I \rho(t,s)h_2(s)ds \leq \frac{h_2(t)}{L}.$$

V. For $\upsilon(t) \in C_h$, there exists $R > 0$ such that $|\upsilon(t)| \leq Rh(t)$, for $t \in I$ and $R = L^{-1}$.

VI. For $f(t;\upsilon(t)) \in C_h$, there exists $R^* > 0$ such that

$$|f(t;\upsilon(t))| \leq R^* h(t) + |f(t)| \text{ if } f(t) \in C_h.$$

With respect to the above, result concerning the exponential decay of equation (4.3) is presented as follow:

Theorem 4.1. *Suppose that all hypotheses (I-VI) are fulfilled with $h_1(t) = 1-e^{-\kappa t}$ and $h_2(t) = \ln(1+\kappa t)$ for $\kappa > 0$. Then, there exists a unique solution of equation (4.3) belonging to C_h whenever R is small enough.*

Proof. Let $\mathrm{T} : u(t) \to \upsilon(t)$ from C_h to C_h, where $\upsilon(t)$ is the solution of (4.3) and $u(t)$ is such that

$$u(t) \in f(t;\upsilon(t)) + \int_I \rho(t,s)\upsilon(s)ds$$

Let $\mathrm{T} : w(t) \to \omega(t)$ be such that

$$w(t) \in f(t;\omega(t)) + \int_I \rho(t,s)\nu(s)ds$$

where $f(t;\omega(t)) \in f(t) + \int_I \varrho(t,s,\omega(s))ds$ is the perturbed operator associated with $\nu(t) \in \mathrm{T}\upsilon(t)$. Going by the conditions (I-IV), it follows that

$$\begin{aligned}
|u(t)-w(t)| &= \sup_{t\in I}\left| f(t;\upsilon(t)) - f(t;\omega(t)) + \int_I \rho(t,s)\upsilon(s)ds - \int_I \rho(t,s)\nu(s)ds \right| \\
&\le \sup_{t\in I}\left\{ |f(t;\upsilon(t)) - f(t;\omega(t))| + \left| \int_I \rho(t,s)\upsilon(s)ds - \int_I \rho(t,s)\nu(s)ds \right| \right\} \\
&= \sup_{t\in I}\left\{ \left| \int_I \left(\varrho(t,s,\upsilon(s)) - \varrho(t,s,\omega(s))\right) ds \right| + \left| \int_I \rho(t,s)(\upsilon(s)-\nu(s))ds \right| \right\} \\
&\le \sup_{t\in I}\left\{ \int_I |\varrho(t,s,\upsilon(s)) - \varrho(t,s,\omega(s))|\, ds + \int_I \rho(t,s)\,|\upsilon(s)-\nu(s)|\, ds \right\} \\
&\le \sup_{t\in I}\left\{ \int_I \rho_0(t,s)(1-e^{-|\upsilon(s)-\omega(s)|/2})ds \right. \\
&\qquad\qquad \left. + \int_I \rho(t,s)\ln(1+|\upsilon(s)-\nu(s)|/2)ds \right\} \\
&\le \frac{1-e^{-|\upsilon(t)-\omega(t)|/2}}{L|\upsilon(t)-\omega(t)|}\sup_{t\in I}\{|\upsilon(t)-\omega(t)|\} \\
&\qquad\qquad + \frac{\ln(1+|\upsilon(t)-\nu(t)|/2)}{L|\upsilon(t)-\nu(t)|}\sup_{t\in I}\{|\upsilon(t)-\mathrm{T}\upsilon(t)|\}
\end{aligned}$$

By replacing $u = \mathrm{T}\upsilon$ and $w = \mathrm{T}\omega$ and letting $\vartheta(t_i) = Rt_i$ for each $t_i \in \mathbb{R}^+$, $i \in \{1,2\}$ and $R = \frac{1}{L}$. The last inequality is resulted to

$$d(\mathrm{T}\upsilon, \mathrm{T}\omega) \le \alpha(t_{n,1})\vartheta(t_1) + \beta(t_{n,2})\vartheta(t_2)$$

Clearly, $\alpha, \beta \in \mathcal{F}^*$ and $\vartheta \in \Phi$. Thus, the map T satisfies condition (2.6) and by the application of Theorem 3.1., the perturbed equation (4.3) has a unique solution in C_h. □

Remark 4.2. Result concerning boundedness could be proved for the perturbed equation (4.3) in C_h given that $h(t)$ is a constant function. Other forms can be sought in the literature herein.

4.2. Application II

Here, consider the generalized hypergeometric function denoted and defined by

$$ {}_{m}H_{n}(\gamma_1,\gamma_2,\ldots,\gamma_m;\delta_1,\delta_2,\ldots,\delta_n;y)=\sum_{r=0}^{\infty}\frac{(\gamma_1)_r(\gamma_2)_r\cdots(\gamma_m)_r y^r}{(\delta_1)_r(\delta_2)_r\cdots(\delta_n)_r r!} \tag{4.5} $$

where $(\gamma_i)_r$ and $(\delta_j)_r$, $i=1(1)m$; $j=1(1)n$ are Pochhammer symbols with $(\gamma_i)_0=(\delta_j)_0=1$ for each i,j. If $m,n=1$, then (4.5) gives the confluent hypergeometric function and the conventional hypergeometric function if $m=2$ and $n=1$. The existence of a case of hypergeometric operators is presented as follow:

Practical Example 4.3. *Let $T:\mathcal{K}\to\mathcal{K}$ be defined by an hypergeometric operator $\mathrm{T}y=y_2H_1(1,1;2;y)$, for all $y\in\mathcal{K}$. Let $\mathcal{K}=[0,1]$ be furnished with metric $d(y,z)=|y-z|$ and define $\vartheta(s)=\frac{2}{p_0}s$, for all $s\in\mathbb{R}^+$ and $p_0>2$.*
Also, let $s_n,t_n:=\inf\left\{d(y,\varsigma):d(y,\varsigma)\geq\frac{1}{n},\ \text{for } y,\varsigma\in\mathcal{K};n\in\mathbb{N}\right\}$.
Then, the hypergeometric operator T satisfies (2.6) *with $\alpha,\beta\in\mathcal{F}^*$ and has a unique fixed point in $\mathcal{K}$.*

Proof. Let $y,z\in\mathcal{K}$ with $y>z$. From (4.5) with the Pochhammer condition, there results

$$\begin{aligned}
\mathrm{T}y-\mathrm{T}z&=y_2H_1(1,1;2;y)-z_2H_1(1,1;2;z)\\
&=y\sum_{r=0}^{\infty}\frac{(1)_k(1)_r y^r}{(2)_r r!}-z\sum_{r=0}^{\infty}\frac{(1)_r(1)_r z^r}{(2)_r r!}\\
&=\sum_{r=0}^{\infty}\frac{(1)_r(1)_r\left(y^{r+1}-z^{r+1}\right)}{(2)_r r!}\\
&=y-z-\frac{(y^2-z^2)}{2}+\frac{(y^3-z^3)}{3}-\frac{(y^4-z^4)}{4}+\frac{(y^5-z^5)}{5}+\cdots
\end{aligned}$$

For $y>z$, then $y^p-z^p\geq(y-z)^p\geq(y-z)^{p+1}$ and

$$\frac{(y-z)^{p+1}}{p+1}-\frac{(y-z)^p}{p}\geq\frac{(y^{p+1}-z^{p+1})}{p+1}-\frac{(y^p-z^p)}{p}.$$

hold for $p>1$. Then,

$$\begin{aligned}
\mathrm{T}y-\mathrm{T}z&\leq y-z-\frac{(y-z)^2}{2}+\frac{(y-z)^3}{3}-\frac{(y-z)^4}{4}+\frac{(y-z)^5}{5}+\cdots\\
&=y-z-(y-z)^2+\frac{(y-z)^2}{2}+\frac{(y-z)^3}{3}+\cdots\\
&\qquad-\left[\frac{(y-z)^4}{2}+\frac{(y-z)^6}{3}+\frac{(y-z)^8}{4}+\cdots\right]\\
&=\sum_{r=1}^{\infty}\frac{(y-z)^r}{r}-\sum_{r=1}^{\infty}\frac{(y-z)^{2r}}{r}
\end{aligned}$$

Since $y-z>0$, then

$$\mathrm{T}y-\mathrm{T}z \leq y-z-(y-z)^2$$

On other hand, if $y<z$, then

$$\begin{aligned}
\mathrm{T}y-\mathrm{T}z &\leq \sum_{r=2}^{\infty}\frac{(y-z)^r}{r}-\sum_{r=2}^{\infty}\frac{(y-z)^{2r}}{r}\\
&\leq \sum_{r=2}^{\infty}\frac{(y-z)^r}{r}-\sum_{r=2}^{\infty}(y-z)^r\sum_{r=2}^{\infty}\frac{(y-z)^r}{r}\\
&=\left(1-\sum_{r=2}^{\infty}(y-z)^r\right)\sum_{r=2}^{\infty}\frac{(y-z)^r}{r}
\end{aligned}$$

Observe that the similitude of the latter is in manifolds. Let $p_0>2$ be a positive integer and $|y-z|\leq y$, then there results

$$\begin{aligned}
d(\mathrm{T}y,\mathrm{T}z) &\leq \frac{(1-|y-z|)}{p_0}|y-z|+\frac{1}{p_0}\left(1-\sum_{r=2}^{\infty}\frac{y^r}{r}\right)\left|\sum_{r=1}^{\infty}\frac{y^r}{r}-y\right|\\
&\equiv \left(\frac{1-d(y,z)}{2}\right)\vartheta\left(d(y,z)\right)+\left(\frac{1-d(\mathrm{T}y,y)}{2}\right)\vartheta\left(d(\mathrm{T}y,y)\right)
\end{aligned}$$

where $\mathrm{T}y=y_2H_1(1,1;2;y)$ and $\vartheta(s)=\frac{2}{p_0}s$. By letting $\alpha(s)=\frac{1-s}{2}$ and $\beta(t)=\frac{1-t}{2}$, then $\alpha,\beta\in\mathcal{F}^*$ since $\alpha(s_n),\beta(t_n)\to\frac{1}{2}$ as $s_n,t_n\to 0$. By the hypothesis of Theorem 3.1., the sequence $y_{n+1}=y_{n2}H_1(1,1;2;y_n)$ converges to the fixed point 0 for any initial seed $y_0\in\mathcal{K}$. □

The estimates $\xi_{n,1}$, $\xi_{n,2}$ and $\xi_{n,3}$ (see Theorem 3.8. and Remark 3.9.) when $x_0=\frac{1}{6}$, $p_0\in\mathbb{N}$ and few generations are presented in Table 1. It is seen in Table 1 that the ϑ-quasi-Geraghty map has better convergent rate.

5. Concluding Remarks

This study discussed the existence properties, stability and convergent rate of the operator satisfying one of the ${}^3C_2$ cases of ϑ-quasi-Geraghty contractive maps. The contractive condition is weaker than previous conditions and efficacies are quantified in Theorem 3.8, 3.10 and shown on Table 1. But without prejudice on results concerning other forms in inequality (2.2), that is, the cases $\{t_1;t_{p_0}\}$ and $\{t_2;t_{p_0}\}$ given by

$$d(\mathrm{T}x,\mathrm{T}y)\leq\alpha_1(t_{n,1})\vartheta(t_1)+\alpha_2(t_{2,p_0})\vartheta(t_{p_0})$$

and

$$d(\mathrm{T}x,\mathrm{T}y)\leq\alpha_2(t_{n,2})\vartheta(t_2)+\alpha_{p_0}(t_{n,p_0})\vartheta(t_{p_0})$$

respectively, will be investigated in future studies.

Table 1: Estimates for Practical Example 4.3.

	Estimates					
	ϑ-quasi-Geraghty map		ϑ-Geraghty map		Geraghty map	
n	$\alpha \in \mathfrak{F}^*$	$\xi_{n,1}$	$\alpha \in \mathfrak{F}$	$\xi_{n,2}$	$\alpha \in \mathfrak{F}$	$\xi_{n,3}$
1	0.4166667	0.2777778	0.8333333	0.4166667	0.8333333	0.8333333
5	0.4500000	0.0024300	0.900000	0.0184528	0.900000	0.5904900
10	0.4666667	8.495×10^{-6}	0.9333333	0.00048986	0.9333333	0.5016118
20	0.4800000	1.268×10^{-10}	0.960000	4.215×10^{-7}	0.9600000	0.4420024
50	0.4909091	5.565×10^{-25}	0.9818182	3.549×10^{-16}	0.9818182	0.3995338
100	0.4952381	7.452×10^{-49}	0.9904762	3.030×10^{-31}	0.9904762	0.3840644
1000	0.4995025	0	0.999005	3.449×10^{-302}	0.9990050	0.3695311

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