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TWO LINEAR POLYNOMIALS SHARED BY AN ENTIRE FUNCTION AND ITS LINEAR DIFFERENTIAL POLYNOMIALS

Goutam Kumar Ghosh

Abstract. In this paper, we study a uniqueness problem of entire functions that share two linear polynomials with its linear differential polynomial. We deduce two theorems which improve some previous results given by I. Lahiri [7].

1. Introduction, definitions and results

Let f and g be two nonconstant entire functions, and let a be a polynomial. We denote by $E(a; f)$ the set of zeros of $f-a$ counted with multiplicities and by $\overline{E}(a; f)$ the set of distinct zeros of $f-a$. Then we say that f and g share a CM (counting multiplicities) if $E(a; f) = E(a; g)$. Also we say that f and g share a IM (ignoring multiplicities) if $\overline{E}(a; f) = \overline{E}(a; g)$. For standard definitions and results we refer the reader to [4]. By meromorphic functions we shall always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [13]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ as $r \to \infty$ and $r \notin E$.

In 1977, L. A. Rubel and C. C. Yang [11] first investigated the uniqueness of entire functions, which share certain values with their derivatives. They proved that if a non-constant entire function f and its first derivative $f^{(1)}$ share two distinct finite numbers a, b CM, then $f \equiv f^{(1)}$. Considering $f(z) = e^{e^z} \int_0^z e^{-e^t}(1-e^t)dt$ [13, p. 386], one can easily verify that sharing of two values is essential.

In 1979, E. Mues and N. Steinmetz [10] improved the result of Rubel and Yang [11] replacing CM shared values by IM shared values. In 1990, Yang [12] extended the result of Rubel and Yang to any k^{th} order derivative $f^{(k)}$ of the

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entire function f. In 2000, Li and Yang [9] improved the result of Yang [12] and settled a conjecture of Frank [2] affirmatively. Their result can be stated as follows.

Theorem 1.1 ([9]). *Let f be a nonconstant entire function, k be a positive integer and a and b two distinct finite values. If f and $f^{(k)}$ share a and b IM, then $f \equiv f^{(k)}$.*

The natural extension of a derivative of an entire function f is a linear differential polynomial generated by f. In 1994, Gu [3] extended the result of Rubel and Yang [11] to a linear differential polynomial. The result of Gu stated as follows:

Theorem 1.2 ([3]). *Let f be a nonconstant entire function a and b be two distinct finite complex numbers and $L(f) = f^{(n)} + a_1 f^{(n-1)} + \cdots + a_n f$, where a_j $(j = 1, 2, \ldots, n)$ are small entire functions of f. If f and $L(f)$ share a and b CM and $a + b \neq 0$ or $a_n \not\equiv -1$, then $f \equiv L(f)$.*

The following theorem of Bernstein et al. [1] is an improvement of Theorem 1.2.

Theorem 1.3 ([1]). *Let f be a nonconstant entire function, a and b be two distinct finite complex numbers and $L(f) = b_n f^{(n)} + b_{n-1} f^{(n-1)} + \cdots + b_1 f^{(1)} + b_0 f$, where b_j $(j = 0, 1, 2, \ldots, n)$ are small meromorphic functions of f. If f and $L(f)$ share a and b CM, then $f \equiv L(f)$.*

In contrast to the derivative of an entire function, we see in the following examples that it is not possible in the case of a linear differential polynomial to replace any CM shared value by an IM shared value.

Example 1.4. Let $f = 1 + (e^z - 1)^2$ and $L(f) = \frac{1}{2} f^{(2)} - f^{(1)}$. Then f and $L(f)$ share 1 IM and 2 CM but $f \not\equiv L(f)$.

Example 1.5 ([8]). Let $f = \frac{1}{2} e^z + \frac{1}{2} e^{-z}$ and $L(f) = f^{(2)} + f^{(1)}$. Then f and $L(f)$ share 1 and -1 IM but $f \not\equiv L(f)$.

Although one IM shared value and one CM shared value cannot ensure the equality of an entire function with a linear differential polynomial generated by it, Li and Yang [8] exhibited two possibilities in the following theorem.

Theorem 1.6 ([8]). *Let f be a nonconstant entire function and*

$$L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}, \tag{1.1}$$

where b_j $(j = -1, 0, 1, \ldots, n)$ are small meromorphic functions of f. Let a and b be two distinct finite values. If f and $L(f)$ share a CM and b IM, then either $f \equiv L(f)$ or f and $L(f)$ have the following forms: $f = b + (a - b)(e^{\alpha} - 1)^2$ and $L(f) = b + (a - b)(e^{\alpha} - 1)$, where α is an entire function.

If we look at the above theorem, then we see that in the case of non-equality of f and $L(f)$, almost all the b-points of f and $L(f)$ are double and simple, respectively, whereas the a-points of f and $L(f)$ are almost all simple. In fact, we shall show that the simple a-points and b-points of f play a role to ascertain the equality of f and $L(f)$. Also, we shall see that the simple a-points of f still play a crucial role even if the other value b shared IM. To this end, we need the following idea of value sharing.

Let $A \subset \mathbb{C}$ and k be a nonnegative integer or infinity. We denote by $E_k(a; f, A)$ the collection of those a-points of f that belong to A, where an a-point of f with multiplicity p is counted p times if $p \leq k$ and $k+1$ times if $p \geq k+1$.

Also by $\overline{N}_A(r, a; f)$ we denote the reduced counting function of those a-points of f that lie in A. We now put $A = \overline{E}(a; f) \cap \overline{E}(a; g)$ and $B = \overline{E}(a; f) \Delta \overline{E}(a; g)$, where Δ denotes the symmetric difference of sets.

We shall say that f and g share the value a with weight k in the weak sense, written symbolically f, g share $(a, k)^*$, if $E_k(a; f, A) = E_k(a; g, A)$ and $\overline{N}_B(r, a; f) = S(r, f)$ and $\overline{N}_B(r, a; g) = S(r, g)$.

It is clear that if f, g share $(a, k)^*$, then f, g share $(a, p)^*$ for every integer p with $0 \leq p < k$. Further f, g share $(a, 0)^*$ if and only if f, g share the value a IM^* and f, g share the value a CM^* if f, g share $(a, \infty)^*$. For the definitions of IM^* and CM^* we refer to [8]. We further note that the notion of weighted sharing in the weak sense coincides with that of weighted sharing (see [5,6] for the definition) if $B = \emptyset$.

If $a = a(z)$ is a small function of f and g, then we shall say that f, g share $(a, k)^*$ if $f - a$ and $g - a$ share $(0, k)^*$.

In 2018, I. Lahiri [7] improved Theorem 1.6 in the following manner.

Theorem 1.7 ([7])**.** *Let f be a nonconstant entire function and $L(f)$ be defined by* (1.1)*. Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share $(a, 1)^*$ and $(b, 1)^*$, then $f \equiv L(f)$.*

In the same paper I. Lahiri [7] gave another theorem.

Theorem 1.8 ([7])**.** *Let f be a nonconstant entire function and $L(f)$ be defined by* (1.1)*. Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share $(a, 1)^*$ and $(b, 0)^*$, then the conclusion of Theorem 1.6 holds.*

We now state the main results of the paper which improve Theorem 1.7 and Theorem 1.8 by considering shared linear polynomials instead of shared values.

Theorem 1.9. *Let f be a transcendental entire function and $L(f)$ be defined by* (1.1)*. Suppose that $a(z) = \alpha z + \beta_1$, $b(z) = \alpha z + \beta_2$ are two distinct linear polynomials where, $\alpha(\neq 0), \beta_1(\neq \beta_2)$ are constants. If f and $L(f)$ share $(a, 1)^*$ and $(b, 1)^*$, then $f \equiv L(f)$.*

Theorem 1.10. *Let f be a transcendental entire function and $L(f)$ be defined by* (1.1)*. Suppose that $a(z) = \alpha z + \beta_1$, $b(z) = \alpha z + \beta_2$ are two distinct linear* *polynomials, where $\alpha(\neq 0), \beta_1(\neq \beta_2)$ are constants. If f and $L(f)$ share $(a,1)^*$ and $(b,0)^*$, then the conclusion of Theorem 1.6 holds.*

Li and Yang [8] exhibited by an example that Theorem 1.6 is not valid for meromorphic functions. However, they proved the following extension of Theorem 1.6.

Theorem 1.11 ([8])**.** *Let f be a nonconstant meromorphic function with $N(r,f) = S(r,f)$ and $L(f)$ be defined by (1.1). Let $a(\not\equiv \infty)$ and $b(\not\equiv \infty)$ be two distinct small functions of f. If f and $L(f)$ share a CM^* and b IM^*, then either $f \equiv L(f)$ or f and $L(f)$ have the following forms: $f = b+(a-b)(e^\alpha - 1)^2$ and $L(f) = b + (a-b)(e^\alpha - 1)$, where α is an entire function.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. *Let f and g be two transcendental entire functions sharing $(a,0)^*$, $(b,0)^*$ and $(\infty,0)^*$, where $a(z) = \alpha_1 z + \beta_1$, $b(z) = \alpha_2 z + \beta_2$ are two distinct linear polynomials and $\alpha_1(\neq 0), \beta_1, \alpha_2(\neq 0), \beta_2$ are constants. Then*

$$T(r,f) \leq 3T(r,g) + S(r,f)$$

and

$$T(r,g) \leq 3T(r,f) + S(r,g).$$

The lemma is a consequence of the Second Fundamental Theorem.

Note. Lemma 2.1 implies that $S(r,f) = S(r,g)$.

Lemma 2.2. *Let f be a transcendental entire function and $L(f)$ be defined by (1.1). Let $a(z) = \alpha z + \beta_1$, $b(z) = \alpha z + \beta_2$ be two distinct linear polynomials, where $\alpha(\neq 0), \beta_1(\neq \beta_2)$ are constants. If f and $L(f)$ share $(a,0)^*$ and $(b,0)^*$, then*

$$T(r,f) = \overline{N}(r,a;f) + \overline{N}(r,b;f) + S(r,f)$$

provided $f \not\equiv L(f)$.

Proof. Let $\phi = \frac{(f^{(1)}-\alpha)(f-L)}{(f-a)(f-b)}$. By Lemma 2.1, $S(r,L) = S(r,f)$. We suppose that $f \not\equiv L$. Then by the hypothesis $N(r,\phi) = S(r,f)$. Since

$$\begin{aligned}\phi = &\frac{1-b_0}{a-b}\left(\frac{a(f^{(1)}-\alpha)}{f-a} - \frac{b(f^{(1)}-\alpha)}{f-b}\right) - \frac{b_{-1}+b_1\alpha}{a-b}\left(\frac{f^{(1)}-\alpha}{f-a} - \frac{f^{(1)}-\alpha}{f-b}\right)\\ &- \frac{f^{(1)}-\alpha}{f-a}\left(\frac{b_1(f^{(1)}-\alpha)}{f-b} - \frac{b_2 f^{(2)}}{f-b} - \cdots - \frac{b_n f^{(n)}}{f-b}\right),\end{aligned}$$

from the lemma of logarithmic derivative we see that $m(r,\phi) = S(r,f)$ and so $T(r,\phi) = S(r,f)$. We have

$$T(r, f-L) = T\left(r, \frac{\phi(f-a)(f-b)}{f^{(1)}-\alpha}\right)$$

$$
\begin{aligned}
&= T\left(r, \frac{f^{(1)}-\alpha}{(f-a)(f-b)}\right) + S(r,f)\\
&= T\left(r, \frac{1}{a-b}\left(\frac{f^{(1)}-\alpha}{f-a} - \frac{f^{(1)}-\alpha}{f-b}\right)\right) + S(r,f)\\
&= N\left(r, \frac{1}{a-b}\left(\frac{f^{(1)}-\alpha}{f-a} - \frac{f^{(1)}-\alpha}{f-b}\right)\right) + S(r,f)\\
&= N\left(r, \frac{f^{(1)}-\alpha}{f-a}\right) + N\left(r, \frac{f^{(1)}-\alpha}{f-b}\right) + S(r,f)\\
&= \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f).
\end{aligned}
$$

From the expression of L, it is clear that $T(r, f-L) \le T(r,f) + S(r,f)$. Thus $\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) \le T(r,f) + S(r,f)$. According to Nevanlinna's Second Fundamental Theorem and above inequality, we have $T(r,f) = \overline{N}(r,a;f) + \overline{N}(r,b;f) + S(r,f)$. □

3. Proofs of the theorems

Proof of Theorem 1.9. Set

$$
\phi = \frac{(f^{(1)}-\alpha)(f-L)}{(f-a)(f-b)}, \psi = \frac{L^{(1)}-\alpha}{L-b} - \frac{f^{(1)}-\alpha}{f-b} \text{ and } \gamma = \frac{L^{(1)}-\alpha}{L-a} - \frac{f^{(1)}-\alpha}{f-a}.
$$

Then

$$
\phi\frac{f-a}{f^{(1)}-\alpha} = 1 - \frac{L-b}{f-b}. \tag{3.1}
$$

By taking the derivative in both sides of the above identity and using it again, we deduce that

$$
(\phi+\psi)\frac{f^{(1)}-\alpha}{f-a} - \phi\frac{f^{(2)}}{f^{(1)}-\alpha} + \phi^{(1)} - \phi\psi = 0. \tag{3.2}
$$

Since f and L share $(a,1)^*$, $(b,1)^*$ and $(\infty,0)^*$, by Lemma 2.1, $S(r,L) = S(r,f)$. We suppose that $f \not\equiv L$. Then by the hypothesis $N(r,\phi) = S(r,f)$. Since

$$
\begin{aligned}
\phi = \frac{1-b_0}{a-b}&\left(\frac{a(f^{(1)}-\alpha)}{f-a} - \frac{b(f^{(1)}-\alpha)}{f-b}\right) - \frac{b_{-1}+b_1\alpha}{a-b}\left(\frac{f^{(1)}-\alpha}{f-a} - \frac{f^{(1)}-\alpha}{f-b}\right)\\
&- \frac{f^{(1)}-\alpha}{f-a}\left(\frac{b_1(f^{(1)}-\alpha)}{f-b} - \frac{b_2 f^{(2)}}{f-b} - \cdots - \frac{b_n f^{(n)}}{f-b}\right),
\end{aligned}
$$

from the lemma of the logarithmic derivative we see that $m(r,\phi) = S(r,f)$ and so $T(r,\phi) = S(r,f)$.

Now we verify $\overline{N}_{(2}(r,a;f) = S(r,f)$ and $\overline{N}_{(2}(r,b;f) = S(r,f)$. Let z_0 be a zero of $f-a$ with multiplicity $p(\ge 2)$ and a zero of $L-a$ with multiplicity

$q(\geq 2)$. Then z_0 is a zero of ϕ with multiplicity at least $\min\{p,q\}-1 \geq 1$. Hence

$$\overline{N}_{(2}(r,a;f \mid L=a,\geq 2) \leq N(r,0;\phi) = S(r,f),$$

where $\overline{N}_{(2}(r,a;f \mid L=a,\geq 2)$ denotes the reduced counting function of those a-points of f whose multiplicity(≥ 2) which are also multiple a-points of L. Since f and L share $(a,1)^*$,

$$\begin{aligned}\overline{N}_{(2}(r,a;f) &= \overline{N}_{(2}(r,a;f \mid L=a,\geq 2) + \overline{N}_{(2}(r,a;f \mid L=a,=1)\\ &= S(r,f),\end{aligned}$$

where $\overline{N}_{(2}(r,a;f \mid L=a,=1)$ denotes the reduced counting function of multiple a-points of f are also simple a-pionts of L. Similarly, $\overline{N}_{(2}(r,b;f) = S(r,f)$.

Since by Lemma 2.2 we have

$$T(r,f) = \overline{N}(r,a;f) + \overline{N}(r,b;f) + S(r,f),$$

so $\overline{N}(r,a;f)$ and $\overline{N}(r,b;f)$ not simultaneously $S(r,f)$. Now we consider the following cases.

Case I. $\overline{N}(r,a;f) \neq S(r,f)$. Since

$$\psi = \frac{L^{(1)}-\alpha}{L-b} - \frac{f^{(1)}-\alpha}{f-b}$$

and f and L share $(b,1)^*$,

$$\begin{aligned}N(r,\psi) = \overline{N}(r,\psi) &\leq \overline{N}_{(2}(r,b;f) + S(r,f)\\ &= S(r,f),\end{aligned}$$

since simple zeros of $f-b$ are not the poles of ψ. Since $m(r,\psi) = S(r,f)$, we obtain $T(r,\psi) = S(r,f)$.

Since $\overline{N}(r,a;f) \neq S(r,f)$ and $\overline{N}_{(2}(r,a;f) = S(r,f)$, it follows from (3.2) that

$$\phi + \psi \equiv 0$$

and so,

$$\frac{f^{(2)}}{f^{(1)}-\alpha} - \frac{\phi^{(1)}}{\phi} + \frac{L^{(1)}-\alpha}{L-b} - \frac{f^{(1)}-\alpha}{f-b} = 0.$$

Integrating the above equation we get

$$\phi(f-b) = c(f^{(1)}-\alpha)(L-b),$$

where c is a nonzero constant. Now using the definition of ϕ,

$$(f-L) = c(f-a)(L-b). \tag{3.3}$$

From (3.3) $f-b = (L-b) + c(f-a)(L-b)$ which implies

$$\frac{f-b}{L-b} = c\left(f - \frac{ac-1}{c}\right) \tag{3.4}$$

and

$$\frac{L-a}{f-a} = -c\left(L - \frac{bc+1}{c}\right). \tag{3.5}$$

Since f and L share $(a,1)^*$ and $(b,1)^*$, it follows from (3.4) and (3.5) that

$$\begin{aligned}\overline{N}\left(r, \frac{ac-1}{c}; f\right) &= \overline{N}\left(r, 0; \frac{f-b}{L-b}\right)\\ &\le \overline{N}_{(2}(r, b; f) + S(r,f)\\ &= S(r,f)\end{aligned}$$

and

$$\begin{aligned}\overline{N}\left(r, \frac{bc+1}{c}; L\right) &= \overline{N}\left(r, 0; \frac{L-a}{f-a}\right)\\ &\le \overline{N}_{(2}(r, a; L) + S(r,L)\\ &= \overline{N}_{(2}(r, a; f) + S(r,L)\\ &= S(r,L)\end{aligned}$$

and by the Second Fundamental Theorem,

$$T(r,f) = \overline{N}(r, a; f) + S(r,f) \tag{3.6}$$

and

$$T(r,L) = \overline{N}(r, b; L) + S(r,L). \tag{3.7}$$

From (3.6) and (3.7) and Lemma 2.2, we find that $T(r,L) = S(r,L)$, which is a contradiction.

Case II. $\overline{N}(r, b; f) \ne S(r,f)$. Since

$$\gamma = \frac{L^{(1)} - \alpha}{L-a} - \frac{f^{(1)} - \alpha}{f-a}$$

and f and L share $(a,1)^*$,

$$\begin{aligned}N(r,\gamma) = \overline{N}(r,\gamma) &\le \overline{N}_{(2}(r, a; f) + S(r,f)\\ &= S(r,f),\end{aligned}$$

since simple zeros of $f-a$ are not the poles of γ. Since $m(r,\gamma) = S(r,f)$, we obtain $T(r,\gamma) = S(r,f)$. From the definition of ϕ,

$$\phi \frac{f-b}{f^{(1)} - \alpha} = 1 - \frac{L-a}{f-a}. \tag{3.8}$$

By taking the derivative in both sides of the above identity and using it again, we deduce that

$$(\phi + \gamma)\frac{f^{(1)} - \alpha}{f-b} - \phi\frac{f^{(2)}}{f^{(1)} - \alpha} + \phi^{(1)} - \phi\gamma = 0. \tag{3.9}$$

Since $\overline{N}(r,b;f) \neq S(r,f)$ and $\overline{N}_{(2}(r,b;f) = S(r,f)$, from (3.9) we get $\phi+\gamma \equiv 0$. So,

$$\frac{f^{(2)}}{f^{(1)}-\alpha} - \frac{\phi^{(1)}}{\phi} + \frac{L^{(1)}-\alpha}{L-a} - \frac{f^{(1)}-\alpha}{f-a} = 0.$$

Proceeding as in Case I, $\overline{N}(r, \frac{ac+1}{c}; L) = S(r,L)$ and $\overline{N}(r, \frac{bc-1}{c}; f) = S(r,f)$. By the Second Fundamental Theorem, $T(r,f) = \overline{N}(r,b;f)+S(r,f)$ and $T(r,L) = \overline{N}(r,a;L) + S(r,L)$. Since $\overline{N}(r,b;L) = \overline{N}(r,a;f) + S(r,L)$ it follows from Lemma 2.2 that $T(r,L) = S(r,L)$, which is a contradiction. This proves the theorem. □

Proof of Theorem 1.10. Let $L = L(f)$ and define ϕ as in the proof of Theorem 1.9. Since f and L share $(a,1)^*$, $(b,0)^*$ and $(\infty,0)^*$, by Lemma 2.1, $S(r,f) = S(r,L)$. Suppose $f \not\equiv L$. By the hypothesis, $T(r,\phi) = S(r,f)$. Since f and L share $(a,1)^*$, as in the proof of Theorem 1.9, $\overline{N}_{(2}(r,a;f) = S(r,f)$.

We first suppose that $\overline{N}(r,b;f) = S(r,f)$. Then by Lemma 2.2, $\overline{N}(r,a;f) \neq S(r,f)$. Proceeding as the proof of Case I of Theorem 1.9,

$$\begin{aligned} T(r,L) &= \overline{N}(r,b;L) + S(r,L) \\ &= \overline{N}(r,b;f) + S(r,L) \\ &= S(r,L), \end{aligned}$$

which is a contradiction. Therefore $\overline{N}(r,b;f) \neq S(r,f)$. Now, proceeding as the proof of Case II of Theorem 1.9, we obtain (3.9).

Suppose that $\phi + \gamma \equiv 0$. Then, from (3.9),

$$\frac{f^{(2)}}{f^{(1)}-\alpha} - \frac{\phi^{(1)}}{\phi} + \frac{L^{(1)}-\alpha}{L-a} - \frac{f^{(1)}-\alpha}{f-a} = 0. \tag{3.10}$$

Integrating the above and using the definition of ϕ, we get

$$c_1(f-L) = (L-a)(f-b), \tag{3.11}$$

where c_1 is a nonzero constant. Let z_1 be a b-point of f with multiplicity p and a b-point of L with multiplicity q. From (3.11), it follows that $p \leq q$. By the Taylor expansion in some neighbourhood of z_1, we get

$$f(z) - b = \alpha_p(z-z_1)^p + O(z-z_1)^{p+1}$$

and

$$L(z) - b = \beta_q(z-z_1)^q + O(z-z_1)^{q+1},$$

where $\alpha_p\beta_q \neq 0$.

We suppose that $p < q$. Then, in some neighbourhood of z_1,

$$\frac{f(z)-L(z)}{f(z)-b} = \frac{\alpha_p + O(z-z_1)}{\alpha_p + O(z-z_1)}.$$

Therefore, putting $z = z_1$ in (3.11) we get $c_1 = b-a$. Now from (3.11) we get for $c_1 = b-a$, $(b-a)(f-L) = (L-a)(f-b)$ implies $(L-b)(f-a) = 0$, which is a contradiction. Therefore $p = q$ and so f and L share $(b,\infty)^*$. Then, by Theorem 1.9, $f \equiv L$, which is a contradiction. Hence, $\phi + \gamma \not\equiv 0$. So, from (3.9), we have

$$\begin{aligned}\overline{N}_{1)}(r, b; f) &\le N(r, 0; \phi + \gamma) + S(r, f)\\ &= S(r, f).\end{aligned}$$

Let z_2 be a b-point of f with multiplicity greater than or equal to $n + 2$. If z_2 is a b-point of L, then from (1.1) and the hypothesis, $b = b_{-1}(z_2) + bb_0(z_2) + b_1(z_2)\alpha$.

If $b(z) \not\equiv b_{-1}(z) + bb_0(z) + b_1(z)\alpha$, then

$$\begin{aligned}\overline{N}_{(n+2}(r, b; f) &\le N(r, b; b_{-1} + bb_0 + b_1\alpha) + S(r, f)\\ &= S(r, f).\end{aligned}$$

If $b \equiv b_{-1}(z) + bb_0(z) + b_1(z)\alpha$, then from (1.1), $L - f = (b_0 - 1)(f - b) + b_1(f^{(1)} - \alpha) + b_2 f^{(2)} + \cdots + b_n f^{(n)}$. Hence, if z_2 is not a pole of any one of $b_j (j = 0, 1, 2, \ldots, n)$, then z_2 is a zero of $L - f$ with multiplicity ≥ 2 and so is a zero of ϕ.

Therefore,

$$\begin{aligned}\overline{N}_{(n+2}(r, b; f) &\le N(r, 0; \phi) + \sum_{j=0}^{n} N(r, \infty; b_j)\\ &= S(r, f).\end{aligned}$$

Hence in any case,

$$\overline{N}_{(n+2}(r, b; f) = S(r, f).$$

Next let z_3 be a b-point of f with multiplicity p $(2 \le p \le n+1)$. If z_3 is not a pole of $\phi^{(1)} - \phi\gamma$, then from (3.9) that $\phi(z_3) + p\gamma(z_3) = 0$.

We suppose that $\phi(z) + p\gamma(z) \not\equiv 0$ for any $p \in \{2, 3, \ldots, n + 1\}$. Then, from above,

$$\begin{aligned}\overline{N}_{n+1)}(r, b; f) - \overline{N}_{1)}(r, b; f) &\le \sum_{p=2}^{n+1} N(r, 0; \phi + p\gamma) + N(r, \infty; \phi^{(1)} - \phi\gamma)\\ &= S(r, f)\end{aligned}$$

and so

$$\overline{N}_{n+1)}(r, b; f) = S(r, f).$$

Therefore

$$\begin{aligned}\overline{N}(r, b; f) &= \overline{N}_{n+1)}(r, b; f) + \overline{N}_{(n+2}(r, b; f)\\ &= S(r, f),\end{aligned}$$

which is a contradiction. Therefore, there exists a $p \in \{2, 3, \ldots, n + 1\}$ such that $\phi(z) + p\gamma(z) \equiv 0$. Then from (3.9), we get

$$\left(1 - \frac{1}{p}\right)\frac{f^{(1)} - \alpha}{f - b} - \frac{f^{(2)}}{f^{(1)} - \alpha} + \frac{\phi^{(1)}}{\phi} - \frac{L^{(1)} - \alpha}{L - a} + \frac{f^{(1)} - \alpha}{f - a} = 0.$$

Integrating and using the definition of ϕ,

$$(f-L)^p = c_2(f-b)(L-a)^p, \tag{3.12}$$

where c_2 is a nonzero constant.

Suppose that $\overline{N}(r,b;f) = S(r,f)$. Since f and L share $(a,1)^*$, we have $\overline{N}(r,a;L) = S(r,f) = S(r,L)$. So, f and L share the value a CM^*. Then by Theorem 1.11, there exists an entire function δ such that $f = b+(a-b)(e^\delta - 1)^2$.

Hence $f-a = (a-b)e^\delta(e^\delta - 2)$ and so

$$\begin{aligned}\overline{N}(r,a;f) &= \overline{N}(r,2;e^\delta) + S(r,e^\delta)\\ &= T(r,e^\delta) + S(r,e^\delta)\\ &= \frac{1}{2}T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction. Therefore, $\overline{N}(r,a;f) \neq S(r,f)$.

Let z_4 be an a-point of f and L with respective multiplicities q and s. From (3.12), we see that $s \leq q$. We suppose that $s < q$. From (3.12), $c_2 = (-1)^p/(a-b)$. So again from (3.12), we get

$$f = b + (-1)^p(a-b)(h-1)^p, \tag{3.13}$$

and

$$L = b + \frac{(a-b)(h-1)}{h}[(-1)^p(h-1)^{p-1}+1],$$

where $h = \frac{f-a}{L-a}$. Since f is an entire function, from (3.13), we see that h is also entire. Also, (3.13) implies that

$$pT(r,h) = T(r,f) + S(r,f).$$

Further, we see that $\overline{N}(r,0;h) \leq \overline{N}_{(2}(r,a;f) + S(r,f) = S(r,f) = S(r,h)$. Therefore, by the Second Fundamental Theorem, $\overline{N}(r,d;h) \neq S(r,f)$ for a complex number $d(\neq 0,\infty)$ with $(-1)^p(d-1)^{p-1}+1 = 0$. Since f and L share $(b,0)^*$, we must have $p = 2$. Hence $f-a = (a-b)h(h-2)$ and $L-a = (a-b)(h-2)$. Since z_4 is a common zero of $f-a$ and $L-a$, we have $s = q$, which is a contradiction to the supposition. Therefore, f and L share $(a,\infty)^*$. Now we achieve the result by Theorem 1.11. This proves the theorem. □

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Goutam Kumar Ghosh
Department of Mathematics
Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya (The University of Burdwan)
Hatgobindapur, Burdwan, W.B., India
Email address: g80g@rediffmail.com