Communications for Statistical Applications and Methods
2023, Vol. 30, No. 2, 191–213

https://doi.org/10.29220/CSAM.2023.30.2.191
Print ISSN 2287-7843 / Online ISSN 2383-4757

Different estimation methods for the unit inverse exponentiated weibull distribution

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Abstract

Unit distributions are frequently used in probability theory and statistics to depict meaningful variables having values between zero and one. Using convenient transformation, the unit inverse exponentiated weibull (UIEW) distribution, which is equally useful for modelling data on the unit interval, is proposed in this study. Quantile function, moments, incomplete moments, uncertainty measures, stochastic ordering, and stress-strength reliability are among the statistical properties provided for this distribution. To estimate the parameters associated to the recommended distribution, well-known estimation techniques including maximum likelihood, maximum product of spacings, least squares, weighted least squares, Cramer von Mises, Anderson–Darling, and Bayesian are utilised. Using simulated data, we compare how well the various estimators perform. According to the simulated outputs, the maximum product of spacing estimates has lower values of accuracy measures than alternative estimates in majority of situations. For two real datasets, the proposed model outperforms the beta, Kumaraswamy, unit Gompartz, unit Lomax and complementary unit weibull distributions based on various comparative indicators.

Keywords: inverse exponentiated weibull distribution, stochastic ordering, Arimoto measure, stress strength model, maximum product spacing

1. Introduction

The weibull distribution is well-known for being the most common distribution for modelling lifetimes (see Murthy *et al.*, 2004), and it has been widely employed in engineering, reliability, and biological research throughout the last few decades. The inadequacy of this distribution to accommodate non-monotone hazard rates is its most significant flaw. As a result, it's become clear that this approach has to be more generalized. Mudholkar and Srivastava (1993) proposed a generalization to the weibull distribution with an extra shape parameter that allows for non-monotone hazard rates. Because of their simplicity and flexibility, the weibull and exponentiated weibull models are frequently studied in survival analysis.

The relevance of inverted distributions may be seen in a variety of domains, including biological sciences, life testing challenges, and so on. Inverted distributions have a different structure than non-inverted distributions in terms of density and hazard rate shapes. Several writers have dedicated a great deal of time and effort to discuss inverted distributions and their applications; such as, the inverse weibull distribution (Keller and Kamath, 1982), the inverted Lindley distribution (Sharma *et al.*,

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Published 31 March 2023 / journal homepage: http://csam.or.kr

2015), the inverted power Lindley distribution (Barco *et al.*, 2017), the inverted Kumaraswamy distribution (Abd AL-Fattah *et al.*, 2017), inverted exponentiated weibull distribution (Lee *et al.*, 2017), inverted Nadarajah–Haghighi distribution (Tahir *et al.*, 2018), inverse power Lomax distribution (Hassan and Abd-Allah, 2019), inverse exponentiated Lomax distribution (Hassan and Mohamed, 2019), and inverted Topp–Leone distribution (Hassan *et al.*, 2020, 2022) among others.

Our interest here with inverted exponentiated weibull (IEW), presented by Lee *et al.* (2017) depend on the transformation $Y = 1/X$, where X has the EW distribution. The IEW distributions can have decreasing, increasing, or bathtub-shaped hazard functions. The probability density function (PDF) and cumulative distribution function (CDF) of the IEW distribution are as below:

$$G(x) = 1 - \left(1 - e^{-\varepsilon x^{-\delta}}\right)^{\phi}; x, \delta, \varepsilon, \phi > 0, \tag{1.1}$$

$$g(x) = \delta\varepsilon\phi x^{-\delta-1} e^{-\varepsilon x^{-\delta}} \left(1 - e^{-\varepsilon x^{-\delta}}\right)^{\phi-1}; x, \delta, \varepsilon, \phi > 0, \tag{1.2}$$

where, ε is the scale parameter, δ and ϕ are the shape parameters. The IEW distribution includes several distributions. For $\delta = 1$, the PDF (1.2) reduces to generalised inverted exponential distribution. For $\delta = 2$, the PDF (1.2) reduces to generalised inverted Rayleigh distribution. For $\delta = \phi = 1$, the PDF (1.2) provides inverted exponential distribution. For $\phi = 1$, the PDF (1.2) gives the inverted weibull distribution. For $\phi = 1, \delta = 2$, provides inverted Rayleigh distribution.

The creation of new flexible probability distributions to give well-fitting models to datasets with values ranging from zero to one has recently peaked statisticians' interest. Statistical distributions with a bound of (0,1) are useful for modelling proportions, percentages, indices, rates, and ratios. The beta distribution is the most well-known unit distribution in statistical literature, as it provides a practical and helpful model in many fields of statistics. Its data modelling capacity, on the other hand, may be insufficient to explain the data. In addition, unit distributions provide additional flexibility over the unit interval without adding new parameters to the basic distribution. In the literature, many of the bounded distributions were created using an appropriate transformation, and they performed better in data modelling than the beta distribution such as; Johnson SB distribution (Johnson, 1949), Topp–Leone distribution (Topp and Leone, 1955), log gamma distribution (Consul and Jain, 1971), unit gamma distribution (Grassia, 1977), the Kumaraswamy distribution (Kumaraswamy, 1980), the simplex distribution (Barndorff-Nielsen and Jørgensen, 1991), the log-Lindley distribution (Gómez-Déniz *et al.*, 2014), unit-Birnbaum-Saunders distribution (Mazucheli *et al.*, 2018), unit-Lindley and unit-Gompertz distributions (Mazucheli *et al.*, 2019a,b), unit-inverse Gaussian distribution (Ghitany *et al.*, 2019), unit-weibull distribution (Mazucheli *et al.*, 2020), unit-Burr-XII distribution (Korkmaz and Chesneau, 2021), the unit generalized log Burr XII distribution (Bhatti *et al.*, 2021), the unit-Gamma/Gompertz distribution (Bantan *et al.*, 2021) and unit exponentiated half logistic distribution (Hassan *et al.*, 2022).

Parameter estimation is essential in the study of any probability distribution. The maximum likelihood (ML) estimation is frequently used to estimate any model's parameters due to its desirable qualities. They are asymptotically consistent, unbiased, and normally distributed. Other specific estimation techniques that have been developed over time rely on a variety of methodologies, such as the methods least squares (LS), weighted LS (WLS), maximum product spacing (MPS), Cramer von Mises (CM), Anderson–Darling (AD), and Bayesian.

The goal of this study is to offer a novel unit flexible probability distribution termed the unit IEW (UIEW) distribution, which has particular sub models on the (0,1) interval and is based on a type transformation $Z = e^{-X}$, where X is the IEW distribution. We provide a thorough comparison of seven methodologies for estimating the UIEW model's parameters as well as an analysis of how these estimators performed for various parameter values and sample sizes. We particularly contrast ML, LS, WLS, MPS, CM, AD, and Bayesian estimates. It is difficult to theoretically examine the characteristics of different estimating approaches, thus we carry out extensive simulation studies to evaluate the behaviours of different estimates with a bias and mean squared error (MSE). We are encouraged to create the UIEW distribution because of the following aspects;

(i) The distribution function and quantile function of the UIEW distribution have simple and closed form expressions.

(ii) It can fit better than other well-known unit interval distributions.

(iii) To derive statistical properties such as random number generators, sub models, moments and incomplete moments, reliability measures, uncertainty measures, and stochastic ordering.

(iv) Six conventional estimating techniques as well as Bayesian method are used to assess the UIEW distribution parameters.

(v) Use simulation studies to examine the precision of different estimators.

(vi) To demonstrate the utility of the UIEW model compared with some other models.

The following is an overview of the structure of the paper. The suggested distribution is defined in Section 2. Section 3 discusses the distributional characteristics that are most important to it. Section 4 discusses the strategies for estimating unknown parameters using various estimation procedures. A simulation study is undertaken, also in Section 4, to assess the parameter estimates. The results of two real data study implementations are shown in Section 5, and the conclusion is offered in Section 6.

2. The unit inverse exponentiated weibull model

Here, we offer the UIEW distribution, a new bounded distribution with support on (0, 1) that emerges from the transformation of the type $Z = e^{-X}$ where X is the IEW distribution. The IEW distribution's CDF may therefore be derived as follows:

$$H(z) = P(Z \leq z) = P\left(e^{-X} \leq z\right) = P(-X \leq \ln(z)) = 1 - P(X \leq -\ln(z)) = 1 - F_X(-\ln(z)),$$

which simply provides

$$H(z) = \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi}; \delta, \varepsilon, \phi > 0, \quad 0 < z < 1. \tag{2.1}$$

Hence, we have $H(z) = 0$, for $z \leq 0$, and $H(z) = 1$, for $z \leq 1$. The UIEW distribution PDF may be acquired as follows:

$$h(z) = \delta\varepsilon\phi z^{-1} (-\ln z)^{-\delta-1} e^{-\varepsilon(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi-1}; \delta, \varepsilon, \phi > 0, \quad 0 < z < 1. \tag{2.2}$$

A random variable with PDF (2.2) will be denoted by UIEW$(\delta, \varepsilon, \phi)$. The following are the survival function and hazard function of PDF (2.2)

$$\bar{H}(z) = 1 - \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi}, \quad 0 < z < 1,$$

$$\eta(z) = \frac{\delta\varepsilon\phi z^{-1} (-\ln z)^{-\delta-1} e^{-\varepsilon(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi-1}}{1 - \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi}}.$$

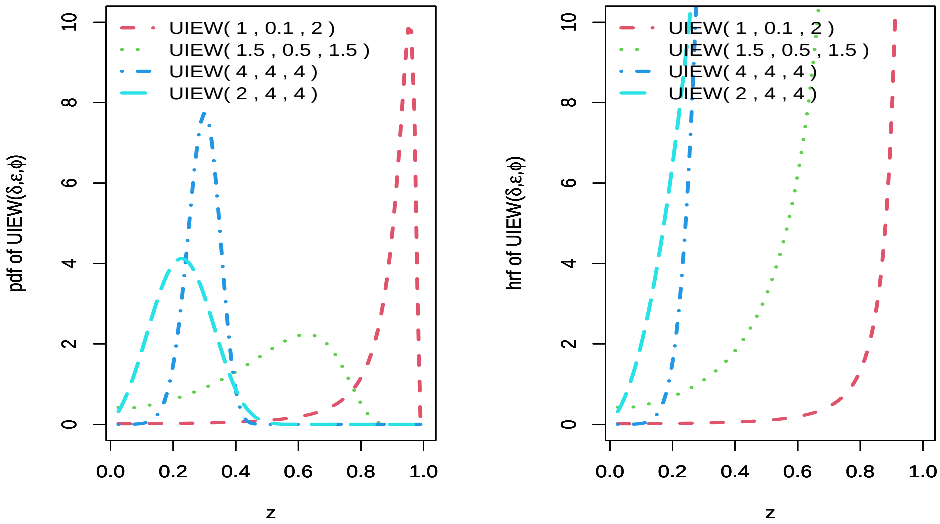

Figure 1: *Density and hazard rate of the UIEW distribution with different values.*

For a few chosen values of the parameters, Figure 1 illustrates the various forms of the PDF and hazard function of the UIEW distribution. We can note from Figure 1 that the PDF can be left skewed, right skewed, asymmetric and unimodal shaped. The hazard function of the UIEW distribution can be, increasing and J-shaped.

The UIEW distribution is a highly flexible model that recognizes several distributions as particular sub-models:

(i) For $\phi = 1$, the UIEW distribution reduces to the unit inverse weibull as new sub-model.

(ii) For $\phi = 1$ and $\delta = 1$, the UIEW distribution reduces to the unit inverse exponential distribution as new sub-model.

(iii) For $\phi = 1$ and $\delta = 2$, the UIEW distribution reduces to the unit inverse Rayleigh distribution as new sub-model.

3. The UIEW distribution's features

We looked at some of the structural properties of the UIEW distribution, including quantile function, moments and incomplete moments, entropy measures, stochastic ordering and stress-strength (SS) reliability, in this part.

3.1. Quantile function

For $q \in (0, 1)$, the quantile function of Z is obtained by inverting (2.1) as follows:

$$z_q = \exp\left(-\left(\frac{-1}{\varepsilon}\left[\ln\left(1 - q^{\frac{1}{\phi}}\right)\right]\right)^{-\frac{1}{\delta}}\right), \quad 0 < q < 1. \tag{3.1}$$

Table 1: Moments values of the UIEW distribution

μ_n'	(i)	(ii)	(iii)	(iv)	(v)	(vi)
μ_1'	0.545	0.277	0.391	0.417	0.391	0.218
μ_2'	0.326	0.123	0.172	0.180	0.172	0.072
μ_3'	0.206	0.062	0.081	0.080	0.081	0.028
μ_4'	0.134	0.033	0.040	0.036	0.040	0.012
σ^2	0.029	0.046	0.019	0.006	0.019	0.024
α_3	−0.820	0.219	−0.421	−0.507	−0.421	0.407
α_4	3.278	1.779	2.697	3.302	2.697	2.252

Putting $q = 0.25, 0.5$, and 0.75 in (3.1) yields the first, median, and third quantiles. It's simple to simulate the random variable. If Q is a uniform variate on the unit interval $(0, 1)$, then $Z = z_q$ at q follows (3.1).

3.2. Moments measures

The n^{th} moment for Z with PDF (2.2) is calculated as

$$\mu_n' = \delta\varepsilon\phi \int_0^1 z^{n-1} (-\ln z)^{-\delta-1} e^{-\varepsilon(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi-1} dz. \tag{3.2}$$

Let $u = (-\ln z)^{-\delta}$, and using the binomial expansion, then the n^{th} moment of Z, can be expressed as

$$\mu_n' = \sum_{r=0}^{\infty} (-1)^r \binom{\phi-1}{r} \varepsilon\phi \int_0^{\infty} e^{-nu^{\frac{-1}{\delta}}} e^{-\varepsilon(r+1)u} du.$$

Use the exponential expansion then, the previous equation is

$$\mu_n' = \sum_{r,i=0}^{\infty} \binom{\phi-1}{r} \frac{(-1)^{r+i}\, n^i \varepsilon\phi}{i!\,(\varepsilon(r+1))^{1-\frac{i}{\delta}}} \Gamma\left(1 - \frac{i}{\delta}\right),$$

where, $\Gamma(\cdot)$ is gamma function. Furthermore, the n^{th} central moment of Z, is defined by

$$\mu_n = E(Z - \mu_1')^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\mu_1')^j \mu_{n-j}'.$$

For specific parameter settings, such as (i) $(\delta = 1.5, \varepsilon = 0.5, \phi = 2)$, (ii) $(\delta = 2, \varepsilon = 0.5, \phi = 0.5)$, (iii) $(\delta = 2, \varepsilon = 1, \phi = 2)$, (iv) $(\delta = 3, \varepsilon = 1, \phi = 3)$, (v) $(\delta = 2, \varepsilon = 2, \phi = 2)$, (vi) $(\delta = 1.5, \varepsilon = 2, \phi = 1.5)$, are provided. Table 1 contains values of some moments, variance (σ^2), coefficient of skewness (α_3) and coefficient of kurtosis (α_4) for the UIEW distribution. As seen in Table 1, the UIEW distribution is right and left skewed according to values of (α_3). Also, the distribution is leptokurtic and platykurtic according to the values of (α_4). Furthermore, the n^{th} lower incomplete moment, say $\mathrm{I}_n(t)$ of the UIEW distribution is given by:

$$\mathrm{I}_n(t) = \delta\varepsilon\phi \int_0^t z^{n-1} (-\ln z)^{-\delta-1} e^{-\varepsilon(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{\phi-1} dz.$$

Let $u = (-\ln z)^{-\delta}$, and using the binomial expansion, then the n^{th} incomplete moment of Z, can be expressed as

$$\mathrm{I}_n(t) = \sum_{r=0}^{\infty} (-1)^r \binom{\phi - 1}{r} \varepsilon\phi \int_0^{(-\ln t)^{-\delta}} e^{-nu^{\frac{-1}{\delta}}} e^{-\varepsilon(r+1)u} du.$$

After simplification the n^{th} moment is as below:

$$\mathrm{I}_n(t) = \sum_{r,i=0}^{\infty} \binom{\phi - 1}{r} \frac{(-1)^{r+i} n^i \varepsilon\phi}{i!\,(\varepsilon(r+1))^{1-\frac{i}{\delta}}} \gamma\left(1 - \frac{i}{\delta},\ \left(-\ln\left(\frac{t}{\varepsilon(r+1)}\right)\right)^{-\delta}\right),$$

where $\gamma(\ldots, t)$ is lower incomplete gamma function. The first incomplete moment is well-known in applications such as the Lorenz and Bonferroni curves, which are described as $Lo(t) = \mathrm{I}_1(t)/\mu_1'$ and $Bo(t) = Lo(t)/F(t)$ respectively. These curves are particularly useful in economics, demography, insurance, engineering, and medicine.

3.3. Uncertainty measures

We looked at certain information measures such as, Rényi, Arimoto, Havrda and Charvat, and Tsallis entropies. All these measures indicate the total quantity of data in the system. The Rényi (see Rényi, 1960) entropy, denoted by $\Re(c)$ of Z is defined by

$$\Re(c) = (1 - c)^{-1} \log\left(\int_0^{\infty} (h(z))^c dz\right). \tag{3.3}$$

Using PDF (2.2) in (3.3) gives

$$\Re(c) = (1 - c)^{-1} \log\left((\delta\varepsilon\phi)^c \int_0^1 z^{-c} (-\ln z)^{-c(\delta+1)} e^{-c\varepsilon(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon(-\ln z)^{-\delta}}\right]^{c(\phi-1)} dz\right). \tag{3.4}$$

Using the binomial and exponential expansions in (3.4), then $\Re(c)$ transformed to

$$\Re(c) = (1 - c)^{-1} \log\left(\sum_{k,j=0}^{\infty} \Delta_{k,j} \Gamma\left(\frac{c(\delta + 1)}{\delta} - \frac{1}{\delta} - \frac{j}{\delta}\right)\right),$$

where

$$\Delta_{k,j} = \binom{c(\phi - 1)}{k} \frac{(\varepsilon\phi)^c \delta^{c-1} (-1)^{j+k} (1 - c)^j}{j!\,(\varepsilon(k + c))^{\frac{c(\delta+1)}{\delta} - \frac{1}{\delta} - \frac{j}{\delta}}}.$$

The Havrda and Charvat (Havrda and Charvat, 1967) entropy, denoted by $\mathrm{H}(c)$, of the UIEW distribution is calculated as:

$$\begin{aligned} \mathrm{H}(c) &= \frac{1}{2^{1-c} - 1}\left[\left(\int_0^1 (h(z))^c dz\right)^{\frac{1}{c}} - 1\right], \quad c \neq 1,\ c > 0 \\ &= \frac{1}{2^{1-c} - 1}\left[\left(\sum_{k,j=0}^{\infty} \Delta_{k,j} \Gamma\left(\frac{c(\delta + 1)}{\delta} - \frac{1}{\delta} - \frac{j}{\delta}\right)\right)^{\frac{1}{c}} - 1\right]. \end{aligned}$$

Table 2: Entropy measures of UIEW distribution

c	Measure	(i)	(ii)	(iii)	(iv)	(v)	(vi)
0.2	I(c)	−0.166	−0.080	−0.175	−0.217	−0.171	−0.286
	A(c)	−0.121	−0.069	−0.126	−0.145	−0.124	−0.170
	H(c)	−0.656	−0.370	−0.678	−0.783	−0.669	−0.919
	T(c)	−0.156	−0.078	−0.163	−0.199	−0.160	−0.255
0.9	I(c)	−4.184	−0.719	−0.773	−1.319	−0.318	−0.646
	A(c)	−3.346	−0.691	−0.741	−1.227	−0.312	−0.624
	H(c)	−5.180	−1.070	−1.147	−1.899	−0.483	−0.966
	T(c)	−3.419	−0.694	−0.744	−1.235	−0.313	−0.626

The Arimoto (see Arimoto, 1971) entropy measure. Represented by A(c), of the UIEW distribution is calculated as:

$$\begin{aligned} \mathrm{A}(c) &= \frac{c}{1-c}\left[\left(\int_0^1 (h(z))^c\,dz\right)^{\frac{1}{c}} - 1\right], \quad c \neq 1,\ c > 0 \\ &= \frac{c}{1-c}\left[\left(\sum_{k,j=0}^{\infty} \Delta_{k,j}\Gamma\left(\frac{c(\delta+1)}{\delta} - \frac{1}{\delta} - \frac{j}{\delta}\right)\right)^{\frac{1}{c}} - 1\right]. \end{aligned}$$

The Tsallis (see Tsallis, 1988) entropy measure, represented by T(c), of the UIEW distribution is as below:

$$\begin{aligned} \mathrm{T}(c) &= \frac{1}{c-1}\left[1 - \int_0^1 (h(z))^c\,dz\right], \quad c \neq 1,\ c > 0 \\ &= \frac{1}{c-1}\left[1 - \left(\sum_{k,j=0}^{\infty} \Delta_{k,j}\Gamma\left(\frac{c(\delta+1)}{\delta} - \frac{1}{\delta} - \frac{j}{\delta}\right)\right)\right]. \end{aligned}$$

Numerical values of $\mathrm{I}(c), \mathrm{A}(c), \mathrm{H}(c)$, and $\mathrm{T}(c)$ are given for given parameter values (i) ($\delta = 0.5, \varepsilon = 0.5, \phi = 0.5$), (ii) ($\delta = 0.5, \varepsilon = 0.5, \phi = 1.5$), (iii) ($\delta = 0.5, \varepsilon = 0.5, \phi = 3$), (iv) ($\delta = 1.5, \varepsilon = 0.5, \phi = 0.5$), (v) ($\delta = 1.5, \varepsilon = 0.5, \phi = 1.5$), (vi) ($\delta = 1.5, \varepsilon = 0.5, \phi = 3$) (Table 2). The following may be seen from Table 2:

1. For all values of the parameters except set (v), the H(c) entropy measure picks the lowest values among the others, yielding less information.

2. For all considered values of the parameters except at ($\delta = 0.5, \varepsilon = 0.5, \phi = 0.5$), the A(c) entropy measure yields the largest values, suggesting more uncertainty.

3. The entropy measures include; $\mathrm{I}(c), \mathrm{A}(c)$, and $\mathrm{T}(c)$ values decrease as the value of c increases, indicating that we have more knowledge. As the value of c increases, H(c) measure decreases for all sets except set (v).

3.4. Stochastic ordering

The stochastic ordering is an extensively researched notion in probability distributions and is an essential tool in reliability theory and other domains to examine comparative behaviour of random variables. Assume that $Z_i; i = 1, 2$ has the UIEW($\delta_i, \varepsilon_i, \phi_i$) distribution with PDF $h(z_i)$ and CDF $H(z_i)$ of

Table 3: The Biases and MSEs of different estimates of UIEW distribution at ($\delta = 1.5, \varepsilon = 1, \phi = 1$)

n	Measures	ML	MPS	LS	WLS	AD	CM	Bayesian
100	Bias($\hat{\delta}$)	0.1594	0.0652	0.2124	0.1214	0.1097	0.2195	0.0676
	MSE($\hat{\delta}$)	0.3893	0.2864	0.7165	0.3743	0.3271	0.7561	0.0368
	Bias($\hat{\varepsilon}$)	0.2295	0.0402	0.0811	0.0643	0.0682	0.1105	−0.2037
	MSE($\hat{\varepsilon}$)	0.2295	0.2107	0.4398	0.2867	0.2680	0.4798	0.0665
	Bias($\hat{\phi}$)	0.1593	0.1662	0.4341	0.2525	0.2407	0.5327	−0.1488
	MSE($\hat{\phi}$)	0.5786	0.5680	2.8697	1.0149	0.9493	4.5507	0.0410
200	Bias($\hat{\delta}$)	0.0612	0.0095	0.0927	0.0564	0.0482	0.0972	−0.0423
	MSE($\hat{\delta}$)	0.1207	0.1028	0.278	0.1529	0.1379	0.2817	0.0245
	Bias($\hat{\varepsilon}$)	0.0209	0.0285	0.0475	0.0313	0.0373	0.0568	0.1182
	MSE($\hat{\varepsilon}$)	0.0285	0.0953	0.2109	0.1318	0.1261	0.2151	0.0410
	Bias($\hat{\phi}$)	0.0687	0.0795	0.1863	0.1031	0.1058	0.1985	0.0292
	MSE($\hat{\phi}$)	0.1745	0.1695	0.6876	0.2731	0.2630	0.7258	0.0318
300	Bias($\hat{\delta}$)	0.0412	0.0015	0.0723	0.0449	0.0381	0.0752	0.1410
	MSE($\hat{\delta}$)	0.0721	0.0638	0.1699	0.0968	0.0878	0.1715	0.0282
	Bias($\hat{\varepsilon}$)	0.0125	0.0217	0.0224	0.0151	0.0202	0.0285	−0.1184
	MSE($\hat{\varepsilon}$)	0.0647	0.0618	0.1384	0.0862	0.0830	0.1402	0.0239
	Bias($\hat{\phi}$)	0.0426	0.0548	0.1068	0.0596	0.0632	0.1138	−0.1488
	MSE($\hat{\phi}$)	0.0990	0.0973	0.3657	0.1531	0.1472	0.3786	0.0312

Z_i, respectively. If $h_{Z_1}(z)/h_{Z_2}(z)$ is a decreasing function for all values of z, we may state that Z_1 is stochastically smaller than Z_2 in terms of likelihood ratio order (denoted by $Z_1 \leq_{lr} Z_2$).

Let $Z_1 \sim \text{UIEW}(\delta_1, \varepsilon_1, \phi_1)$ and $Z_2 \sim \text{UIEW}(\delta_2, \varepsilon_2, \phi_2)$ when relevant assumptions are met, we shall prove that the UIEW distributions are ordered with regard to likelihood ratio ordering. The density ratio is

$$\frac{h_{Z_1}(z)}{h_{Z_2}(z)} = \frac{\delta_1 \varepsilon_1 \phi_1 (-\ln z)^{-\delta_1 - 1} e^{-\varepsilon_1 (-\ln z)^{-\delta_1}} \left[1 - e^{-\varepsilon_1 (-\ln z)^{-\delta_1}}\right]^{\phi_1 - 1}}{\delta_2 \varepsilon_2 \phi_2 (-\ln z)^{-\delta_2 - 1} e^{-\varepsilon_2 (-\ln z)^{-\delta_2}} \left[1 - e^{-\varepsilon_2 (-\ln z)^{-\delta_2}}\right]^{\phi_2 - 1}},$$

therefore,

$$\begin{aligned} \frac{d}{dz} \log \frac{h_{Z_1}(z)}{h_{Z_2}(z)} &= \frac{-(\delta_1 + 1)}{z \ln(z)} + \frac{(\delta_2 + 1)}{z \ln(z)} - \varepsilon_1 \delta_1 z^{-1} (-\ln z)^{-\delta_1 - 1} + \varepsilon_2 \delta_2 z^{-1} (-\ln z)^{-\delta_2 - 1} \\ &+ \frac{(\phi_1 - 1) \varepsilon_1 \delta_1 z^{-1} (-\ln z)^{-\delta_1 - 1}}{e^{\varepsilon_1 (-\ln z)^{-\varepsilon_1}} - 1} - \frac{(\phi_2 - 1) \varepsilon_2 \delta_2 z^{-1} (-\ln z)^{-\delta_2 - 1}}{e^{\varepsilon_2 (-\ln z)^{-\delta_2}} - 1} < 0. \end{aligned}$$

If $\delta_1 \leq \delta_2$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and $\phi_1 \leq \phi_2$, we get $h_{Z_1}(z)/h_{Z_2}(z)$ is decreasing in z and hence $Z_1 \leq_{lr} Z_2$. Furthermore, we can conclude for other different ordering, $Z_1 \leq_{hr} Z_2$ (hazard rate order), $Z_1 \leq_{mrl} Z_2$ (mean residual life order) and $Z_1 \leq_{sr} Z_2$ (stochastic order). According to Shaked and Shanthikumar (2007), these stochastic orders are connected to one another, and the following pattern occurs

$$Z_1 \leq_{lr} Z_2 \;\Rightarrow\; Z_1 \leq_{hr} Z_2 \;\Rightarrow\; Z_1 \leq_{mrl} Z_2 \;\Rightarrow\; Z_1 \leq_{sr} Z_2.$$

3.5. Stress-Strength reliability

The notion "SS reliability" is used in statistical literature to describe the reliability of a system with random strength Z_1 that is exposed to random stress Z_2, with the system failing if Z_2 exceeds Z_1. Let $Z_1 \sim \text{UIEW}(\delta_1, \varepsilon_1, \phi_1)$ and $Z_2 \sim \text{UIEW}(\delta_2, \varepsilon_2, \phi_2)$ are two independent random variables. The UIEW

Table 4: The Biases and MSEs of different estimates of UIEW distribution at ($\delta = 1, \varepsilon = 1, \phi = 1.5$)

n	Measures	ML	MPS	LS	WLS	AD	CM	Bayesian
100	Bias($\hat{\delta}$)	0.0862	0.0371	0.1615	0.0730	0.1097	0.1738	0.0500
	MSE($\hat{\delta}$)	0.1457	0.1168	0.3962	0.1560	0.3271	0.6834	0.0278
	Bias($\hat{\varepsilon}$)	0.0608	0.0631	0.0756	0.0887	0.0682	0.0932	−0.0752
	MSE($\hat{\varepsilon}$)	0.2745	0.2595	0.4845	0.3618	0.2680	0.4892	0.0629
	Bias($\hat{\phi}$)	0.3787	0.3664	0.7010	0.5676	0.2407	0.7344	0.2959
	MSE($\hat{\phi}$)	2.4998	2.5798	7.3016	4.7552	0.9493	6.9530	0.2119
200	Bias($\hat{\delta}$)	0.0335	0.0062	0.0677	0.0366	0.0482	0.0806	−0.0425
	MSE($\hat{\delta}$)	0.0505	0.0448	0.1211	0.0704	0.1379	0.2803	0.0054
	Bias($\hat{\varepsilon}$)	0.0325	0.0374	0.0473	0.0415	0.0373	0.0536	0.0959
	MSE($\hat{\varepsilon}$)	0.1177	0.1126	0.2498	0.1611	0.1261	0.2438	0.0176
	Bias($\hat{\phi}$)	0.1536	0.1536	0.3362	0.2156	0.1058	0.3329	0.1403
	MSE($\hat{\phi}$)	0.5677	0.5434	2.2895	0.9693	0.2630	1.9155	0.0340
300	Bias($\hat{\delta}$)	0.0228	0.0020	0.0666	0.0277	0.0381	0.0651	−0.0014
	MSE($\hat{\delta}$)	0.0304	0.0277	0.1764	0.0422	0.0878	0.1859	0.0019
	Bias($\hat{\varepsilon}$)	0.0196	0.0258	0.0160	0.0217	0.0202	0.0239	0.0145
	MSE($\hat{\varepsilon}$)	0.0743	0.0716	0.1624	0.1032	0.0830	0.1644	0.0035
	Bias($\hat{\phi}$)	0.0938	0.0993	0.1742	0.1252	0.0632	0.1933	0.0247
	MSE($\hat{\phi}$)	0.3016	0.2911	0.9977	0.5008	0.1472	1.0433	0.0009

distribution's SS reliability is then calculated as follows:

$$R = P(Z_2 < Z_1) = \delta\varepsilon_1\phi_1 \int_0^1 z^{-1}(-\ln z)^{-\delta-1} e^{-\varepsilon_1(-\ln z)^{-\delta}} \left[1 - e^{-\varepsilon_1(-\ln z)^{-\delta}}\right]^{\phi_1 - 1} \left[1 - e^{-\varepsilon_2(-\ln z)^{-\delta}}\right]^{\phi_2}. \quad (3.5)$$

Using the binomial expansions in (3.5), we get

$$R = \sum_{u_1,u_2=0}^{\infty} \delta\varepsilon_1\phi_1 (-1)^{u_1+u_2} \binom{\phi_1 - 1}{u_1}\binom{\phi_2}{u_2} \int_0^1 z^{-1}(-\ln z)^{-(\delta+1)} e^{-(\varepsilon_1+\varepsilon_1 u_1+\varepsilon_2 u_2)(-\ln z)^{-\delta}} dz. \quad (3.6)$$

Hence, the UIEW distribution's SS reliability is as below:

$$R = \sum_{u_1,u_2=0}^{\infty} \frac{\varepsilon_1\phi_1 (-1)^{u_1+u_2}}{(\varepsilon_1 + \varepsilon_1 u_1 + \varepsilon_2 u_2)} \binom{\phi_1 - 1}{u_1}\binom{\phi_2}{u_2}.$$

4. Parameter estimation

In this section, we discuss the estimation of UIEW distribution parameters using ML, MPS, LS, WLS, AD, CM, and Bayesian methods. Furthermore, the performance of such estimators will be examined through simulation study.

4.1. Maximum likelihood estimation

Suppose that a random sample of size n is drawn from a population having a UIEW distribution given by (2.2) with unknown parameter vector $\vartheta = (\delta, \varepsilon, \phi)'$. Then the log likelihood function for ϑ will be

$$\ln l(\vartheta) = n\ln(\delta\varepsilon\phi) - \sum_{i=1}^{n} \ln z_i - (\delta+1)\sum_{i=1}^{n}\left[\ln(-\ln z_i) + \varepsilon(-\ln z_i)^{\delta}\right] + (\phi - 1)\sum_{i=1}^{n} \ln\left[1 - e^{-\varepsilon(-\ln z_i)^{-\delta}}\right].$$

Table 5: The Biases and MSEs of different estimates of UIEW distribution at ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$)

n	Measures	ML	MPS	LS	WLS	AD	CM	Bayesian
100	Bias($\hat{\delta}$)	0.1304	0.0565	0.2177	0.1072	0.1123	0.2373	−0.0260
	MSE($\hat{\delta}$)	0.3286	0.2631	0.6642	0.3477	0.3237	0.6920	0.0602
	Bias($\hat{\varepsilon}$)	0.0598	0.0623	0.0642	0.0890	0.0746	0.0769	−0.0883
	MSE($\hat{\varepsilon}$)	0.2750	0.2600	0.4181	0.3565	0.3133	0.4419	0.0571
	Bias($\hat{\phi}$)	0.3771	0.3652	0.5516	0.5559	0.4569	0.6136	−0.1225
	MSE($\hat{\phi}$)	2.4995	2.5863	4.3676	4.4056	3.1798	4.7414	0.1484
200	Bias($\hat{\delta}$)	0.0520	0.0111	0.1028	0.0528	0.0658	0.1135	0.0575
	MSE($\hat{\delta}$)	0.1143	0.1014	0.2790	0.1480	0.1967	0.2917	0.0186
	Bias($\hat{\varepsilon}$)	0.0308	0.0356	0.0446	0.0410	0.0383	0.0479	0.0723
	MSE($\hat{\varepsilon}$)	0.1182	0.1131	0.2336	0.1599	0.1550	0.2377	0.0167
	Bias($\hat{\phi}$)	0.1506	0.1505	0.3003	0.2139	0.1986	0.3125	0.1688
	MSE($\hat{\phi}$)	0.5681	0.5433	1.6691	0.9713	0.8628	1.8008	0.0675
300	Bias($\hat{\delta}$)	0.0366	0.0053	0.0890	0.0425	0.0568	0.0887	−0.0450
	MSE($\hat{\delta}$)	0.0692	0.0631	0.1928	0.0952	0.1390	0.1884	0.0126
	Bias($\hat{\varepsilon}$)	0.0171	0.0234	0.0163	0.0207	0.0162	0.0232	0.0351
	MSE($\hat{\varepsilon}$)	0.0747	0.0721	0.1579	0.1031	0.1032	0.1608	0.0128
	Bias($\hat{\phi}$)	0.0893	0.0948	0.1672	0.1233	0.1125	0.1855	0.1363
	MSE($\hat{\phi}$)	0.3014	0.2912	0.8526	0.4996	0.4731	0.9326	0.0673

Differentiate the log likelihood function with respect to δ, ε, and ϕ respectively,

$$\frac{\partial \ln l(\vartheta)}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \left[\ln(-\ln z_i) + \varepsilon(-\ln z_i)^{\delta}\right] + \frac{(\phi-1)\,\varepsilon\,(-\ln z_i)^{-\delta}\ln(-\ln z_i)}{\left(e^{\varepsilon(-\ln z_i)^{-\delta}} - 1\right)}, \tag{4.1}$$

$$\frac{\partial \ln l(\vartheta)}{\partial \varepsilon} = \frac{n}{\varepsilon} - (\delta+1)\sum_{i=1}^{n}(-\ln z_i)^{\delta} + (\phi-1)\sum_{i=1}^{n}\frac{(-\ln z_i)^{-\delta}}{\left[e^{\varepsilon(-\ln z_i)^{-\delta}} + 1\right]}, \tag{4.2}$$

$$\frac{\partial \ln(\vartheta)}{\partial \phi} = \frac{n}{\phi} + \sum_{i=1}^{n}\ln\left[1 - e^{-\varepsilon(-\ln z_i)^{-\delta}}\right]. \tag{4.3}$$

Equate Equations (4.1)-(4.3) to zero and solve for $\delta = \hat{\delta}_{\mathrm{ML}}, \varepsilon = \hat{\varepsilon}_{\mathrm{ML}}$, and $\phi = \hat{\phi}_{\mathrm{ML}}$ simultaneously. However, the obtained equations cannot be solved analytically to obtain $\hat{\delta}_{\mathrm{ML}}, \hat{\varepsilon}_{\mathrm{ML}}$, and $\hat{\phi}_{\mathrm{ML}}$.

4.2. Maximum product of spacings estimation

The MPS estimation method was proposed by Cheng and Amin (1979) and justified by Ranneby (1984) as an alternative to ML method in the case of having continuous univariate distributions. It was demonstrated that the MPS technique generates consistent and asymptotically effective estimators for various distributions where the ML method fails due to unboundedness of the likelihood, such as a three-parameter gamma, lognormal, or weibull distribution. The MSP estimators are consistent in situations like mixtures of normal distributions where the ML approach is known to give inconsistent estimators (see Ranneby, 1984). Ekström (2006) is a good resource for in-depth information. The MPS method is used primarily for maximising the geometric mean of spacings in the data, which are the differences between the values of the cumulative distribution function at adjacent data points. In this method, an ordered sample $z_{(1)}, \ldots, z_{(n)}$ of size n is drawn from a population having a UIEW distribution given by (2.2), the MPS estimator of set of parameters $\vartheta = (\delta, \varepsilon, \phi)'$ is the value that maximizes the following MPS(ϑ) where $D_i(\vartheta) = H(z_{(i)}, \vartheta) - H(z_{(i-1)}, \vartheta), i = 1, \ldots, n+1, z_0 = -\infty, z_{(n+1)} = \infty$ and

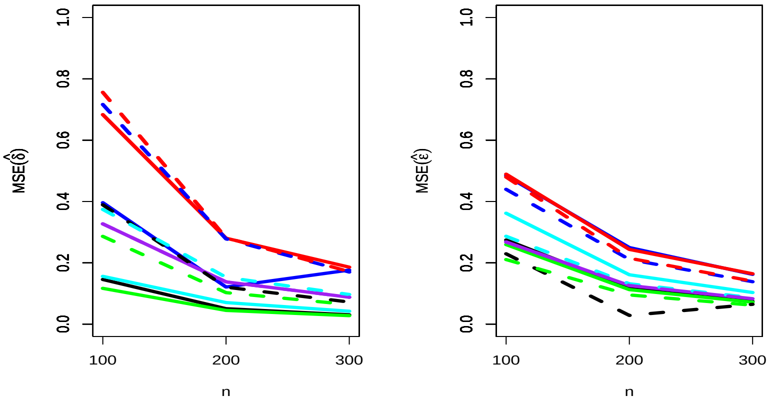

Figure 2: *Plots of MSE($\hat{\delta}$) (left panel) and MSE($\hat{\varepsilon}$) (right panel) against n for ML(black), MPS(green), LS(blue), WLS(cyan), AD(purple), and CM(red) when ($\delta = 1, \varepsilon = 1, \phi = 1.5$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (dashed).*

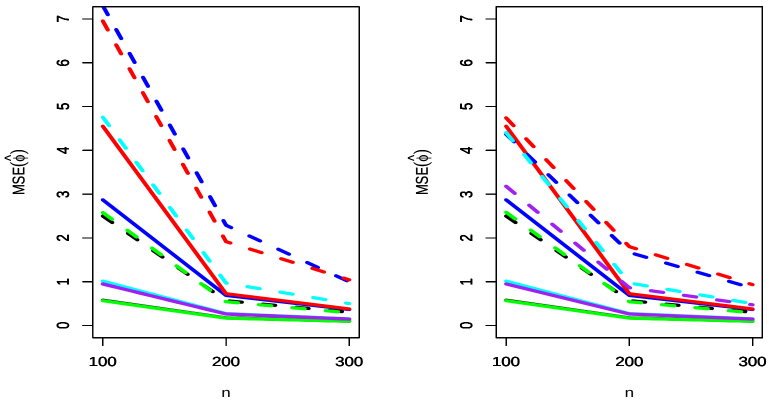

Figure 3: *Plots of MSE($\hat{\phi}$) against n for ML(black), MPS(green), LS(blue), WLS(cyan), AD(purple), and CM(red) when ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1, \varepsilon = 1, \phi = 1.5$) (dashed) (left panel) and ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$) (dashed) (right panel).*

$H(z_{(i)}, \vartheta)$ is given by (2.1). Hence, MPS estimates, $\hat{\delta}_{\text{MPS}}, \hat{\varepsilon}_{\text{MPS}}$, and $\hat{\phi}_{\text{MPS}}$, are obtained by solving simultaneously the following non-linear equations:

$$\frac{\partial \text{MPS}(\vartheta)}{\partial \delta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{(\partial/\partial\delta)\left[H\left(z_{(i)}, \vartheta\right)\right] - (\partial/\partial\delta)\, H\left(z_{(i-1)}, \vartheta\right)}{H\left(z_{(i)}, \vartheta\right) - H\left(z_{(i-1)}, \vartheta\right)} \right] = 0,$$

$$\frac{\partial \text{MPS}(\vartheta)}{\partial \varepsilon} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{(\partial/\partial\varepsilon)\left[H\left(z_{(i)}, \vartheta\right)\right] - (\partial/\partial\varepsilon)\, H\left(z_{(i-1)}, \vartheta\right)}{H\left(z_{(i)}, \vartheta\right) - H\left(z_{(i-1)}, \vartheta\right)} \right] = 0,$$

$$\frac{\partial \text{MPS}(\vartheta)}{\partial \phi} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{(\partial/\partial\phi)\left[H\left(z_{(i)}, \vartheta\right)\right] - (\partial/\partial\phi)\, H\left(z_{(i-1)}, \vartheta\right)}{H\left(z_{(i)}, \vartheta\right) - H\left(z_{(i-1)}, \vartheta\right)} \right] = 0,$$

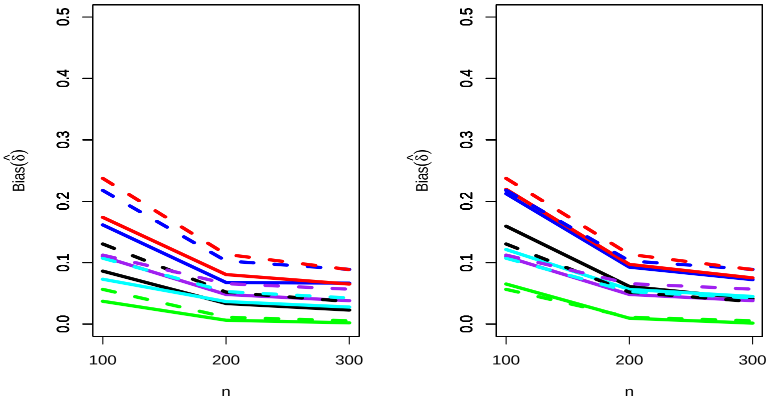

Figure 4: *Plots of Bias($\hat{\delta}$) against n for ML(black), MPS(green), LS(blue), WLS(cyan), AD(purple), and CM(red) when ($\delta = 1, \varepsilon = 1, \phi = 1.5$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$) (dashed) (left panel) and ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$) (dashed) (right panel).*

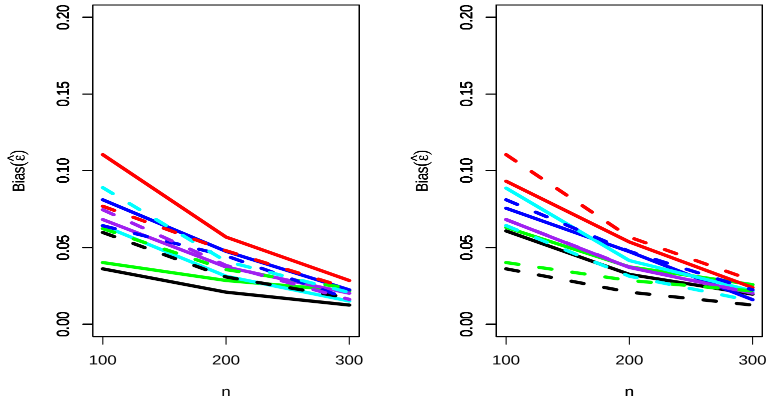

Figure 5: *Plots of Bias($\hat{\varepsilon}$) against n for ML(black), MPS(green), LS(blue), WLS(cyan), AD(purple), and CM(red) when ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$) (dashed) (left panel) and ($\delta = 1, \varepsilon = 1, \phi = 1.5$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (dashed) (right panel).*

where

$$\frac{\partial}{\partial \delta}\left[H\left(z_{(i)}, \vartheta\right)\right] = -\phi\varepsilon\left(1 - e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\right)^{\phi-1} e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\left(-\ln z_{(i)}\right)^{-\delta}\ln\left(-\ln z_{(i)}\right), \tag{4.4}$$

$$\frac{\partial}{\partial \varepsilon}\left[H\left(z_{(i)}, \vartheta\right)\right] = \phi\left(1 - e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\right)^{\phi-1} e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\left(-\ln z_{(i)}\right)^{-\delta}, \tag{4.5}$$

$$\frac{\partial}{\partial \phi}\left[H\left(z_{(i)}, \vartheta\right)\right] = \left(1 - e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\right)^{\phi}\ln\left(1 - e^{-\varepsilon\left(-\ln z_{(i)}\right)^{-\delta}}\right). \tag{4.6}$$

However, the obtained equations cannot be solved analytically, so numerical technique will be employed using non-linear optimization algorithms to obtain, $\hat{\delta}_{\text{MPS}}, \hat{\varepsilon}_{\text{MPS}}$, and $\hat{\phi}_{\text{MPS}}$.

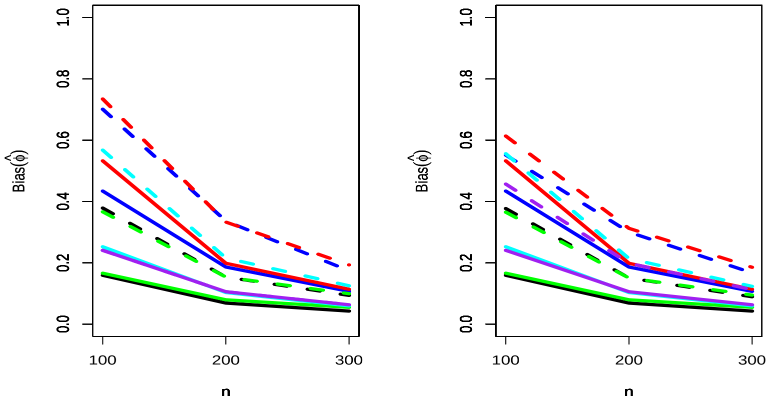

Figure 6: *Plots of Bias($\hat{\phi}$) against n for ML(black), MPS(green), LS(blue), WLS(cyan), AD(purple), and CM(red) when ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1, \varepsilon = 1, \phi = 1.5$) (dashed) (left panel) and ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (solid) and ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$) (dashed) (right panel).*

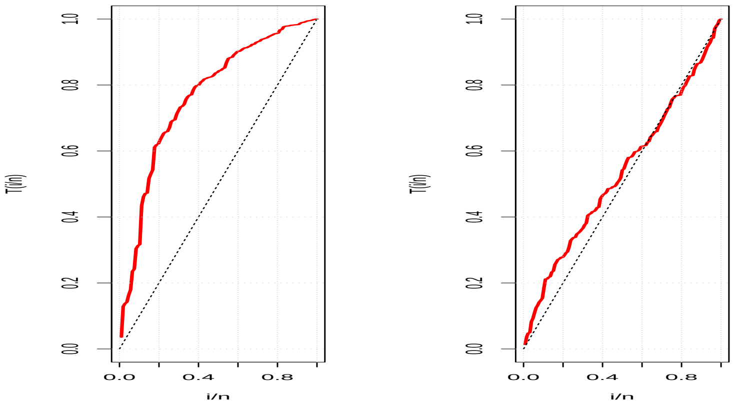

Figure 7: *TTT plot for the UIEW distribution for data I (left panel) and data II (right panel).*

4.3. LS and WLS estimation

Here, the LS and the WLS (Swain *et al.*, 1988) for estimating the unknown parameters are regarded. Let $z_{(1)}, \ldots, z_{(n)}$ an ordered sample of size n is drawn from UIEW distribution (2.2), so the LS estimators of δ, ε, and ϕ for the UIEW distribution, denoted by $\hat{\delta}_{\mathrm{LS}}, \hat{\varepsilon}_{\mathrm{LS}}$, and $\hat{\phi}_{\mathrm{LS}}$ respectively, are obtained by minimizing the following function:

$$\mathrm{B}(\vartheta) = \sum_{i=1}^{n} S_i \left[H\left(z_{(i)}, \vartheta\right) - E\left(H\left(z_{(i-1)}, \vartheta\right)\right)\right]^2, \tag{4.7}$$

with respect to ϑ, where, $E(H(z_{(i-1)}, \vartheta)) = i/(n+1), i = 1, 2, \ldots, n$. We can get the LS estimators designated by $\hat{\delta}_{\mathrm{LS}}, \hat{\varepsilon}_{\mathrm{LS}}$, and $\hat{\phi}_{\mathrm{LS}}$ by setting $S_i = 1$, whereas we can get the WLS estimators denoted by $\hat{\delta}_{\mathrm{WLS}}, \hat{\varepsilon}_{\mathrm{WLS}}$, and $\hat{\phi}_{\mathrm{WLS}}$ by setting $S_i = ((n+2)(n+1)^2)/i(n-i+1)$, $i = 1, 2, \ldots, n$. These

Table 6: ML estimates, SE, CAIC, BIC, HQIC and KS with its p-value for the both data

Data	Model	$\hat{\delta}_{ML}$	$\hat{\varepsilon}_{ML}$	$\hat{\phi}_{ML}$	AIC	CAIC	HQIC	KS (p-value)
I	UIEW	0.6928 (0.1855)	2.0870 (0.7614)	8.9635 (7.0311)	**−50.6131**	**−50.3801**	**−47.3625**	**0.0595 (0.8216)**
	Kum	2.1949 (0.2224)	3.4363 (0.5820)	-	−46.7894	−46.6740	−44.6223	0.0763 (0.5372)
	Beta	2.4125 (0.3145)	2.8297 (0.3744)	-	−43.5545	−43.4391	−41.3874	0.0910 (0.3189)
	UG	2.1194 (0.8675)	0.3877 (0.1144)	-	−6.9774	−6.8620	−4.81035	0.1835 (0.0013)
	UL	0.5405 (0.0324)	44.2958 (42.9907)	-	3.8927	4.0081	6.0597	0.2461 (0.000004)
	CUW	0.4738 (0.0214)	1.7809 (0.1355)	-	−51.0080	−50.8926	−48.8410	0.0663 (0.7343)
II	UIEW	0.5727 (0.0532)	8.3027 (0.4928)	91.3615 (33.433)	**−302.7727**	**−302.576**	**−299.3159**	**0.0573 (0.8028)**
	Kum	1.0544 (0.0801)	8.8952 (1.5727)	-	−300.2396	−300.142	−297.935	0.0809 (0.3813)
	Beta	1.1412 (0.1282)	9.0719 (1.2295)	-	−301.1047	−301.007	−298.8001	0.0832 (0.3477)
	UG	0.1386 (0.0362)	0.6857 (0.0510)	-	−272.4598	−272.362	−270.1552	0.1214 (0.0488)
	UL	0.1634 (0.0266)	112.6454 (72.5011)	-	−162.1871	−162.089	−159.8825	0.3381 (6×10^{-3})
	CUW	0.0846 (0.0082)	1.0264 (0.0667)	-	−299.9292	−299.831	−297.6246	0.0779 (0.4285)

estimators can also be obtained by solving the equations stated below:

$$\frac{\partial \mathrm{B}(\vartheta)}{\partial \delta} = 2\sum_{i=1}^{n} S_i \left[H(z_{(i)}, \vartheta) - \frac{i}{n+1} \right]^2 \frac{\partial}{\partial \delta} H(z_{(i)}, \vartheta) = 0,$$

$$\frac{\partial \mathrm{B}(\vartheta)}{\partial \varepsilon} = 2\sum_{i=1}^{n} S_i \left[H(z_{(i)}, \vartheta) - \frac{i}{n+1} \right]^2 \frac{\partial}{\partial \varepsilon} H(z_{(i)}, \vartheta) = 0,$$

$$\frac{\partial \mathrm{B}(\vartheta)}{\partial \phi} = 2\sum_{i=1}^{n} S_i \left[H(z_{(i)}, \vartheta) - \frac{i}{n+1} \right]^2 \frac{\partial}{\partial \phi} H(z_{(i)}, \vartheta) = 0,$$

where $\partial/\partial\delta\ H(z_{(i)}, \vartheta), \partial/\partial\varepsilon\ H(z_{(i)}, \vartheta)$ and $\partial/\partial\phi\ H(z_{(i)}, \vartheta)$ are given in (4.4)-(4.6). As can be observed, the resultant equations cannot be solved analytically, hence a numerical method will be used with non-linear optimization techniques.

4.4. Anderson–Darling estimation

The AD method was introduced by Anderson and Darling (1952). Let $z_{(1)}, \ldots, z_{(n)}$ an ordered sample of size n is drawn from UIEW distribution (2.2), so the estimators $\hat{\delta}_{\mathrm{AD}}, \hat{\varepsilon}_{\mathrm{AD}}$, and $\hat{\phi}_{\mathrm{AD}}$ of δ, ε, and ϕ respectively, are obtained by minimizing the following function:

$$\mathrm{AD}(\vartheta) = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left[\log H(z_{(i)}, \vartheta) + \log \left(H(1 - z_{(n+1-i)}, \vartheta) \right) \right],$$

with respect to ϑ. Hence $\hat{\delta}_{\mathrm{AD}}, \hat{\varepsilon}_{\mathrm{AD}}$, and $\hat{\phi}_{\mathrm{AD}}$ are solutions for the following equations:

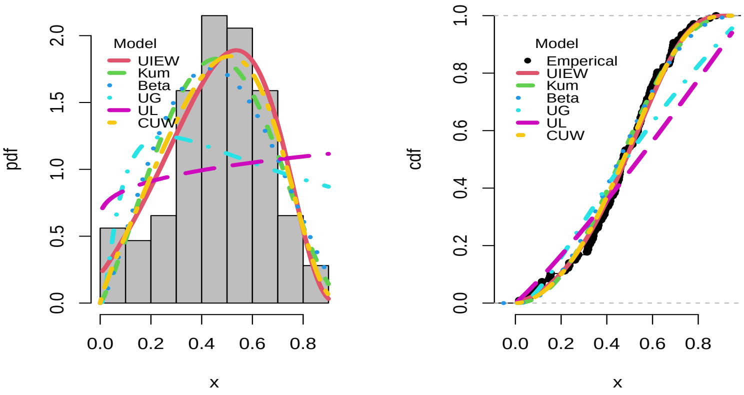

Figure 8: *Plots of the estimated pdfs (left panel) and estimated cdfs (right panel) for the data I.*

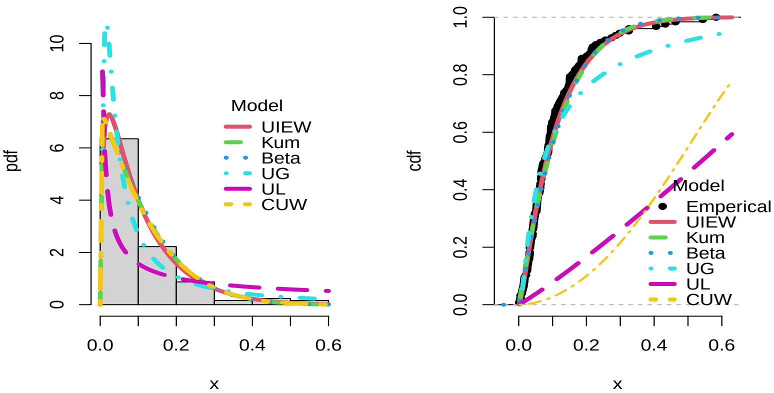

Figure 9: *Plots of the estimated pdfs (left panel) and estimated cdfs (right panel) for the data II.*

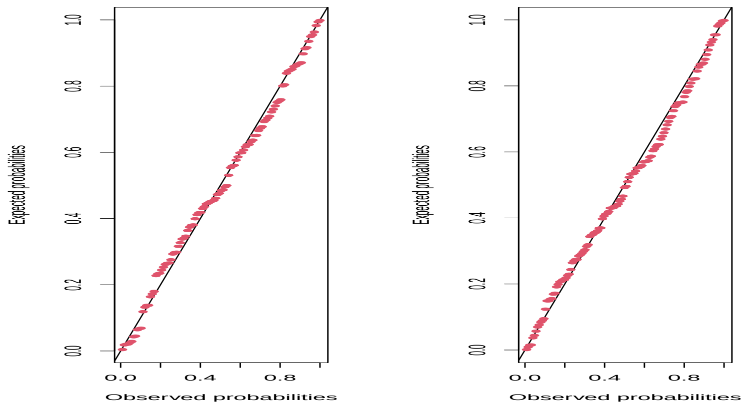

Figure 10: *PP plots for the UIEW distribution for data I (left panel) and data II (right panel).*

$$\frac{\partial \text{AD}(\vartheta)}{\partial \delta} = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\frac{(\partial/\partial\delta) H(z_{(i)}, \vartheta)}{H(z_{(i)}, \vartheta)} - \frac{(\partial/\partial\delta)(H(1-z_{(n+1-i)}, \vartheta))}{H(1-z_{(n+1-i)}, \vartheta)} \right] = 0,$$

$$\frac{\partial \text{AD}(\vartheta)}{\partial \varepsilon} = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\frac{(\partial/\partial\varepsilon) H(z_{(i)}, \vartheta)}{H(z_{(i)}, \vartheta)} - \frac{(\partial/\partial\varepsilon)(H(1-z_{(n+1-i)}, \vartheta))}{H(1-z_{(n+1-i)}, \vartheta)} \right] = 0,$$

$$\frac{\partial \text{AD}(\vartheta)}{\partial \phi} = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\frac{(\partial/\partial\phi) H(z_{(i)}, \vartheta)}{H(z_{(i)}, \vartheta)} - \frac{(\partial/\partial\phi)(H(1-z_{(n+1-i)}, \vartheta))}{H(1-z_{(n+1-i)}, \vartheta)} \right] = 0,$$

where $\partial/\partial\delta\ H(z_{(i)}, \vartheta), \partial/\partial\varepsilon\ H(z_{(i)}, \vartheta)$, and $\partial/\partial\phi\ H(z_{(i)}, \vartheta)$ are given in (4.4)-(4.6). Due to the difficulty of solving the resulting equations, non-linear optimization strategies will be used.

4.5. The Cramér-von Mises estimation

Let $z_{(1)}, \ldots, z_{(n)}$ an ordered sample of size n is drawn from UIEW distribution (2.2), hence the CM estimators $\hat{\delta}_{\text{CM}}, \hat{\varepsilon}_{\text{CM}}$, and $\hat{\phi}_{\text{CM}}$ of δ, ε, and ϕ, respectively, are obtained by minimizing the following function:

$$\text{CM}(\vartheta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[H(z_{(i)}, \vartheta) - \frac{2i-1}{2n} \right]^2,$$

with respect to ϑ. Hence $\hat{\delta}_{\text{CM}}, \hat{\varepsilon}_{\text{CM}}$, and $\hat{\phi}_{\text{CM}}$ are obtained by solving simultaneously the following non-linear equations after putting them with zero

$$\frac{\partial \text{CM}(\vartheta)}{\partial \delta} = 2\sum_{i=1}^{n} \frac{\partial}{\partial \delta} (H(z_{(i)}, \vartheta)) \left[H(z_{(i)}, \vartheta) - \frac{2i-1}{2n} \right] = 0,$$

$$\frac{\partial \text{CM}(\vartheta)}{\partial \varepsilon} = 2\sum_{i=1}^{n} \frac{\partial}{\partial \varepsilon} (H(z_{(i)}, \vartheta)) \left[H(z_{(i)}, \vartheta) - \frac{2i-1}{2n} \right] = 0,$$

$$\frac{\partial \text{CM}(\vartheta)}{\partial \phi} = 2\sum_{i=1}^{n} \frac{\partial}{\partial \phi} (H(z_{(i)}, \vartheta)) \left[H(z_{(i)}, \vartheta) - \frac{2i-1}{2n} \right] = 0,$$

where $\partial/\partial\delta\ H(z_{(i)}, \vartheta), \partial/\partial\varepsilon\ H(z_{(i)}, \vartheta)$, and $\partial/\partial\phi\ H(z_{(i)}, \vartheta)$ are provided in (4.4)-(4.6). non-linear optimization techniques will be applied due to the complexity of the resultant equations.

4.6. Bayesian estimators

The Bayesian estimator of the UIEW distribution parameters is provided here. The Bayesian estimators of δ, ε and ϕ are taken into consideration, under the squared error loss function (SELF), which is defined by the following formulas,

$$L(\tilde{\delta}, \delta) = (\tilde{\delta} - \delta)^2, \quad L(\tilde{\varepsilon}, \varepsilon) = (\tilde{\varepsilon} - \varepsilon)^2, \quad L(\tilde{\phi}, \phi) = (\tilde{\phi} - \phi)^2.$$

Suppose that the prior distributions of parameters have an independent gamma distribution. The joint gamma prior density of δ, ε and ϕ can be written as

$$\pi_0(\delta, \varepsilon, \phi) \propto \delta^{c_1-1} \varepsilon^{c_2-1} \phi^{c_3-1} e^{-d_1\delta} e^{-d_3\phi} e^{-d_2\varepsilon}; \; c_i, d_i > 0, \quad i = 1, 2, 3. \tag{4.8}$$

Table 7: The ML, MPS, LS, WLS, AD, CM and Bayesian estimates for both data sets

	Method	$\hat{\delta}$	$\hat{\varepsilon}$	$\hat{\phi}$	KS
Data I	ML	0.6928	2.0870	8.9635	0.0595
	MPS	0.6520	2.1547	9.4299	0.0691
	LS	0.8072	1.8464	7.2607	0.0468
	WLS	0.8778	1.5611	5.1940	0.0499
	AD	0.8343	1.6756	5.8797	0.0508
	CM	0.8083	1.8753	7.5551	0.0499
	Bayesian	0.7166	2.0227	8.6662	0.9986
Data II	ML	0.5727	8.3027	91.3615	0.0573
	MPS	......	......	......	......
	LS	0.6450	8.8541	87.6791	0.0353
	WLS	0.6557	8.4966	69.9389	0.0407
	AD	0.6269	8.5694	82.8390	0.0428
	CM	0.6596	8.7633	78.4349	0.0352
	Bayesian	0.5789	8.2724	91.4097	0.0672

The joint posterior of the UIEW distribution with parameters δ, ε and ϕ is obtained using the likelihood function and joint prior density (4.8) as:

$$\pi^{\bullet}\left(\delta, \varepsilon, \phi \mid \underline{z}\right) \propto \pi_0\left(\delta, \varepsilon, \phi\right) L\left(\underline{z} \mid \delta, \varepsilon, \phi\right).$$

Then the joint posterior can be written as

$$\pi^{\bullet}\left(\delta, \varepsilon, \phi \mid \underline{z}\right) \propto \delta^{c_1+n-1}\varepsilon^{c_2+n-1}\phi^{c_3+n-1}e^{-\delta\left(d_1+\sum_{i=1}^{n}\ln(-\ln z_i)\right)}e^{-\varepsilon\left(d_2+\sum_{i=1}^{n}(-\ln z_i)^{-\delta}\right)}e^{-d_3\phi+(\phi-1)\sum_{i=1}^{n}\ln\left[1-e^{-z(-\ln z)^{-\delta}}\right]}.$$

The following processes result in the conditional posterior densities of δ, ε and ϕ

$$\pi_1^{\bullet\bullet}\left(\delta \mid \underline{z}\right) = K^{-1}\delta^{n+c_1-1}e^{-\delta\left(d_1+\sum_{i=1}^{n}\ln(-\ln z_i)\right)} \times \int_0^{\infty}\int_0^{\infty}\phi^{n+c_3-1}\varepsilon^{n+c_2-1}e^{-\varepsilon\left(d_2+\sum_{i=1}^{n}(-\ln z_i)^{-\delta}\right)}e^{-d_3\phi+(\phi-1)\sum_{i=1}^{n}\ln\left[1-e^{-z(-\ln z)^{-\delta}}\right]}d\phi\, d\varepsilon, \tag{4.9}$$

$$\pi_2^{\bullet\bullet}\left(\varepsilon \mid \underline{z}\right) = K^{-1}\varepsilon^{n+c_2-1}e^{-\varepsilon d_2} \times \int_0^{\infty}\int_0^{\infty}\delta^{n+c_1-1}\phi^{n+c_3-1}e^{-\delta\left(d_1+\sum_{i=1}^{n}\ln(-\ln z_i)\right)}e^{-\varepsilon\sum_{i=1}^{n}(-\ln z_i)^{-\delta}}e^{-c_3\phi+(\phi-1)\sum_{i=1}^{n}\ln\left[1-e^{-z(-\ln z)^{-\delta}}\right]}d\phi\, d\delta, \tag{4.10}$$

$$\pi_3^{\bullet\bullet}\left(\phi \mid \underline{z}\right) = K^{-1}\phi^{n+c_3-1}e^{-d_3\phi} \times \int_0^{\infty}\int_0^{\infty}\delta^{n+c_1-1}\varepsilon^{n+c_2-1}e^{-\delta\left(d_1+\sum_{i=1}^{n}\ln(-\ln z_i)\right)}e^{-\varepsilon\left(d_2+\sum_{i=1}^{n}(-\ln z_i)^{-\delta}\right)}e^{(\phi-1)\sum_{i=1}^{n}\ln\left[1-e^{-z(-\ln z)^{-\delta}}\right]}d\delta\, d\varepsilon, \tag{4.11}$$

where

$$K = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\delta^{n+c_1-1}\varepsilon^{n+c_2-1}\phi^{n+c_3-1} \times e^{-\delta\left(d_1+\sum_{i=1}^{n}\ln(-\ln z_i)\right)}e^{-\varepsilon\left(d_2+\sum_{i=1}^{n}(-\ln z_i)^{-\delta}\right)}e^{-d_3\phi+(\phi-1)\sum_{i=1}^{n}\ln\left[1-e^{-z(-\ln z)^{-\delta}}\right]}d\phi\, d\varepsilon\, d\delta.$$

Equations (4.9)-(4.11) are hard to be solved analytically, hence the Markov chain Monte Carlo (MCMC) method can be used to obtain such Bayesian estimates numerically based on SELF.

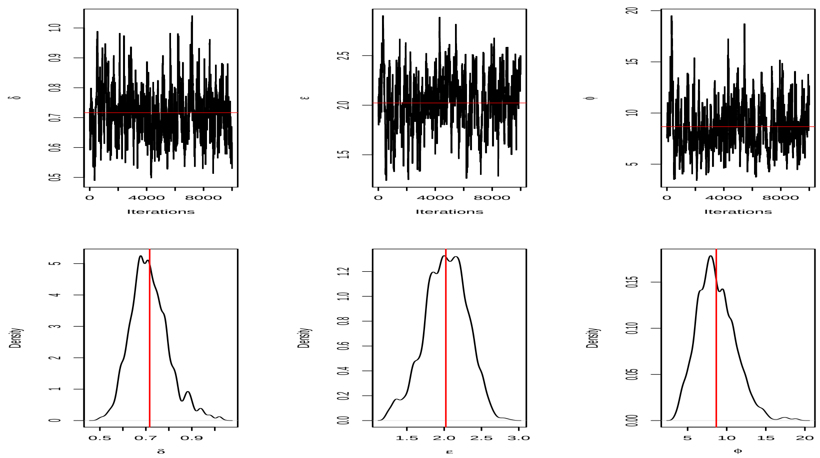

Figure 11: *MCMC trace (upper panel) and posterior distribution (lower panel) of the UIEW parameters for data I.*

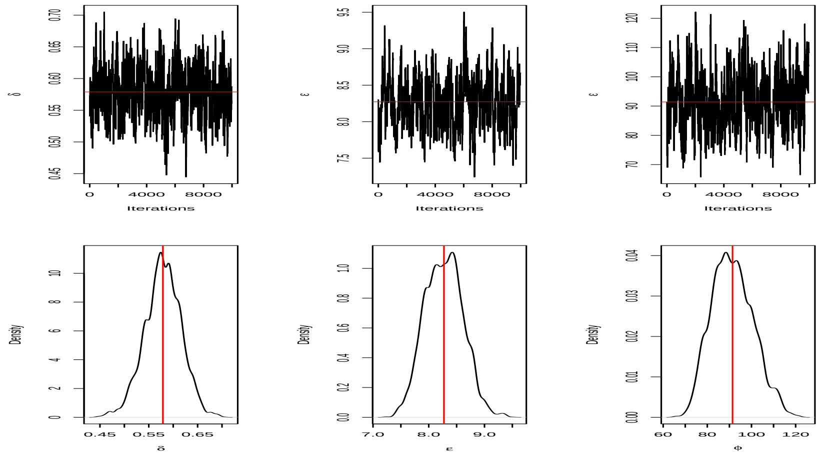

Figure 12: *MCMC trace (upper panel) and posterior distribution (lower panel) of the UIEW parameters for data II.*

4.7. Simulation study

Here, the ML, MPS, LS, WLS, AD, and CM estimates of the UIEW distribution are calculated numerically by using non-linear optimization algorithms to examine their performance with varying sample size n. For each method, one thousand random samples of sizes $n = 100, 200$, and 300 are generated from the UIEW distribution. The true parameter values are (a) ($\delta = 1.5, \varepsilon = 1, \phi = 1$) (b) ($\delta = 1, \varepsilon = 1, \phi = 1.5$) and (c) ($\delta = 1.5, \varepsilon = 1, \phi = 1.5$). For the ML, MPS, LS, WLS, AD, and CM estimates, the average of each parameter estimate is calculated by computing the mean of the one thousand replicates. Moreover, MSE and the bias of each estimate were calculated as well. The MSE of each estimate is calculated by computing the mean of squares of the differences of the one thousand replicates of the estimates from the true value of the parameters, whereas the estimate of the bias is calculated by computing the difference between the average estimate and the true value of the parameter.

The MCMC method is used to generate Bayesian estimates. The MCMC approaches like Gibbs sampling and the more general metropolis within Gibbs samplers are crucial. Two well-known MCMC applications are the metropolis hastings (MH) algorithm and Gibbs sampling. In this context, samples of δ, ε, and ϕ can be easily generated by their joint posterior distribution using the metropolis (M) algorithm which is a special case of the MH algorithm when the proposal function is symmetric (Metropolis *et al.*, 1953; Robert, 2015). The M algorithm is as follows:

Step 1 : Initialize starting state $(\delta^{\circ}, \varepsilon^{\circ}, \phi^{\circ})$ and set $t = 1$, where $t = 1$: N (number of iterations).

Step 2 : Sample proposal values $\delta^{*}, \varepsilon^{*}$, and ϕ^{*} from symmetric proposal distributions which are normal for δ, and ε, and uniform for ϕ.

Step 3 : Compute acceptance probability r as follows

$$r = \frac{\pi^{\bullet}\left(\delta^{*}, \varepsilon^{*}, \phi^{*} \mid \underline{z}\right)}{\pi^{\bullet}\left(\delta^{t-1}, \varepsilon^{t-1}, \phi^{t-1} \mid \underline{z}\right)}.$$

Step 4 : Generate random value u from uniform $(0, 1)$.

Step 5 : Compare u and $\min(r, 1)$: If $u < \min(r, 1)$, then $\delta^{t+1} = \delta^{*}$, $\varepsilon^{t+1} = \varepsilon^{*}$ and $\phi^{t+1} = \phi^{*}$, otherwise $\delta^{t+1} = \delta^{t}$, $\varepsilon^{t+1} = \varepsilon^{t}$ and $\phi^{t+1} = \phi^{t}$,

Step 6 : $t = t + 1$, and repeat Steps 2–5 until reach the needed iterations number which is N in this context.

Step 7 : We obtain $(\delta^{1}, \delta^{2}, \ldots, \delta^{T})$, $(\varepsilon^{1}, \varepsilon^{2}, \ldots, \varepsilon^{T})$, and $(\phi^{1}, \phi^{2}, \ldots, \phi^{T})$.

Step 8 : The Bayes estimates of δ, ε, and ϕ under SELF are the average of $(\delta^{1}, \delta^{2}, \ldots, \delta^{T})$, $(\varepsilon^{1}, \varepsilon^{2}, \ldots, \varepsilon^{T})$, and $(\phi^{1}, \phi^{2}, \ldots, \phi^{T})$, in Step 7 respectively.

Step 9 : The bias and MSE are computed as previously mentioned.

The MSE and bias of each estimate utilising various methodologies are provided in Tables 3–5 for different values of parameters.

Tables 3–5 show that the MSE and biasedness of all estimates decrease with increasing sample size as expected except for the Bayesian method there is no clear pattern in the biasedness. In addition, according to the simulation studies, the amount of MSEs of the ML, LS, WLS, AD, CM, and MPS estimates are bigger than those of Bayesian estimates except possibly for some . Hence, we conclude that the Bayesian method performs quite well in estimating the model parameters for these sample sizes. Furthermore, Figures 2–6 illustrate the following observations:

- The ML, MPS, WLS methods produce larger MSE(δ) with increasing δ and decreasing ϕ, whereas there is no change when applying AD method.

- If δ increases and ϕ decreases, MSE($\hat{\varepsilon}$) decreases except for AD method, there is no change.

- All methods of estimation produce larger MSE($\hat{\phi}$) with increasing ϕ and decreasing δ except AD method, there is no change and Bayesian method when the sample size is 300.
- If δ is fixed, all methods of estimation produce larger MSE($\hat{\phi}$) with increasing ϕ.
- If ϕ is fixed and δ increases, Bias($\hat{\delta}$) increases for sample sizes 200 and 300 .
- If δ is fixed and ϕ increases, Bias($\hat{\delta}$) increases for LS, AD and CM.
- If δ is fixed and ϕ increases, Bias($\hat{\varepsilon}$) increases for ML, MPS and WLS whereas decreases for Bayesian method.
- If δ increases and ϕ decreases, Bias($\hat{\varepsilon}$) decreases for ML, MPS, and WLS and increases for Bayesian method, whereas it does not change for AD.
- If ϕ increases and δ decreases, Bias($\hat{\phi}$) increases except for Bayesian method. Furthermore, there is no change in Bias($\hat{\phi}$) for AD method.
- If δ is fixed and ϕ increases, Bias($\hat{\phi}$) increases for all methods except for Bayesian method.
- A negative relationship between ϕ and δ suggests no change in the MSE of $\hat{\delta}$, $\hat{\varepsilon}$, and $\hat{\phi}$. In addition, there is no change in the biasedness of $\hat{\varepsilon}$ and $\hat{\phi}$.

5. Data analysis

This section will analyze two actual data sets to demonstrate the applicability of the suggested distribution. The first application is about the proportion of total milk production in the first birth of 107 cows (Carnaúba farm, Brazil, Cordeiro and dos Santos (2012)). Whereas the second data set represents remission times (in months) of a random sample of 126 bladder cancer patients reported in Lee and Wang (2003). The two data sets are converted to the interval (0, 1) by applying the transformation (data-data$_{\min}$)/(data$_{\max}$-data$_{\min}$). For a comparison, we shall consider five models in addition to the UIEW, namely, Kumaraswamy (Kum), beta, unit Gompertz (UG), unit Lomax (UL) and the complementary unit weibull distribution (CUW). We compute the estimate of parameters for all models accompanied with their standard errors (SEs), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC). Moreover, the values for the Kolmogorov Smirnov (KS) statistic and its p-value are presented.

Figure 7 shows the total time on test (TTT) graph for data sets I and II. We can detect a concave curve in both TTT graphs, which suggests that the hazard function behind the data could be increasing. For some values, this specificity also applies to the hazard function for the UIEW distribution, which supports its consideration for these data sets (for more information on the TTT plots, see (Aarset, 1987)). It is shown from Table 6 below that the UIEW distribution is the best model for data I and II since all its statistics values are smaller than the others. This result is confirmed in Figures 8 and 9. Additionally, Figure 10 includes the PP-graphs of the UIEW model for both sets of data. The plots of PP therefore closely fit the proposed model. Further, as shown in Table 7 and using both real data, several approaches are employed to estimate the UIEW parameters. Due to several observations' equivalence in data II, the MPS method is not applicable. The results in Table 7 indicate that all estimates provide a good fit to data I. In addition, it is observed that the LS has the lowest value of KS. For data II, the CM and LS give the lowest value of KS. Figures 11 and 12 illustrated trace plots of the posterior distributions of the parameters to track the convergence of the MCMC outputs. Additionally, they display the histograms for the marginal posterior density estimates of the parameters to show how well the MCMC process converges.

6. Summary and conclusion

In probability theory and statistics, relevant variables with values between zero and one are often represented by unit distributions. The unit-inverse exponentiated weibull distribution, which is useful for modelling data on the unit interval, is proposed in this study. The characteristics of this distribution, such as quantile function, moments, incomplete moments, uncertainty measures, stochastic ordering, and stress-strength reliability are provided. Utilizing Bayesian and several classical techniques to the suggested distribution are calculated. The classical methods include ML, MPS, LS, WLS, AD, and CM. We assessed the performance of different estimates in terms of their bias and MSE. The results of the simulation show that the MPS method performs well most of the time. The Bayesian method performs quite well in estimating the model parameters for these sample sizes. The MSE reduced for each estimate as the sample size grew, indicating demonstrating the consistency of the estimates. Two real data applications demonstrate that the UIEW model frequently offers superior fits than some other competing models.

Acknowledgement

We gratefully acknowledge the editor and referees for their meaningful suggestions and comments relating to the improvement of the paper.

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Received September 29, 2022; Revised December 02, 2022; Accepted January 03, 2023