KYUNGPOOK Math. J. 63(2023), 105-122
https://doi.org/10.5666/KMJ.2022.63.1.105
pISSN 1225-6951 eISSN 0454-8124

Spectra of Higher Spin Operators on the Sphere

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Abstract. We present explicit formulas for the spectra of higher spin operators on the subbundle of the bundle of spinor-valued trace free symmetric tensors that are annihilated by Clifford multiplication over the standard sphere in odd dimension. In the even dimensional case, we give the spectra of the square of such operators. The Dirac and Rarita-Schwinger operators are zero-form and one-form cases, respectively. We also give eigenvalue formulas for the conformally invariant differential operators of all odd orders on the subbundle of the bundle of spinor-valued forms that are annihilated by Clifford multiplication in both even and odd dimensions on the sphere.

1. Introduction

The higher spin operators are generalized gradients like the Dirac and Rarita-Schwinger operators ([9], [13]). They are defined on the subbundle of the bundle of spinor-valued trace free symmetric tensors that are annihilated by Clifford multiplication (5.5). On the standard sphere S^n with n odd, they act as a constant on each $\mathrm{Spin}(n+1)$ irreducible summand of the section space. We apply the spectrum generating technique of [7] to get the eigenvalue quotients between $\mathrm{Spin}(n+1)$ summands and so to get the spectral function. From here, Theorem 5.3 and an easy computation of the eigenvalue on a single $\mathrm{Spin}(n+1)$ summand leads us to complete eigenvalue formulas for the operators. In the even dimensional case, these operators map positive spinors to negative spinors and vice versa. So we consider the square of the operators and get eigenvalue formulas.

Similarly, we consider the spin operators on the subbundle of spinor-forms that are annihilated by Clifford multiplication. In this setting, only two different $\mathrm{Spin}(n+1)$ isotypic types (4.1) appear and we give eigenvalue formulas for all odd order conformally invariant differential operators in both even and odd dimensions.

2. Conformally Covariant Operators

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. If f is a (possibly

Received January 7, 2022; revised September 25, 2022; accepted September 29, 2022.
2020 Mathematics Subject Classification: 53C27, 53C18.
Key words and phrases: conformally invariant operator, higher spin operator.

local) diffeomorphism on M, we denote by $f\cdot$ the natural action of f on tensor fields. It acts on vector fields as $f\cdot X = (df)X$ and on covariant tensors as $f\cdot\phi = (f^{-1})^*\phi$.

A vector field T is said to be *conformal* with *conformal factor* $\omega \in C^\infty(M)$ if

$$\mathcal{L}_T\, g = 2\omega g\,,$$

where $\mathcal{L}$ is the Lie derivative and g is the metric tensor. The conformal vector fields form a Lie algebra $\mathfrak{c}(M,g)$. A conformal transformation on (M,g) is a (possibly local) diffeomorphism h for which $h\cdot g = \Omega^2 g$ for some positive function $\Omega \in C^\infty(M)$. The global conformal transformations form a group $\mathscr{C}(M,g)$. Let $\mathscr{T}$ be a space of C^∞ tensor fields of some fixed type over M. For example, we can take 2-forms or trace-free symmetric covariant three-tensors. We have representations ([2]) defined by

$$\begin{aligned} \mathfrak{c}(M,g) &\xrightarrow{U_a} \text{End}\ \mathscr{T}, \quad U_a(T) = \mathcal{L}_T + a\omega \ \text{ and} \\ \mathscr{C}(M,g) &\xrightarrow{u_a} \text{Aut}\ \mathscr{T}, \quad u_a(h) = \Omega^a h\cdot \end{aligned} \tag{2.1}$$

for $a \in \mathbb{C}$.

Note that if a conformal vector field T integrates to a one-parameter group of global conformal transformation $\{h_\varepsilon\}$, then

$$\{U_a(T)\phi\}(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\{u_a(h_{-\varepsilon})\phi\}(x)\,.$$

In this sense, U_a is the infinitesimal representation corresponding to u_a.
A differential operator $D : C^\infty(M) \to C^\infty(M)$ is said to be *infinitesimally conformally covariant of bidegree* (a,b) if

$$DU_a(T)\phi = U_b(T)D\phi$$

for all $T \in \mathfrak{c}(M,g)$ and D is said to be *conformally covariant of bidegree* (a,b) if

$$Du_a(h)\phi = u_b(h)D\phi$$

for all $h \in \mathscr{C}(M,g)$.

To relate conformal covariance to conformal invariance, we recall that the conformal weight of a bundle V with the bundle metric g_V induced from g is r if and only if

$$\widehat{g} = \Omega^2 g \Longrightarrow \widehat{g}_V = \Omega^{-2r} g_V\,.$$

The tangent bundle, for instance, has conformal weight -1. Let us denote a bundle V with conformal weight r by V^r. Then we can impose new a conformal weight s on V^r by taking the tensor product of it with the bundle $I^{(s-r)/n}$ of scalar $((s-r)/n)$-densities ([3]). Now if we look at an operator of bidegree (a,b) as an operator from the bundle with conformal weight $-a$ to the bundle with conformal weight $-b$, the operator becomes conformally invariant.

As an example, let us consider the conformal Laplacian on M:

$$Y = \triangle + \frac{n-2}{4(n-1)}\mathrm{Scal},$$

where $\triangle = -g^{ab}\nabla_a\nabla_b$ and Scal is the scalar curvature. Note that $Y : C^\infty(M) \to C^\infty(M)$ is conformally covariant of bidegree $((n-2)/2, (n+2)/2)$. That is,

$$\widehat{Y} = \Omega^{-\frac{n+2}{2}} Y \mu(\Omega^{\frac{n-2}{2}}),$$

where $\widehat{Y}$ is Y evaluated in $\widehat{g}$ and $\mu(\Omega^{\frac{n-2}{2}})$ is multiplication by $\Omega^{\frac{n-2}{2}}$. If we let $V = C^\infty(M)$ and view Y as an operator

$$Y : V^{-\frac{n-2}{2}} \to V^{-\frac{n+2}{2}},$$

we have, for $\phi \in V^{-\frac{n-2}{2}}$,

$$\widehat{Y}\ \widehat{\phi} = \widehat{Y\phi},$$

where $\widehat{Y}$, $\widehat{\phi}$, and $\widehat{Y\phi}$ are Y, ϕ, and $Y\phi$ computed in $\widehat{g}$, respectively.

3. Dominant Weights

Let λ be a dominant weight of an irreducible Spin(n) representation. That is, let

$$\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l \cup (1/2 + \mathbb{Z})^l, \quad l = [n/2]$$

satisfy the inequality constraint (dominant condition)

$$\begin{array}{ll} \lambda_1 \geq \ldots \geq \lambda_l \geq 0, & n \text{ odd}, \\ \lambda_1 \geq \ldots \geq \lambda_{l-1} \geq |\lambda_l|, & n \text{ even}. \end{array}$$

Here, λ is identified with the highest weight of the irreducible representation of Spin(n) ([10]). We shall denote by $V(\lambda)$ the representation with the highest weight λ. Those $\lambda \in \mathbb{Z}^l$ are exactly the representations that factor through SO(n). For example, $V(1, 0, \ldots, 0)$ and $V(1, 1, 1, 0, \ldots, 0)$ are the defining representation and the three-form representation of SO(n), respectively and $V(\frac{1}{2}, \ldots, \frac{1}{2})$ is the spinor representation in the odd dimensional case.

If M is an n-dimensional smooth manifold with Spin(n) structure and $\mathcal{F}$ is the bundle of spin frames, we denote by $\mathbb{V}(\lambda)$ the associated vector bundle $\mathcal{F} \times_\lambda V(\lambda)$.

4. Intertwining Relation

Let $G = \mathrm{Spin}_0(n+1, 1)$ be the identity component of the Spin$(n+1, 1)$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Then, in an Iwasawa decomposition $G = KAN$, the maximal compact subgroup K of G is isomorphic to Spin$(n+1)$. Let M be the centralizer of the Lie algebra $\mathfrak{a}$ of A in K. Then M is isomorphic to $\mathrm{Spin}(n)$ and $P = MAN$ is a maximal parabolic subgroup of G. Note that $G/P = K/M$ is diffeomorphic to the sphere S^n ([7]).

Let $V(\lambda)$ be a finite dimensional irreducible representation of M with maximum weight λ. Consider the G module $\mathcal{E}(G; \lambda, \nu)$ of C^∞ functions

$$F : G \to V(\lambda) \text{ with } F(gman) = a^{-\nu-\rho}\lambda(m)^{-1}F(g), \quad g \in G, m \in M, a \in A, n \in N,$$

where ρ is half the sum of the positive $(\mathfrak{g}, \mathfrak{a})$ roots. This space is in one-to-one correspondence with the space of smooth sections of $\mathbb{V}(\lambda)$, the K module $\mathcal{E}(K; \lambda|_{K\cap M})$ of C^∞ functions

$$f : K \to V(\lambda) \text{ with } f(km) = \lambda(m)^{-1}f(k), \quad k \in K, m \in M.$$

The K-finite subspace $\mathcal{E}_K(G; \lambda, \nu) \cong_K \mathcal{E}_K(K; \lambda|_{K\cap M})$ is defined as

$$\bigoplus_{\alpha \in \hat{K},\, \alpha \downarrow \lambda} \mathcal{V}(\alpha), \tag{4.1}$$

where $\hat{K}$ is the set of dominant $\mathrm{Spin}(n+1)$ weights and $\mathcal{V}(\alpha)$ is the α-isotypic component satisfying the classical branching rule of K and M ([1]):

$$\alpha \downarrow \lambda \text{ iff } \alpha_1 - \lambda_1 \in \mathbb{Z} \text{ and } \begin{cases} \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \cdots \geq \lambda_l \geq |\alpha_{l+1}|, & \text{n odd} \\ \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \cdots \geq \lambda_{l-1} \geq \alpha_l \geq |\lambda_l|, & \text{n even.} \end{cases}$$

The conformal action of G and its infinitesimal representation correspond to those in (2.1).

Let $A = A_{2r}$ be an intertwinor of order $2r$ of the $(\mathfrak{g}, K)$ representation. If $X \in \mathfrak{g}$ with its conformal factor ω, A is a K-map satisfying the intertwining relation

$$A\left(\tilde{\mathcal{L}}_X + \left(\frac{n}{2} - r\right)\omega\right) = \left(\tilde{\mathcal{L}}_X + \left(\frac{n}{2} + r\right)\omega\right)A, \tag{4.2}$$

where $\tilde{\mathcal{L}}_X$ is the *reduced Lie derivative* ([3]), $\tilde{\mathcal{L}}_X = \mathcal{L}_X + (l - m)\omega$ on tensors of $\begin{pmatrix} l \\ m \end{pmatrix}$-type ($l$ contravariant and m covariant).

5. Higher Spin Operators

Let $\mathbb{V}(\lambda)$ be an irreducible associated vector bundle (3). The covariant derivative takes sections of $\mathbb{V}(\lambda)$ to sections of ([1])

$$T^*M \otimes \mathbb{V}(\lambda) \cong_{\mathrm{Spin}(n)} \mathbb{V}(\mu_1) \oplus \cdots \oplus \mathbb{V}(\mu_N), \tag{5.1}$$

where

$$\mu_i = \lambda \pm e_a, \text{ for some } a \in \{1, \ldots, [n/2]\}$$
or
$$\mu_i = \lambda \text{ if } n \text{ is odd and } \lambda_{[n/2]} \neq 0.$$

Here e_a is $(0,\dots,0,1,0,\dots,0)$ with "1" in the a-th slot.

Let us consider the fundamental tensor-spinor ([6])

$$\gamma : T^*S^n \to \mathrm{End}(\Sigma) \tag{5.2}$$

satisfying the Clifford relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2g_{ij}\,\mathrm{id}_\Sigma, \tag{5.3}$$

where Σ is the spinor bundle:

$$\begin{aligned}&\Sigma = \mathbb{V}(1/2,\dots,1/2), \quad n \text{ odd},\\ &\Sigma = \Sigma_+ \oplus \Sigma_-, \quad \Sigma_\pm = \mathbb{V}(1/2,\dots,1/2,\pm 1/2), \quad n \text{ even}.\end{aligned}$$

The higher spin operators are compositions of projections and the above covariant derivatives (5.1):

$$\begin{aligned}&\mathbb{V}\left(\frac{1}{2}+k,\frac{1}{2},\dots,\frac{1}{2}\right) \to \mathbb{V}\left(\frac{1}{2}+k,\frac{1}{2},\dots,\frac{1}{2}\right), \quad n \text{ odd}\\ &\mathbb{V}\left(\frac{1}{2}+k,\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2}\right) \to \mathbb{V}\left(\frac{1}{2}+k,\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2}\right), \quad n \text{ even}\\ &\text{for } k = 1,2,\dots,\end{aligned}$$

Taking a normalization, we can express these operators as

$$(\mathcal{R}^{(k)}\varphi)_{a_1\cdots a_k} = \gamma^b\nabla_b\varphi_{a_1\cdots a_k} - \frac{2k}{n+2k-2}\gamma_{(a_1}\nabla^b\varphi_{a_2\cdots a_k)b}, \tag{5.4}$$

where indices enclosed in parentheses are symmetrized and φ is a spinor-valued trace-free symmetric k-tensor annihilated by Clifford multiplication. i.e.

$$\gamma^{a_1}\varphi_{a_1a_2\cdots a_k} = 0. \tag{5.5}$$

Case I: n odd

$\mathcal{R}^{(k)}$ is an intertwinor of order 1 with $\begin{pmatrix} 0 \\ k \end{pmatrix}$ tensor type. So (4.2) becomes

$$\mathcal{R}^{(k)}\left(\mathcal{L}_X + \left(\frac{n}{2}-\frac{1}{2}-k\right)\omega\right) = \left(\mathcal{L}_X + \left(\frac{n}{2}+\frac{1}{2}-k\right)\omega\right)\mathcal{R}^{(k)}.$$

Now we specialize to the case of S^n with its standard metric. Let $Y = \sin\rho\,\partial_\rho$ be a conformal vector field and $\omega = \cos\rho$ its corresponding conformal factor ([12]).

The following lemma compares the Lie derivative and covariant derivative on the spinor-valued k-tensor bundle on S^n.

Lemma 5.1. *For any spinor-k-tensor* Φ*,*

$$((\mathcal{L}_Y - \nabla_Y)\Phi)_{a_1\cdots a_k} = k\,\omega\Phi_{a_1\cdots a_k}\,.$$

Proof. Note that for a 1-form η and a vector field X,

$$\langle(\mathcal{L}_Y - \nabla_Y)\eta, X\rangle = -\langle\eta, (\mathcal{L}_Y - \nabla_Y)X\rangle,$$

since $\mathcal{L}_Y - \nabla_Y$ kills scalar functions. But by the symmetry of the Riemannian connection,

$$[Y,X] - \nabla_Y X = -\nabla_X Y.$$

We conclude that

$$(\mathcal{L}_Y - \nabla_Y)\eta = \langle\eta, \nabla Y\rangle,$$

where in the last expression, $\langle\cdot,\cdot\rangle$ is the pairing of a 1-form with the contravariant part of a $\begin{pmatrix}1\\1\end{pmatrix}$-tensor:

$$((\mathcal{L}_Y - \nabla_Y)\eta)_\lambda = \eta_\mu\nabla_\lambda Y^\mu.$$

Since Y is a conformal vector field,

$$(\nabla Y_\flat)_{\lambda\mu} = (\nabla Y_\flat)_{(\lambda\mu)} + (\nabla Y_\flat)_{[\lambda\mu]} = (\omega g + \frac{1}{2}dY_\flat)_{\lambda\mu} = \omega g_{\lambda\mu},$$

where $(Y_\flat)_\alpha = g_{\alpha\beta}Y^\beta$, the contraction of Y with the metric tensor g. Thus

$$((\mathcal{L}_Y - \nabla_Y)\eta)_\lambda = \omega\eta_\lambda\,.$$

Since $\mathcal{L}_Y - \nabla_Y$ is a derivation, for any k-tensor $\varphi_{a_1\cdots a_k}$,

$$((\mathcal{L}_Y - \nabla_Y)\varphi)_{a_1\cdots a_k} = k\omega\varphi_{a_1\cdots a_k}.$$

On the spinor bundle Σ, on the other hand, ([11, eq(16)])

$$\mathcal{L}_Y - \nabla_Y = -\tfrac{1}{4}\nabla_{[a}Y_{b]}\gamma^a\gamma^b = -\tfrac{1}{8}(dY_\flat)_{ab}\gamma^a\gamma^b = 0,$$

where γ is the fundamental tensor-spinor. Thus the lemma follows. □

The spectrum generating relation is given in the following lemma.

Lemma 5.2. *On spinor-tensors of any type,*

$$[\nabla^*\nabla, \omega] = 2\left(\nabla_Y + \frac{n}{2}\omega\right)\,,$$

where [,] is the operator commutator.

Proof. If φ is any smooth section of a tensor-spinor bundle, then

$$[\nabla^*\nabla, \omega]\varphi = (\triangle\omega)\varphi - 2(d\omega)^\alpha\nabla_\alpha\varphi = (n\omega + 2Y^\alpha\nabla_\alpha)\varphi = (n\omega + 2\nabla_Y)\,\varphi.$$

□

Thus, by Lemma 5.1 and Lemma 5.2, the intertwining relation (4.2) on spinor-k-tensors becomes

$$\mathcal{R}^{(k)}\left(\frac{1}{2}[\nabla^*\nabla, \omega] - \frac{1}{2}\omega\right) = \left(\frac{1}{2}[\nabla^*\nabla, \omega] + \frac{1}{2}\omega\right)\mathcal{R}^{(k)}. \tag{5.6}$$

Now we look at the Spin$(n+1)$ types in (4.1) occurring over $\mathbb{V}(1/2+k, 1/2, \ldots, 1/2)$. Define

$$\mathcal{V}_\varepsilon(j,q) := \mathbb{V}\Big(\underbrace{\frac{1}{2}+k+j, \frac{1}{2}+q, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{\varepsilon}{2}}_{[\frac{n+1}{2}]}\Big), \quad \varepsilon = \pm 1,\ q = 0, 1, \ldots, k,\ j = 0, 1, \ldots.$$

For $\varphi \in \mathcal{V}_\varepsilon(j,q)$, we have ([7])

$$\omega\varphi \in \mathcal{V}_\varepsilon(j+1,q) \oplus \mathcal{V}_\varepsilon(j-1,q) \oplus \mathcal{V}_\varepsilon(j,q+1) \oplus \mathcal{V}_\varepsilon(j,q-1) \oplus \mathcal{V}_{-\varepsilon}(j,q).$$

Let $\alpha = \mathcal{V}_\varepsilon(j,q)$ and β be one of the summands in the above direct sum. We consider the compressed intertwining relation (5.6) between α and β:

$$\mu_\beta \cdot \frac{1}{2}\left(\nabla^*\nabla|_\beta - \nabla^*\nabla|_\alpha - 1\right) \cdot |_\beta\omega|_\alpha\varphi = \frac{1}{2}\left(\nabla^*\nabla|_\beta - \nabla^*\nabla|_\alpha + 1\right) \cdot \mu_\alpha \cdot |_\beta\omega|_\alpha\varphi,$$

where μ_α and μ_β are eigenvalues of $\mathcal{R}^{(k)}$ on the isotypic summands α and β, $|_\beta\omega|_\alpha$ is the projection of ω onto the summand β, and $\nabla^*\nabla|_\alpha$ and $\nabla^*\nabla|_\beta$ are evaluations of $\nabla^*\nabla$ on the summands α and β respectively.

Canceling $|_\beta\omega|_\alpha$ from both sides and computing $\nabla^*\nabla$ ([1]), we get μ_β/μ_α transition quantities.

With respect to the diagram of the Spin$(n+1)$ summands based at $\mathcal{V}_1(j,q)$

$$\begin{array}{ccc}
\mathcal{V}_1(j+1,q) & & \mathcal{V}_1(j,q+1) \\
\uparrow & \nearrow & \\
\boxed{\mathcal{V}_1(j,q)} & \rightarrow & V_{-1}(j,q) \\
\downarrow & \searrow & \\
\mathcal{V}_1(j-1,q) & & \mathcal{V}_1(j,q-1)
\end{array}$$

we get the corresponding diagram of the transition quantities μ_β/μ_α

$$\begin{array}{ccc}
(J+1)/J & & (n+2q)/(n+2q-2) \\
\uparrow & \nearrow & \\
\bullet & \rightarrow & -1 \\
\downarrow & \searrow & \\
(J-1)/J & & (n+2q-4)/(n+2q-2)
\end{array} \tag{5.7}$$

where $J = n/2+k+j$. Thus the eigenvalue of $\mathcal{R}^{(k)}$ on $\mathcal{V}_1(1/2+k, 1/2+k)$ completely determines the spectra of $\mathcal{R}^{(k)}$.

Let Λ^k and Sym^k be the spaces of k-forms and symmetric k-tensors, respectively.

Theorem 5.3. *For* $\Phi \in (\Sigma \otimes \Lambda^k) \cup (\Sigma \otimes \mathrm{Sym}^k)$ *with* $\gamma^{a_1}\Phi_{a_1 a_2 \cdots a_k} = 0$,

$$\nabla\!\!\!\!/\,^2 \Phi_{a_1 \cdots a_k} = \left(\nabla^*\nabla + \frac{n(n-1)}{4} + k\right)\Phi_{a_1 \cdots a_k},$$

where $\nabla\!\!\!\!/ = \gamma^a \nabla_a$.

Proof. We write $\nabla\!\!\!\!/\,^2\Phi = \gamma^a\gamma^b(\nabla_b\nabla_a + \mathcal{R}_{ab})\Phi = (-\nabla\!\!\!\!/\,^2 + 2\nabla^*\nabla + \gamma^a\gamma^b\mathcal{R}_{ab})\Phi$ using the Clifford relation (5.3), where $\mathcal{R}_{ab} = [\nabla_a, \nabla_b]$ is the spin curvature ([6]). So $\nabla\!\!\!\!/\,^2\Phi = (\nabla^*\nabla + 1/2\gamma^a\gamma^b\mathcal{R}_{ab})\Phi$. And

$$\gamma^a\gamma^b\mathcal{R}_{ab}\Phi_{a_1\cdots a_k} = \gamma^a\gamma^b(W_{ab}\Phi_{a_1\cdots a_k} - R^\nu{}_{a_1 ab}\Phi_{\nu a_2\cdots a_k} - \cdots - R^\nu{}_{a_k ab}\Phi_{a_1\cdots a_{k-1}\nu}),$$

where $W_{ab} = -1/4 R_{\kappa\nu ab}\gamma^\kappa\gamma^\nu$ is the action on spinors and $R_{\kappa\nu\lambda\mu}$ is the Riemann curvature tensor. On S^n, since $R_{\kappa\nu\lambda\mu} = g_{\kappa\lambda}g_{\nu\mu} - g_{\kappa\mu}g_{\nu\lambda}$,

$$\begin{aligned}
&\gamma^a\gamma^b W_{ab} = \frac{n(n-1)}{2}\\
&\gamma^a\gamma^b R^\nu{}_{a_i ab}\Phi_{a_1\cdots a_{i-1}\nu a_{i+1}\cdots a_k} = (\gamma^\nu\gamma_{a_i} - \gamma_{a_i}\gamma^\nu)\Phi_{a_1\cdots a_{i-1}\nu a_{i+1}\cdots a_k}\\
&\quad = (-2\gamma_{a_i}\gamma^\nu - 2\delta^\nu_{a_i})\Phi_{a_1\cdots a_{i-1}\nu a_{i+1}\cdots a_k}\\
&\quad = -2\Phi_{a_1\cdots a_k}
\end{aligned}$$

and the theorem follows. □

Remark 5.4. When $k = 0$, this is the classical Lichnerowicz formula on the spinor bundle over S^n. In general, $\gamma^a\gamma^b W_{ab} = \mathrm{Scal}/2$, where Scal is the scalar curvature.

Notice that $\mathcal{R}^{(k)}$ reduces to $\nabla\!\!\!\!/$ on $\mathcal{V}_1(1/2+k, 1/2+k)$ since it is of diverging type. That is, if $\varphi \in \mathcal{V}_1(1/2+k, 1/2+k)$, then $\varphi = \nabla^*\psi$ for some section ψ over $\mathbb{V}(1/2+k, 3/2, 1/2, \ldots, 1/2)$.

We can now describe the spectra of the higher spin operators.

Theorem 5.5. *The operator* $\mathcal{R}^{(k)}$ *acts as a constant*

$$\varepsilon \cdot \frac{n+2q-2}{n+2k-2} \cdot \left(\frac{n}{2} + k + j\right) \quad on\ \mathcal{V}_\varepsilon\left(\frac{1}{2} + k + j, \frac{1}{2} + q\right),$$
$$\varepsilon = \pm 1,\ q = 0, 1, \ldots, k,\ j = 0, 1, \ldots .$$

Proof. This is a direct consequence of Theorem 5.3 together with the diagram of transition quantities (5.7). □

Remark 5.6. When $k = 0, 1$, we get eigenvalues of the Dirac and the Rarita-Schwinger operators, respectively ([4]).

Case II: n even

In this case $\mathcal{R}^{(k)}$ changes chirality of spinors. That is,

$$\mathcal{R}^{(k)} : \mathbb{V}\left(\frac{1}{2}+k, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}\right) \to \mathbb{V}\left(\frac{1}{2}+k, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{\varepsilon}{2}\right), \quad \varepsilon = \pm 1.$$

Consider ([3]) the commutative diagram.

$$\begin{array}{ccc} \mathbb{V}(1/2+k, 1/2, \dots, 1/2, -\varepsilon/2) & \xrightarrow{I_2} & \mathbb{V}(1/2+k, 1/2, \dots, 1/2, -\varepsilon/2) \\ \uparrow G & & \downarrow G^* \\ \mathbb{V}(1/2+k, 1/2, \dots, 1/2, \varepsilon/2) & \xrightarrow{I_1} & \mathbb{V}(1/2+k, 1/2, \dots, 1/2, \varepsilon/2) \end{array}$$

Here G is the generalized gradient (projection of the covariant derivative action, (5.1), G^* is the adjoint of G, and I_1, I_2 are intertwinors :

$$\begin{aligned} I_1\left(\tilde{\mathcal{L}}_X + \left(\frac{n-2}{2}\right)\omega\right) &= \left(\tilde{\mathcal{L}}_X + \left(\frac{n+2}{2}\right)\omega\right) I_1 \text{ and} \\ I_2\left(\tilde{\mathcal{L}}_X + \left(\frac{n+2}{2}\right)\omega\right) &= \left(\tilde{\mathcal{L}}_X + \left(\frac{n-2}{2}\right)\omega\right) I_2. \end{aligned}$$

Since $(\mathcal{R}^{(k)})^2$ is a constant multiple of G^*G, we just need the ratios of eigenvalues of I_1 to I_2. But I_2 is a constant multiple of $1/I_1$. Thus $(\mathcal{R}^{(k)})^2 = c \cdot (I_1)^2$ for some constant c.

The following lemma determines the constant c.

Lemma 5.7. $(\mathcal{R}^{(k)})^2 = \not\nabla^2$ *on* $\mathcal{V}(1/2+k, 1/2+k, 1/2, \dots, 1/2)$.

Proof. Let $\varphi \in \mathcal{V}(1/2+k, 1/2+k, 1/2, \dots, 1/2)$. So $\varphi = \nabla^* \psi$ for some ψ over $\mathbb{V}(1/2+k, 3/2, 1/2, \dots, \varepsilon/2)$. Since $\mathcal{R}^{(k)}\varphi = \not\nabla\varphi$ on $\mathcal{V}(1/2+k, 1/2+k, 1/2, \dots, 1/2)$,

$$(\mathcal{R}^{(k)})^2 \varphi_{a_1 \cdots a_k} = \not\nabla^2 \varphi_{a_1 \cdots a_k} - \frac{2k}{n+2k-2} \gamma_{(a_1} \nabla^b \not\nabla \varphi_{a_2 \cdots a_k)b}.$$

But

$$\begin{aligned} \nabla^b \not\nabla \varphi_{a_2 \cdots a_k b} &= \gamma^c (\nabla_c \nabla_b + \mathcal{R}_{bc}) \varphi^b{}_{a_2 \cdots a_k} = \gamma^c \mathcal{R}_{bc} \varphi^b{}_{a_2 \cdots a_k} \\ &= \gamma^c (W_{bc} \varphi^b{}_{a_2 \cdots a_k} - R^{\nu b}{}_{bc} \varphi_{\nu a_2 \cdots a_k} - R^\nu{}_{a_2 bc} \varphi^b{}_{\nu a_3 \cdots a_k} - \cdots - R^\nu{}_{a_k bc} \varphi^b{}_{a_2 \cdots a_{k-1} \nu}) \\ &= \text{constant} \cdot \gamma^c \varphi_{c a_2 \cdots a_k} \\ &= 0, \end{aligned}$$

where $\mathcal{R}_{bc}$ is the spin curvature. Hence the claim follows. □

We define

$$\mathcal{V}(j,q) := \mathbb{V}\Big(\underbrace{\frac{1}{2}+k+j, \frac{1}{2}+q, \frac{1}{2}, \ldots, \frac{1}{2}}_{\left[\frac{n+1}{2}\right]}\Big), \quad q = 0, 1, \ldots, k,\ j = 0, 1, \ldots.$$

For $\varphi \in \mathcal{V}(j,q)$, we have ([7])

$$\omega\varphi \in \mathcal{V}(j+1,q) \oplus \mathcal{V}(j-1,q) \oplus \mathcal{V}(j,q+1) \oplus \mathcal{V}(j,q-1) \oplus \mathcal{V}(j,q).$$

With respect to the diagram

$$\begin{array}{ccc} \mathcal{V}(j+1,q) & & \mathcal{V}(j,q+1) \\ \uparrow & \nearrow & \\ \boxed{\mathcal{V}(j,q)} & & \\ \downarrow & \searrow & \\ \mathcal{V}(j-1,q) & & \mathcal{V}(j,q-1) \end{array}$$

we get

$$\begin{array}{ccc} (J+1)/J & & (n+2q)/(n+2q-2) \\ \uparrow & \nearrow & \\ \bullet & & \\ \downarrow & \searrow & \\ (J-1)/J & & (n+2q-4)/(n+2q-2) \end{array}$$

where $J = n/2 + k + j$.

Thus we have

Theorem 5.8. *The operator* $(\mathcal{R}^{(k)})^2$ *acts as a constant*

$$\left[\frac{n+2q-2}{n+2k-2} \cdot \left(\frac{n}{2}+k+j\right)\right]^2 \quad \textit{on } \ \mathcal{V}\left(\frac{1}{2}+k+j, \frac{1}{2}+q\right),\ q = 0, 1, \ldots, k,\ j = 0, 1, \ldots.$$

6. Spin Operators over Spinor-forms

In this section, we consider the conformally invariant spin operators on $\mathbb{V}(\underbrace{3/2, \ldots, 3/2}_{k}, 1/2, \ldots, \varepsilon/2)$ where $\varepsilon = 1$ for n odd, $\varepsilon = \pm 1$ for n even, and $0 \le k < n/2$. These operators satisfy the intertwining relation (4.2). By Lemma 5.1 and 5.2, we have, for an order $2r$ intertwinor $A = A_{2r}$,

$$A\left(\frac{1}{2}[\nabla^*\nabla, \omega] - r\omega\right) = \left(\frac{1}{2}[\nabla^*\nabla, \omega] + r\omega\right)A.$$

Case I: n odd and k = 0

Let $\mathcal{V}_\varepsilon(j) = \mathcal{V}(1/2+j, 1/2, \ldots, 1/2, \varepsilon/2)$ for $\varepsilon = \pm 1$. With respect to the following diagram

$$\begin{array}{ccc} \mathcal{V}_\varepsilon(j+1) & & \\ \uparrow & & \\ \boxed{\mathcal{V}_\varepsilon(j)} & \rightarrow & V_{-\varepsilon}(j) \\ \downarrow & & \\ \mathcal{V}_\varepsilon(j-1) & & \end{array}$$

we get

$$\begin{array}{ccc} (J+1/2+r)/(J+1/2-r) & & \\ \uparrow & & \\ \bullet & \rightarrow & -1 \\ \downarrow & & \\ (-J+1/2+r)/(-J+1/2-r) & & \end{array}$$

where $J = n/2 + j$. Choosing a normalization $\mu_{\mathcal{V}_\varepsilon(0)} = \varepsilon$, we have

Theorem 6.1. *The unique spectral function $Z_\varepsilon(r, j)$ on $\mathcal{V}_\varepsilon(j)$ is up to normalization*

$$Z_\varepsilon(r,j) = \varepsilon \cdot \frac{\Gamma(J+\frac{1}{2}+r)\Gamma(\frac{n}{2}+\frac{1}{2}-r)}{\Gamma(J+\frac{1}{2}-r)\Gamma(\frac{n}{2}+\frac{1}{2}+r)}, \quad \varepsilon = \pm 1, \quad j = 0, 1, 2, \ldots.$$

If $2r = 1$, $Z_\varepsilon(1/2, j) = \varepsilon \cdot J \cdot \frac{2}{n} = \frac{2}{n} \cdot \nabla\!\!\!/$ on $\mathcal{V}_\varepsilon(j)$ is a constant multiple of the Dirac operator. As a consequence of the spectral function in the theorem, we get

Corollary 6.2. ([8]) *The differential operator $D_{2l+1} : \Sigma \to \Sigma$ defined by*

$$D_{2l+1} := \nabla\!\!\!/ \cdot \prod_{p=1}^{l} (\nabla\!\!\!/^{\,2} - p^2)$$

is conformally invariant of order $2l+1$.

Proof. D_{2l+1} acts as $\varepsilon \cdot J \cdot \prod_{p=1}^{l}(J^2 - p^2)$ on $V_\varepsilon(j)$ so it is a constant multiple of $Z_\varepsilon(l+1/2, j)$. □

Case II: n odd and k ≥ 1

Let, for $\varepsilon = \pm 1$,

$$\mathcal{V}_\varepsilon(j,q) = \begin{cases} \mathcal{V}_\varepsilon(3/2+j, 3/2, \ldots, 3/2, \underbrace{1/2+q}_{(k+1)^{\mathrm{st}}}, 1/2, \ldots, 1/2, \varepsilon/2), & k < (n-1)/2 \\ \mathcal{V}_\varepsilon(3/2+j, 3/2, \ldots, 3/2, \varepsilon(1/2+q)), & k = (n-1)/2. \end{cases}$$

With respect to the diagram

$$\begin{array}{ccccc} & & \mathcal{V}_\varepsilon(j+1) & & \\ & & \uparrow & & \\ \mathcal{V}_\varepsilon(j,0) & \leftarrow & \boxed{\mathcal{V}_\varepsilon(j,1)} & \rightarrow & V_{-\varepsilon}(j,1) \\ & & \downarrow & & \\ & & \mathcal{V}_\varepsilon(j-1) & & \end{array}$$

we get

$$\begin{array}{ccccc} & & (J+\frac{1}{2}+r)/(J+\frac{1}{2}-r) & & \\ & & \uparrow & & \\ (n-2k+1-2r)/(n-2k+1+2r) & \leftarrow & \bullet & \rightarrow & -1 \\ & & \downarrow & & \\ & & (-J+\frac{1}{2}+r)/(-J+\frac{1}{2}-r) & & \end{array}$$

where $J = n/2 + 1 + j$. Choosing a normalization $\mu_{\mathcal{V}_\varepsilon(0,1)} = \varepsilon$, we have

Theorem 6.3. *The unique spectral function $Z_\varepsilon(r,j,q)$ on $\mathcal{V}_\varepsilon(j,q)$ is up to normalization*

$$Z_\varepsilon(r,j,q) = \varepsilon \cdot \frac{n-2k+1+2(2q-1)r}{n-2k+1+2r} \cdot \frac{\Gamma(J+\frac{1}{2}+r)\Gamma(\frac{n}{2}+\frac{3}{2}-r)}{\Gamma(J+\frac{1}{2}-r)\Gamma(\frac{n}{2}+\frac{3}{2}+r)},$$
$$\varepsilon = \pm 1, \quad q = 0,1, \quad j = 0,1,2,\ldots.$$

Remark 6.4. If $2r = 1$ and $k = 1$, $Z_\varepsilon(1/2,j,q) = \varepsilon \cdot \frac{n+2(q-1)}{n} \cdot J \cdot \frac{2}{n+2} = \frac{2}{n+2} \cdot \mathcal{R}^{(1)}$ on $\mathcal{V}_\varepsilon(j,q)$. Here $\mathcal{R}^{(1)}$ is the Rarita-Schwinger operator (5.4).

To get all odd order conformally invariant differential operators for $k \geq 1$, we consider the following convenient operators ([4]) on spinor-forms:

$$\begin{aligned} (\tilde{d}\varphi)_{a_0\cdots a_k} &:= \sum_{i=0}^{k} (-1)^i \nabla_{a_i} \varphi_{a_0\cdots a_{i-1}a_{i+1}\cdots a_k}, \\ (\tilde{\delta}\varphi)_{a_2\cdots a_k} &:= -\nabla^b \varphi_{ba_2\cdots a_k}, \\ (\varepsilon(\gamma)\varphi)_{a_0\cdots a_k} &:= \sum_{i=0}^{k} (-1)^i \gamma_{a_i} \varphi_{a_0\cdots a_{i-1}a_{i+1}\cdots a_k}, \\ (\iota(\gamma)\varphi)_{a_2\cdots a_k} &:= \gamma^b \varphi_{ba_2\cdots a_k}, \\ (\mathbb{D}\varphi)_{a_1\cdots a_k} &:= (\iota(\gamma)\tilde{d} + \tilde{d}\iota(\gamma))\varphi)_{a_1\cdots a_k} = -(\tilde{\delta}\varepsilon(\gamma) + \varepsilon(\gamma)\tilde{\delta})\varphi)_{a_1\cdots a_k} = \nabla\!\!\!\!/\, \varphi_{a_1\cdots a_k}. \end{aligned}$$

The operator

$$P_k := \frac{n-2k+4}{2}\iota(\gamma)\tilde{d} + \frac{n-2k}{2}(\tilde{d}\iota(\gamma) - \tilde{\delta}\varepsilon(\gamma)) - \frac{n-2k-4}{2}\varepsilon(\gamma)\tilde{\delta}$$

on $\Sigma \otimes \wedge^k$ restricted to

$$\mathbb{T}^k := \begin{cases} \mathbb{V}(\underbrace{\tfrac{3}{2}, \dots \tfrac{3}{2}}_{k}, \tfrac{1}{2}, \dots, \tfrac{1}{2}), & k \geq 1 \\ \mathbb{V}(\tfrac{1}{2}, \dots, \tfrac{1}{2}), & k = 0 \end{cases} = \{\varphi \in \Sigma \otimes \wedge^k \mid \gamma^{a_1}\varphi_{a_1 a_2 \cdots a_k} = 0\},$$

is conformally invariant on $\mathbb{T}^k$.

Remark 6.5. $(1/(n-2k+2)) \cdot P_k|_{\mathbb{T}^k} = \mathbb{D} + (2/(n-2k+2)) \cdot \varepsilon(\gamma)\tilde{\delta}$ is the Dirac and Rarita-Schwinger operators when $k = 0, 1$, respectively.

Since $P_k = (n-2k+2)\nabla\!\!\!/$ on $\mathcal{V}_\varepsilon(j,1)$, by Theorems 5.3 and 6.3,

$$P_k \text{ acts as} \quad \varepsilon \cdot (n-2k+2q) \cdot J \quad \text{on } \mathcal{V}_\varepsilon(j,q), \quad q = 0, 1.$$

Consider now the operator $T_{k-1} : \mathbb{T}^{k-1} \to \mathbb{T}^k$ defined by

$$T_{k-1} = \frac{1}{k}\tilde{d} + \frac{1}{k(n-2(k-1))}\varepsilon(\gamma)\mathbb{D} + \frac{1}{k(n-2(k-1))(n-2(k-1)+1)}\varepsilon(\gamma)^2\tilde{\delta}\,.$$

This is the orthogonal projection of ∇ onto $\mathbb{T}^k$ summand ($1/k \cdot \tilde{d}^{\text{top}}_{k-1}$ in [6]):

$$\mathbb{T}^{k-1} \xrightarrow{\nabla} T^*S^n \otimes \mathbb{T}^{k-1} \cong_{\text{Spin}(n)} \mathbb{T}^{k-2} \oplus \mathbb{T}^{k-1} \oplus \mathbb{T}^k \oplus \mathbb{Z}^{k-1}, \quad 2 \leq k \leq (n-2)/2.$$

where $\mathbb{Z}^{k-1} \cong_{\text{Spin}(n)} \mathbb{V}(\tfrac{5}{2}, \underbrace{\tfrac{3}{2}, \dots, \tfrac{3}{2}}_{k-2}, \tfrac{1}{2}, \dots \tfrac{1}{2})$. Note also that the formal adjoint of T_{k-1} is $T^*_{k-1} = \tilde{\delta}$. When $k = 1$, $(T_0\theta)_a = \nabla_a\theta + \frac{1}{n}\gamma_a\nabla\!\!\!/\,\theta$ is the twistor operator ([6]).

Lemma 6.6. *The second order operator $T_{k-1}T^*_{k-1}$ acts as a scalar*

$$\begin{array}{cl} 0 & \text{on } \mathcal{V}_\varepsilon(j,1) \text{ and} \\ \dfrac{(n-2k+1)(L^2-(n/2-k+1)^2)}{k(n-2k+2)} & \text{on } \mathcal{V}_\varepsilon(j,0)\,. \end{array}$$

Proof. $T_{k-1}T^*_{k-1}$ clearly annihilates $\mathcal{V}_\varepsilon(j,1)$ type. Assume that $T_{k-1}T^*_{k-1}\varphi = \lambda\varphi$ for $\varphi \in \mathcal{V}_\varepsilon(j,0)$. Then $T^*_{k-1}T_{k-1}T^*_{k-1}\varphi = \lambda T^*_{k-1}\varphi$ and $T^*_{k-1}\varphi \in \mathcal{V}_\varepsilon(j,1)$ over $\mathbb{T}^{k-1}$. So we take $\psi \in \mathcal{V}_\varepsilon(j,1)$ over $\mathbb{T}^{k-1}$ and compute $T^*_{k-1}T_{k-1}\psi$. We get the following, where we write $\psi_{[a:i]}$ for $\psi_{a_0\cdots a_{i-1}a_{i+1}\cdots a_{k-1}}$.

$$\begin{aligned}
&T^*_{k-1}T_{k-1}\psi_{a_1\cdots a_{k-1}} \\
&\quad = \frac{1}{k}\sum_{i=0}^{k-1}(-1)^{i+1}\left(\nabla^{a_0}\nabla_{a_i}\psi_{[a:i]} + \frac{1}{n-2k+2}\nabla^{a_0}\gamma_{a_i}\not\nabla\psi_{[a:i]}\right) \\
&\quad = \frac{1}{k}\left(\nabla^*\nabla - \frac{1}{n-2k+2}\not\nabla^2\right)\psi_{a_1\cdots a_{k-1}} + \frac{1}{k}\underbrace{\sum_{i=1}^{k-1}(-1)^{i+1}\nabla^{a_0}\nabla_{a_i}\psi_{[a:i]}}_{A} \\
&\qquad + \frac{1}{k(n-2k+2)}\underbrace{\sum_{i=1}^{k-1}(-1)^{i+1}\nabla^{a_0}\gamma_{a_i}\not\nabla\psi_{[a:i]}}_{B}.
\end{aligned}$$

For A, we compute

$$\begin{aligned}
(-1)^{i+1}\nabla^{a_0}\nabla_{a_i}\psi_{[a:i]} &= (-1)^{i+1}(\nabla_{a_i}\nabla^{a_0} + \mathcal{R}^{a_0}{}_{a_i})\psi_{[a:i]} \\
&= (-1)^{i+1}\mathcal{R}^{a_0}{}_{a_i}\psi_{[a:i]} = (n-k+3/2)\psi_{a_1\cdots a_{k-1}},
\end{aligned}$$

where $\mathcal{R}$ is the spin curvature (Theorem 5.3). Similar computation shows that $B=0$. Thus, on $\mathcal{V}_\varepsilon(j,1)$ over $\mathbb{T}^{k-1}$, $T^*_{k-1}T_{k-1}$ is

$$\frac{1}{k}\left(\nabla^*\nabla + \left(n-k+\frac{3}{2}\right)(k-1) - \frac{1}{n-2k+2}\not\nabla^2\right).$$

By Theorem 5.3, this proves our claim. □

Putting the above observations together, we have

Theorem 6.7. *The differential operator* $D_{2l+1,k}:\mathbb{T}^k\to\mathbb{T}^k$ *defined by*

$$D_{2l+1,k} := \frac{1}{n-2k+2}P_k\prod_{i=1}^{l}\left(\frac{1}{(n-2k+2)^2}P_k^2 - i^2\cdot\mathrm{id} - c_i\cdot T_{k-1}T^*_{k-1}\right),$$

where

$$c_i = \frac{16ki^2}{(n-2k+2)(n-2k+2-2i)(n-2k+2+2i)}$$

is conformally invariant of order $2l+1$.

Proof. The operator $D_{2l+1,k}$ acts as a constant

$$\begin{cases} J(J^2-1^2)\cdots(J^2-l^2) & \text{on } \mathcal{V}_\varepsilon(j,1) \\ \dfrac{n-2k-2l}{n-2k+2+2l}J(J^2-1^2)\cdots(J^2-l^2) & \text{on } \mathcal{V}_\varepsilon(j,0) \end{cases}$$

Thus by the Theorem 6.3, $D_{2l+1,k}$ is a constant multiple of $Z_\varepsilon(l+1/2,j,q)$. □

Case III: n even

Let $E = \sqrt{-1}\gamma^1(1 - 2\varepsilon(d\rho)\iota(\partial\rho))$ where $\gamma^1 = \gamma(d\rho)$ from (5.2), ε is the exterior multiplication, and ι is the interior multiplication. E changes chairality of the spinor, since $\gamma^1 : \Sigma_\pm \to \Sigma_\mp$. It is readily verified that $E^2 = \mathrm{id}$. And for $\Phi \in \mathbb{T}^k$,

$$\begin{aligned}\gamma^j(E\Phi)_{ji_2\cdots i_k} &= \sqrt{-1}\gamma^j\gamma^1(\Phi_{ji_2\cdots i_k} - 2\delta_j^1\Phi_{1i_2\cdots i_k})\\ &= \sqrt{-1}(-\gamma^1\gamma^j - 2g^{1j})(\Phi_{ji_2\cdots i_k} - 2\delta_j^1\Phi_{1i_2\cdots i_k})\\ &= \sqrt{-1}(2\gamma^1\gamma^1\Phi_{1i_2\cdot i_k} - 2g^{11}\Phi_{1i_2\cdot i_k} + 4g^{11}\Phi_{1i_2\cdot i_k})\\ &= 0.\end{aligned}$$

Thus $E : \mathbb{T}^k_\pm \to \mathbb{T}^k_\mp$ with $E^2 = \mathrm{id}$. On $\mathbb{T}^0_\pm = \Sigma_\pm$, $E = \sqrt{-1}\gamma^1$. We also compute

$$\begin{aligned}\mathcal{L}_Y E &= \sqrt{-1}(\underbrace{\mathcal{L}_Y\gamma^1}_{=0}\cdot(1 - 2\varepsilon(d\rho)\iota(\partial\rho)) + \gamma^1\mathcal{L}_Y(1 - 2\varepsilon(d\rho)\iota(\partial\rho)))\\ &= \sqrt{-1}\gamma^1\mathcal{L}_Y(1 - 2\varepsilon(d\rho)\iota(\partial\rho)) = -2\sqrt{-1}\gamma^1\mathcal{L}_Y\varepsilon(d\rho)\iota(\partial\rho))\\ &= -2\sqrt{-1}\gamma^1\{\varepsilon(d\rho)\iota([Y,\partial\rho]) + \varepsilon(d(Y\rho))\iota(\partial\rho)\}\\ &= -2\sqrt{-1}\gamma^1(-\cos\rho\,\varepsilon(d\rho)\iota(\partial\rho) + \cos\rho\,\varepsilon(d\rho)\iota(\partial\rho)) = 0.\end{aligned}$$

Thus the intertwining relation for the exchanged operator EA is exactly the same as that of A (4.2).

$$EA\left(\tilde{\mathcal{L}}_X + \left(\frac{n}{2} - r\right)\omega\right) = \left(\tilde{\mathcal{L}}_X + \left(\frac{n}{2} + r\right)\omega\right)EA.$$

We first consider $EA : \Sigma_\pm \to \Sigma_\pm$. Let $\mathcal{V}(j) = \mathcal{V}(1/2 + j, 1/2, \ldots, 1/2)$. With respect to the following diagram

$$\begin{array}{c}\mathcal{V}(j+1)\\ \uparrow\\ \boxed{\mathcal{V}(j)}\\ \downarrow\\ \mathcal{V}(j-1)\end{array} \qquad \text{we get} \qquad \begin{array}{c}(J+1/2+r)/(J+1/2-r)\\ \uparrow\\ \bullet\\ \downarrow\\ (-J+1/2+r)/(-J+1/2-r)\end{array}$$

where $J = n/2 + j$. Choosing a normalization $\mu_{\mathcal{V}(0)} = \varepsilon$, we have

Theorem 6.8. *The unique spectral function $Z(r, j)$ on $\mathcal{V}_\varepsilon(j)$ is up to normalization*

$$Z(r,j) = \cdot\frac{\Gamma(J + \frac{1}{2} + r)\Gamma(\frac{n}{2} + \frac{1}{2} - r)}{\Gamma(J + \frac{1}{2} - r)\Gamma(\frac{n}{2} + \frac{1}{2} + r)}, \quad j = 0, 1, 2, \ldots.$$

Corollary 6.9. ([8]) *The differential operator $D_{2l+1} : \Sigma_\pm \to \Sigma_\mp$ defined by*

$$D_{2l+1} := \nabla\!\!\!/ \cdot \prod_{p=1}^{l}(\nabla\!\!\!/^2 - p^2) : \Sigma_\pm \to \Sigma_\mp$$

is conformally invariant of order $2l+1$.

Proof. $E\not\nabla \cdot \prod_{p=1}^{l}(\not\nabla^2 - p^2)$ is a constant multiple of $Z(l+1/2, j)$. So $D_{2l+1} : \Sigma_{\pm} \to \Sigma_{\mp}$ is a differential intertwinor as well. □

Next we consider $EA : \mathbb{T}^k_{\pm} \to \mathbb{T}^k_{\pm}$ for $k \geq 1$. Let $\mathcal{V}(j,q) = \mathcal{V}(3/2 + j, 3/2, \ldots, 3/2, \underbrace{1/2+q}_{(k+1)^{\mathrm{st}}}, 1/2, \ldots, 1/2)$. With respect to the diagram

$$\begin{array}{ccc} & \mathcal{V}(j+1,1) & \\ & \uparrow & \\ \mathcal{V}(j,0) & \leftarrow & \boxed{\mathcal{V}(j,1)} \\ & \downarrow & \\ & \mathcal{V}(j-1,1) & \end{array} \qquad \text{we get} \qquad \begin{array}{ccc} & \frac{(J+\frac{1}{2}+r)}{(J+\frac{1}{2}-r)} & \\ & \uparrow & \\ \frac{(n-2k+1-2r)}{(n-2k+1+2r)} & \leftarrow & \bullet \\ & \downarrow & \\ & \frac{(-J+\frac{1}{2}+r)}{(-J+\frac{1}{2}-r)} & \end{array}$$

where $J = n/2 + 1 + j$. Choosing a normalization $\mu_{\mathcal{V}(0,1)} = 1$, we have

Theorem 6.10. *The unique spectral function $Z(r,j,q)$ on $\mathcal{V}(j,q)$ is up to normalization*

$$Z(r,j,q) = \frac{n-2k+1+2(2q-1)r}{n-2k+1+2r} \cdot \frac{\Gamma(J+\frac{1}{2}+r)\Gamma(\frac{n}{2}+\frac{3}{2}-r)}{\Gamma(J+\frac{1}{2}-r)\Gamma(\frac{n}{2}+\frac{3}{2}+r)},$$
$$q = 0,1, \quad j = 0,1,2,\ldots.$$

The proof of Lemma 5.7 also shows that P_k^2 is a constant multiple of $\not\nabla^2$ on $\mathcal{V}(j,1)$. Thus P_k^2 acts as

$$(n-2k+2q)^2 \cdot J^2 \quad \text{on } \mathcal{V}(j,q), \quad q = 0,1.$$

Theorem 6.11. *The differential operator $D_{2l+1,k} : \mathbb{T}^k_{\pm} \to \mathbb{T}^k_{\mp}$ defined by*

$$D_{2l+1,k} := \frac{1}{n-2k+2} P_k \prod_{i=1}^{l} \left(\frac{1}{(n-2k+2)^2} P_k^2 - i^2 \cdot \mathrm{id} - c_i \cdot T_{k-1}T_{k-1}^* \right),$$

where

$$c_i = \frac{16ki^2}{(n-2k+2)(n-2k+2-2i)(n-2k+2+2i)}$$

is conformally invariant of order $2l+1$.

Proof. The operator

$$\frac{1}{n-2k+2} E P_k \prod_{i=1}^{l} \left(\frac{1}{(n-2k+2)^2} P_k^2 - i^2 \cdot \mathrm{id} - c_i \cdot T_{k-1}T_{k-1}^* \right)$$

is a constant multiple of $Z(l+1/2,q)$ in Theorem 6.10. $\square$

To show an example of the theorem, let us take $k=1$ and $l=1$. Then we get

$$D_{3,1}=\frac{1}{n}P_1\left(\frac{1}{n^2}P_1^2-\mathrm{id}-\frac{16}{n(n-2)(n+2)}T_0T_0^*\right).$$

The third order conformally invariant differential operator S_3 on the twistor bundle ($\mathbb{T}^1$ for n odd and $\mathbb{T}^1_\pm$ for n even) over a general curved manifold shown in [5] is

$$(S_3\varphi)_i=\left(\left\{\frac{n+2}{4n^2}P_1^3-\frac{4}{n(n-2)}T_0T_0^*\cdot P_1\right\}\varphi\right)_i+\mathrm{LOT},$$

where

$$\begin{aligned}\mathrm{LOT}=&-\frac{n+2}{4}J\gamma_i\nabla^j\varphi_j+V_i{}^j\gamma^k\nabla_k\varphi_j+(n+2)V_i{}^k\gamma_k\nabla^j\varphi_j+(n+1)V^{jk}\gamma_i\nabla_k\varphi_j\\&-\frac{n(n+2)}{2}V^{jk}\gamma_k\nabla_j\varphi_i+(n-1)V^{jk}\gamma_k\nabla_i\varphi_j+V^{jl}\gamma_{ikl}\nabla^k\varphi_j+\frac{n}{2}(\nabla^jJ)\gamma_i\varphi_j\\&-\frac{n(n+2)}{4}(\nabla^jJ)\gamma_j\varphi_i+n(\nabla^kV_i{}^j)\gamma_k\varphi_j.\end{aligned}$$

Here

$$J=\frac{\mathrm{Scal}}{2(n-1)},\quad V=\frac{r-Jg}{n-2},\quad \gamma_{ijk}=\gamma_{(i}\gamma_j\gamma_{k)},$$

where Scal is the scalar curvature, r is the Ricci curvature, g is the metric tensor, and γ_{ijk} is the skew-symmetrization of $\gamma_i\gamma_j\gamma_k$ over i,j,k.

On the standard sphere, LOT simplifies to $-\frac{n+2}{4}P_1$. Thus $S_3=\frac{n(n+2)}{4}D_{3,1}$.

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