Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 401–430
https://doi.org/10.4134/CKMS.c210421
pISSN: 1225-1763 / eISSN: 2234-3024

SINGULAR AND MARCINKIEWICZ INTEGRAL OPERATORS ON PRODUCT DOMAINS

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Abstract. In this paper, we prove L^p estimates of a class of singular integral operators on product domains along surfaces defined by mappings that are more general than polynomials and convex functions. We assume that the kernels are in $L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$. Furthermore, we prove L^p estimates of the related class of Marcinkiewicz integral operators. Our results extend as well as improve previously known results.

1. Introduction and statement of results

Let $\mathbb{R}^n$ $(n\geq 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\sigma$. In addition, let $y'=\frac{y}{|y|}\in\mathbb{S}^{n-1}$ $(y\neq 0)$ and let $\Omega\in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{S}^{n-1}}\Omega(y')d\sigma(y')=0. \tag{1.1}$$

The classical Calderón-Zygmund singular integral operator is defined by

$$\begin{aligned}\mathbf{S}_\Omega f(x)&=\text{p.v.}\int_{\mathbb{R}^n}f(x-y)\frac{\Omega(y')}{|y|^n}dy\\&=f*K(x),\end{aligned}$$

where $f\in\mathcal{S}(\mathbb{R}^n)$, the class of Schwarz functions. Here, we set

$$K(x)=\frac{\Omega(x')}{|x|^n}.$$

In [14], Calderón and Zygmund introduced the method of rotation and prove that the operator S_Ω is bounded on L^p provided that $\Omega\in L(\log^+L)(\mathbb{S}^{n-1})$,

Received December 15, 2021; Revised August 2, 2022; Accepted August 17, 2022.
2020 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B15, 42B25.
Key words and phrases. Singular integral operators on product domains, Marcinkiewicz integral operators on product domains, L^p estimates, maximal functions, Hardy Littlewood maximal function, convex.

where

$$\log^{+}(t)=\begin{cases}\log(t), & t>1,\\ 0, & 0<t\le 1.\end{cases}$$

Since the publication of the papers [13] and [14], several authors have studied the L^p mapping properties of S_Ω, we cite [5], [12], [19], [21], [22], [30], among others. Among several function spaces that are connected to the study of singular integral operators, we recall the definition of Block spaces $B_q^{0,0}(\mathbb{S}^{n-1})$, $q>1$ introduced by Jiang and Lu in [26] (see also [27]). A function $\Omega\in B_q^{0,0}(\mathbb{S}^{n-1})$ if $\Omega=\sum_{\mu=1}^{\infty}c_\mu b_\mu$, where $\{c_\mu\}\subset\mathbb{C}$, b_μ is a measurable function supported in an interval $I_\mu\subset\mathbb{S}^{n-1}$ with the property that $\|b_\mu\|_{L^q}\leqslant| I_\mu |^{-1/q'}$, where $\frac{1}{q}+\frac{1}{q'}=1$ and

$$M_q^{0,0}(\{c_\mu\})=\sum_{\mu=1}^{\infty}|c_\mu|(1+\log^{+}(|I_\mu|^{-1})).$$

It is well known that $B_{q_2}^{0,0}\subset B_{q_1}^{0,0}$ whenever $1<q_1<q_2$, $L^q(\mathbb{S}^{n-1})\subseteq B_q^{0,0}(\mathbb{S}^{n-1})$ and that

$$B_q^{0,0}(\mathbb{S}^{n-1})\nsubseteq\bigcup_{p>1}L^q(\mathbb{S}^{n-1}).$$

In [1], Al-Azriyah considered singular integrals along surfaces defined by mappings that are more general than polynomials and convex functions which were introduced by Al-Salman [9]. In fact, Al-Salman introduced the following class of functions:

Definition 1.1 ([9]). A function $\psi:[0,\infty)\to\mathbb{R}$ is said to belong the class $\mathscr{PC}_\lambda(d)$ if there exist a polynomial P belongs to the class $\mathscr{P}_d$ of all real valued polynomials with degree at most d and a mapping $\varphi\in C^{d+1}[0,\infty)$ such that

(i) $\psi(t)=P(t)+\lambda\varphi(t)$,
(ii) $P(0)=0$ and $\varphi^{(j)}(0)=0$ for $0\leqslant j\leqslant d$,
(iii) $\varphi^{(j)}$ is positive non-decreasing on $(0,\infty)$ for $0\leqslant j\leqslant d+1$.

It was pointed out in [9] that the class $\cup_{d\geqslant 0}(\mathscr{PC}_\lambda(d))$ contains properly the class of polynomials $\mathscr{P}_d$ as well as the class of convex increasing functions. In [1], Al-Azriyah proved the following result:

Theorem 1.1 ([1]). *Let $S_{\Omega,\Phi}$ be given by*

$$S_{\Omega,\Phi}f(x)=p.v.\int_{\mathbb{R}^n}f(x-\Phi(y))\frac{\Omega(y')}{|y|^n}\,dy.$$

Suppose that Ω satisfies (1.1) *and that $\Omega\in B_q^{0,0}(\mathbb{S}^{n-1})$ for some $q>1$. If $\Phi\in\mathscr{PC}_\lambda(d)$, then*

$$\|S_{\Omega,\Phi}(f)\|_{L^p(\mathbb{R}^n)}\le C_p\|f\|_{L^p(\mathbb{R}^n)}$$

for $1<p<\infty$ with L^p bounds independent of $\lambda\in\mathbb{R}$ and the coefficients of the particular polynomial involved in the standard representation of Φ.

Our aim in this paper is to discuss the boundedness of the operator $S_{\Omega,\Phi}$ in the product domains setting. The classical singular integral operator on product domains is defined by

$$(T_\Omega f)(x,y) = p.v.\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x-u,y-v)\frac{\Omega(u',v')}{|u|^n\,|v|^m}\,dudv,$$

where $\Omega \in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(u',\cdot)d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega(\cdot,v')d\sigma(v') = 0, \tag{1.2}$$

and

$$\Omega(tx,sy) = \Omega(x,y) \tag{1.3}$$

for any $t,s>0$.

The study of singular integral operators on product domains was initiated by Fefferman [23] and Fefferman-Stein [25]. Subsequently, such operators have been studied by many authors [6], [4], [18], [20], [25], among others. In particular, Fefferman and Stein proved in [24] that T_Ω is bounded on $L^p(\mathbb{R}^{n+m})$ for $(1<p<\infty)$ if Ω satisfies certain Lipschitz conditions. Consequently, the L^p $(1<p<\infty)$ boundedness of T_Ω was established under various conditions on Ω, first by Duoandikoetxea [18] for $\Omega\in L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ with $q>1$ and then by Fan-Guo-Pan [20] when Ω lies in certain Block spaces. In [11], Al-Salman, Al-Qassem and Pan showed that T_Ω is bounded on L^p $(1<p<\infty)$ provided that $\Omega\in L(\log^+ L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, i.e.,

$$\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega(u,v)|(\log 2+|\Omega(u,v)|)^2\,d\sigma(u)\,d\sigma(v)<\infty. \tag{1.4}$$

It is worth noting that,

$$L(\log^+ L)^s(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})\subset L(\log^+ L)^r(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}) \quad \text{whenever } r<s$$

and

$$L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})\subsetneqq L(\log^+ L)^r(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})\subsetneqq L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$$

whenever $q>1$ and $r\geq 1$. In [11], Al-Salman, Al-Qassem and Pan proved that T_Ω may not be bounded on L^p if we replace the condition $\Omega\in L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ by $\Omega\in L(\log L)^{2-\varepsilon}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ for some $\varepsilon>0$. This shows that the condition $\Omega\in L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ is nearly optimal. In light of this result and the one parameter cases in [1] and [9], we consider the operator

$$(T_{\Phi,\Psi,\Omega}f)(x,y) = \text{p.v.}\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x-\Phi(|u|)u',y-\Psi(|v|)v')\frac{\Omega(u',v')}{|u|^n\,|v|^m}\,dudv,$$

where $\Phi,\Psi:[0,\infty)\to\mathbb{R}$ are of the type in Definition 1.1 above.

Our main result is the following:

Theorem 1.2. *Let* $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying* (1.2)-(1.4). *If* $\Phi \in \mathscr{PC}_\lambda(d)$, $\Psi \in \mathscr{PC}_\alpha(b)$ *for* $d, b > 0$ *and* $\lambda, \alpha \in \mathbb{R}$, *then* $T_{\Phi,\Psi,\Omega}$ *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ *for* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

We remark here that Theorem 1.2 is different from the corresponding result in [11] even in the special case of polynomial mappings. In fact, if $\Phi(t) = P(t^2)$ and $\Psi(t) = Q(t^2)$, where P and Q are real valued polynomials, then the result in Theorem 1.2 can be obtained from Theorem 1.3 in [11]. However, if Φ and Ψ are general polynomials, the result in Theorem 1.2 is not covered by the corresponding result in [11]. Furthermore, it can be easily seen that Theorem 1.2 above generalizes the corresponding results in Corollaries 3.1-3.4 in [10].

By making use of similar estimates in this paper, we will be able to prove the L^p boundedness of the related Marcinkiewicz integral operators. For $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ satisfying (1.2)-(1.3) and mappings $\Phi, \Psi : [0, \infty) \to \mathbb{R}$ as above, we consider the operator

$$\mathcal{M}_{\Omega,\Phi,\Psi} f(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F_{t',s'}^{\Phi,\Psi}(f)(x,y) \right|^2 2^{-2(t'+s')} dt'\, ds' \right)^{\frac{1}{2}},$$

where

$$F_{t',s'}^{\Phi,\Psi}(f)(x,y) = \int\int_{\Lambda(t',s')} f(x - \Phi(|u|)u', y - \Psi(|v|)v') \, \frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}} du\, dv$$

and $\Lambda(t',s') = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : |u| \le 2^{t'} \text{ and } |v| \le 2^{s'}\}$. When $\Phi(t) = \Psi(t) = t$, the operator $\mathcal{M}_{\Omega,\Phi,\Psi}$ is known by the classical Marcinkiewicz integral operator on product domains which is denoted by $\mathcal{M}_{\Omega,c}$. In [17], Ding proved that $\mathcal{M}_{\Omega,c}$ is bounded on L^2 if $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. On the other hand, Chen, Ying and Fan in [15] proved the L^p (for all $1 < p < \infty$) boundedness under the same condition on Ω. In [16], Choi established the L^2 boundedness of $\mathcal{M}_{\Omega,c}$ under the weaker condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Subsequently, Al-Qassem, Al-Salman, Pan and Chang proved the L^p boundedness for all $1 < p < \infty$ provided that $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ [3].

Using estimates similar to those obtained for $T_{\Phi,\Psi,\Omega}$, we prove the following result for the operator $\mathcal{M}_{\Omega,\Phi,\Psi}$:

Theorem 1.3. *Suppose that* $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *and* Φ, Ψ *as in Theorem 1.2. Then* $\mathcal{M}_{\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ *for* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

This paper is organized as follows. In Section 2, we present few introductory lemmas and results. A proof of Theorem 1.2 will be presented in Section 3.

Section 4 is devoted to the estimates concerning the Marcinkiewicz integral operator. Finally, the proof of Theorem 1.3 will be presented in Section 5.

Throughout this paper, the letter C will stand for a constant that may vary at each occurrence but it is independent of the essential variables.

2. Preliminary estimates

We start by recalling the following inequality in [19]:

Lemma 2.1 ([19]). *Suppose that* $P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha$ *is a polynomial of degree* m *on* $\mathbb{R}^n$ *and* $\varepsilon < \frac{1}{m}$. *Then there exists* $A_\varepsilon > 0$ *such that*

$$\int_{\mathbb{S}^{n-1}} |P(y')|^{-\varepsilon} d\sigma(y') \leq A_\varepsilon \|P\|,$$

where

$$\|P\| = \sum_{|\alpha|=m} |a_\alpha|.$$

The bound A_ε *may depend on* ε, m *and* n *but it is independent of the coefficients of the polynomial.*

In order to deal with estimates involving mappings of the type in Definition 1.1, we recall the following lemma in [1]:

Lemma 2.2 ([1]). *If* $\varphi \in C^{d+1}[0,\infty)$ *and satisfies the conditions* (i)-(ii) *in Definition 1.1, then*

(i) $\varphi(\alpha r) \leq \alpha\, \varphi(r)$ *for* $0 \leq \alpha \leq 1$ *and* $r > 0$,
(ii) $\varphi(\alpha r) \geq \alpha\, \varphi(r)$ *for* $\alpha \geq 1$ *and* $r > 0$,
(iii) $\varphi^{d+1}(r) \geqslant r^{-d-1}\varphi(r)$ *for* $r > 0$.

The following result is proved in [9]:

Theorem 2.3 ([9]). *Suppose that* $\Upsilon : \mathbb{R}^n \to \mathbb{R}^d$ *is a non-constant mapping and assume that* $\Psi \in \mathscr{PC}_\lambda(d)$ *for some* $d \geqslant 0$. *Suppose also that* $\lambda \in \mathbb{R}$ *and* $\rho > 0$. *If* $\Omega \in L^1(\mathbb{S}^{n-1})$ *is homogeneous of degree zero in* $\mathbb{R}^n$, *then the maximal function* $\mathscr{M}_{\Psi,\Omega}$ *given by*

$$\mathscr{M}_{\Psi,\Omega}(f)(x) = \sup_{j\in\mathbb{Z}} \left| \int_{\rho^j<|y|<\rho^{j+1}} f(x - \Psi(|y|)\,\Upsilon(y'))\ \frac{\Omega(y')}{|y|^n}\, dy \right|$$

satisfies

$$\|\mathscr{M}_{\Psi,\Omega}(f)\|_p \leq C_p\ \|\Omega\|_1\ \|f\|_p$$

for $1 < p < \infty$. *Here, the constant* C_p *is independent of* $\lambda, \Upsilon(y')$ *and the coefficients of the particular polynomials involved in Definition 1.1 of* Ψ.

We shall need the following result in [4] which is an extension of a result of Duoandikoetxea in [18]:

Theorem 2.4 ([4]). *Let $M, N \in \mathbb{N}$ and $\{\lambda_{k,j}^{(l,s)} : k, j \in \mathbb{Z},\, 0 \le l \le N,\, 0 \le s \le M\}$ be a family of Borel measures on $\mathbb{R}^n \times \mathbb{R}^m$ with $\lambda_{k,j}^{(l,0)} = 0$ and $\lambda_{k,j}^{(0,s)} = 0$ for every $k, j \in \mathbb{Z}$. Let $\{a_l, b_s : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbb{R}^+ \setminus (0,2)$, $\{B(l), D(s) : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbb{N}\{\alpha_l, \beta_s : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbb{R}^+$, and let $\{L_l : 1 \le l \le N\} \subseteq L(\mathbb{R}^n, \mathbb{R}^{B(l)})$ and $\{Q_s : 1 \le s \le M\} \subseteq L(\mathbb{R}^m, \mathbb{R}^{D(s)})$. Suppose that for some $C > 0$, $B > 1$, and $p_0 \in (2, \infty)$, the followings hold for $k, j \in \mathbb{Z}$, $1 \le l \le N$, $1 \le s \le M$, $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, and arbitrary functions $\{g_{k,j}\}$ on $\mathbb{R}^n \times \mathbb{R}^m$:*

(i) $\|\lambda_{k,j}^{(l,s)}\| \le CB^2$;

(ii) $\left|\widehat{\lambda}_{k,j}^{(l,s)}(\xi,\eta)\right| \le CB^2 \left|a_l^{KB} L_l(\xi)\right|^{-\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{-\frac{\beta_s}{B}}$;

(iii) $\left|\widehat{\lambda}_{k,j}^{(l,s)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l-1,s)}(\xi,\eta)\right| \le CB^2 \left|a_l^{KB} L_l(\xi)\right|^{\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{-\frac{\beta_s}{B}}$;

(iv) $\left|\widehat{\lambda}_{k,j}^{(l,s)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l,s-1)}(\xi,\eta)\right| \le CB^2 \left|a_l^{KB} L_l(\xi)\right|^{-\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{\frac{\beta_s}{B}}$;

(v) $\left|\widehat{\lambda}_{k,j}^{(l,s)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l-1,s)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l,s-1)}(\xi,\eta) + \widehat{\lambda}_{k,j}^{(l-1,s-1)}(\xi,\eta)\right|$
$\le CB^2 \left|a_l^{KB} L_l(\xi)\right|^{\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{\frac{\beta_s}{B}}$;

(vi) $\left|\widehat{\lambda}_{k,j}^{(l,s-1)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l-1,s-1)}(\xi,\eta)\right| \le CB^2 \left|a_l^{KB} L_l(\xi)\right|^{\frac{\alpha_l}{B}}$;

(vii) $\left|\widehat{\lambda}_{k,j}^{(l-1,s)}(\xi,\eta) - \widehat{\lambda}_{k,j}^{(l-1,s-1)}(\xi,\eta)\right| \le CB^2 \left|b_s^{jB} Q_s(\eta)\right|^{\frac{\beta_s}{B}}$;

(viii) $\left\|\left(\sum_{k,j\in\mathbb{Z}} |\lambda_{k,j}^{(l,s)} * g_{k,j}|^2\right)^{\frac{1}{2}}\right\|_{p_0} \le CB^2 \left\|\left(\sum_{k,j\in\mathbb{Z}} |g_{k,j}|^2\right)^{\frac{1}{2}}\right\|_{p_0}$.

Then for $p_0' < p < p_0$ there exists a positive constant C_p such that

$$\left\|\sum_{k,j\in\mathbb{Z}} \lambda_{k,j}^{(N,M)} * f\right\|_{p_0} \le CB^2 \|f\|_p;$$

$$\left\|\left(\sum_{k,j\in\mathbb{Z}} |\lambda_{k,j}^{(N,M)} * f|^2\right)^{1/2}\right\|_{p_0} \le CB^2 \|f\|_p$$

hold for all f in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$. The constant C_p is independent of B and the linear transformations $\{L_l\}_{l=1}^N$ and $\{Q_s\}_{s=1}^M$.

For $\omega \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{Z}$, we let $a_\omega = 2^{(\omega+1)}$ and $I_{j,\omega} = [a_\omega^j,\, a_\omega^{j+1}]$. We let

$$I_{j,\omega}^{(n)} = \{y \in \mathbb{R}^n : |y| \in I_{j,\omega}\}$$

and

$$I_{j,\omega}^{(m)} = \{y \in \mathbb{R}^m : |y| \in I_{j,\omega}\}.$$

For a homogeneous function $\Omega : \mathbb{S}^{n-1} \times \mathbb{S}^{m-1} \to \mathbb{R}$ and $\omega \in \mathbb{N} \cup \{0\}$, we define the family of measures $\{\sigma_{\Phi,\,\Psi,\Omega,j,k,\omega} : j,\, k \in \mathbb{Z}\}$ by

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f d\sigma_{\Phi,\,\Psi,\Omega,j,k,\omega} = \int_{I^{(n)}_{j,\omega} \times I^{(m)}_{k,\omega}} f(x - \Phi(|u|)\, u', y - \Psi(|v|)\, v') \frac{\Omega(u', v')}{|u|^n\, |v|^m}\, dudv.$$

The corresponding maximal operator is defined by

$$(\sigma_{\Phi,\,\Psi,\Omega,\omega})^* f(x,y) = \sup_{j,k\in\mathbb{Z}} \left|\, |\sigma_{\Phi,\,\Psi,\Omega,j,k,\omega}| * f(x,y)\right|.$$

For simplicity, we shall write $\sigma_{\omega,j,k}$ to denote $\sigma_{\Phi,\,\Psi,\Omega,j,k,\omega}$ and σ^*_ω to denote $(\sigma_{\Phi,\,\Psi,\Omega,\omega})^*$.

Since $\Phi \in \mathscr{PC}_\lambda(d)$ and $\Psi \in \mathscr{PC}_\alpha(b)$ for some $b,\, d > 0$, there exist $\lambda,\, \alpha \in \mathbb{R}$, $P \in \mathscr{P}_d$, $Q \in \mathscr{P}_b$, $\varphi_1 \in C^{d+1}[0,\infty)$ and $\varphi_2 \in C^{b+1}[0,\infty)$ such that

$$\Phi(t) = P(t) + \lambda\varphi_1(t) \quad \text{and} \quad \Psi(r) = Q(r) + \alpha\varphi_2(r).$$

We assume that $P(t) = \sum_{k=0}^{d} c_{k,1}\, t^k$ and $Q(r) = \sum_{k=0}^{b} c_{k,2}\, r^k$, where $\{c_{k,1}\}$ and $\{c_{k,2}\}$ are constants. For $0 \le l \le d$ and $0 \le s \le b$, let

$$P_l(t) = \sum_{k=0}^{l} c_{k,1}\, t^k \quad \text{and} \quad Q_s(r) = \sum_{k=0}^{s} c_{k,2}\, r^k.$$

Here, we use the notation that $\sum_{j\in\emptyset} = 0$. Now, we define the measure $\sigma^{(d+1,b+1)}_{\omega,j,k}$ via the Fourier transform by

$$\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta) = \int_{I^{(n)}_{j,\omega} \times I^{(m)}_{k,\omega}} e^{-i\left(\Phi(|u|)\,\xi\cdot u' + \Psi(|v|)\,\eta\cdot v'\right)} \frac{\Omega(u',v')}{|u|^n\,|v|^m}\, du\, dv, \tag{2.1}$$

and $\{\sigma^{(l,s)}_{\omega,j,k} : 0 \le l \le d, 0 \le s \le b\}$ is defined by

$$\widehat{\sigma}^{\,(l,s)}_{\,\omega,j,k}(\xi,\eta) = \int_{I^{(n)}_{j,\omega} \times I^{(m)}_{k,\omega}} e^{-i\left(P_l(|u|)\,\xi\cdot u' + Q_s(|v|)\,\eta\cdot v'\right)} \frac{\Omega(u',v')}{|u|^n\,|v|^m}\, du\, dv. \tag{2.2}$$

Notice that if Ω satisfies the cancellation property (1.2), then

$$\widehat{\sigma}^{\,(0,0)}_{\,\omega,j,k} = \widehat{\sigma}^{\,(d+1,0)}_{\,\omega,j,k} = \widehat{\sigma}^{\,(0,b+1)}_{\,\omega,j,k} = 0.$$

We have the following result:

Lemma 2.5. *Let* $\{\sigma^{(d+1,b+1)}_{\omega,j,k} : j,\, k \in \mathbb{Z}\}$ *be as above. Suppose that* $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *for some* $q > 1$. *Let* $B_{\Omega,\omega} = (\omega+1)^2 (\|\Omega\|_{L^1})^{1-\frac{1}{\omega+1}} (\|\Omega\|_{L^q})^{\frac{1}{\omega+1}}$. *Then*

(i) $\|\sigma^{(d+1,b+1)}_{\omega,j,k}\| \le C\, (\omega+1)^2\, \|\Omega\|_{L^1}$;

(ii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)\right|$
$\le C\, B_{\Omega,\omega}\, |\lambda\, \varphi_1\, (a^j_\omega)\, \xi\, |^{-\frac{1}{q'(d+1)(\omega+1)}}\, |\alpha\, \varphi_2\, (a^k_\omega)\, \eta\, |^{-\frac{1}{q'(b+1)(\omega+1)}}$;

(iii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)\right|$
$\leq C\,B_{\Omega,\omega}\,\left|\lambda\varphi_1(a_\omega^{j+1})\,\xi\right|^{\frac{1}{\omega+1}}\left|\alpha\varphi_2(a_\omega^{k})\,\eta\right|^{-\frac{1}{q'(b+1)(\omega+1)}}$;

(iv) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)\right|$
$\leq C\,B_{\Omega,\omega}\,\left|\lambda\,\varphi_1(a_\omega^{j})\,\xi\right|^{-\frac{1}{q'(d+1)(\omega+1)}}\left|\alpha\,\varphi_2(a_\omega^{k+1})\,\eta\right|^{\frac{1}{\omega+1}}$;

(v) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|$
$\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}|\lambda\varphi_1(a_\omega^{j+1})\,\xi|^{\frac{1}{\omega+1}}\,|\alpha\varphi_2(a_\omega^{k+1})\,\eta|^{\frac{1}{\omega+1}}$;

(vi) $\left|\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\leq C\;(\omega+1)^2\,\|\Omega\|_{L^1}|\lambda\varphi_1(a_\omega^{j+1})\,\xi|^{\frac{1}{\omega+1}}$;

(vii) $\left|\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\leq C\;(\omega+1)^2\,\|\Omega\|_{L^1}\left|\,\alpha\,\varphi_2(a_\omega^{k+1})\,\eta\right|^{\frac{1}{\omega+1}}$;

where C is independent of ω and $k,j\in\mathbb{Z}$, $(\xi,\eta)\in(\mathbb{R}^n,\mathbb{R}^m)$.

Proof of Lemma 2.5. The estimate (i) is straightforward. In fact, we have

$$\begin{aligned}\|\sigma^{(d+1,b+1)}_{\omega,j,k}\| &\leq \int_{I^{(n)}_{j,\omega}\times I^{(m)}_{k,\omega}}|\Omega(u',v')|\,|u|^{-n}\,|v|^{-m}\,dudv\\ &\leq \|\Omega\|_{L^1}\;(\ln(a_\omega))^2\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}.\end{aligned}$$

To see (ii), notice that

$$\begin{aligned}&\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(u',v')|\left|\int_{I_{j,\omega}\times I_{k,\omega}}e^{-i\left(\Phi(t)\,\xi\cdot u'+\Psi(r)\,\eta\cdot v'\right)}\frac{dtdr}{tr}\right|d\sigma(u')d\sigma(v'),\end{aligned}$$

which by Hölder's inequality implies that

$$\begin{aligned}&\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq\|\Omega\|_q\left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|A_{j,k,\omega}(\xi,u',\eta,v')|^{q'}\;d\sigma(u')d\sigma(v')\right)^{\frac{1}{q'}},\end{aligned}\tag{2.3}$$

where

$$A_{j,k,\omega}(\xi,u',\eta,v')=\int_{I_{j,\omega}\times I_{k,\omega}}e^{-i\left(\Phi(t)\,\xi\cdot u'+\Psi(r)\,\eta\cdot v'\right)}\frac{dtdr}{tr}.$$

By change of variables and triangle inequality, we get

$$|A_{j,k,\omega}(\xi,u',\eta,v')|\leq J_{j,\omega,\Phi}(\xi\cdot u')\,J_{k,\omega,\Psi}(\eta\cdot v'),$$

where

$$J_{j,\omega,\Phi}(\xi,u')=\left|\int_1^{a_\omega}e^{-i(\Phi(a_\omega^j t)\,\xi\cdot u')}\;\frac{dt}{t}\right|,$$

and

$$J_{k,\omega,\Psi}(\eta,v')=\left|\int_1^{a_\omega}e^{-i(\Psi(a_\omega^k r)\,\eta\cdot v')}\;\frac{dr}{r}\right|.$$

Notice that $\Phi^{(d+1)}(t) = \lambda\,\varphi_1^{(d+1)}(t)$ and $\Psi^{(b+1)}(r) = \alpha\,\varphi_2^{(b+1)}(r)$. Thus, by Lemma 2.2, we have

$$\begin{aligned}\left|\Phi^{(d+1)}(a_\omega^j\ t)\right| &= \left|\lambda\,\varphi_1^{(d+1)}(a_\omega^j\ t)\right| \\ &\geq \left|\lambda\,(a_\omega^j\ t)^{-d-1}\varphi_1(a_\omega^j\ t)\right| \\ &\geq C\,|\lambda\,\varphi_1\,(a_\omega^j)|,\end{aligned}$$

whenever $a_\omega^j < t < a_\omega^{j+1}$. Similarly, we have

$$\left|\Psi^{(b+1)}(a_\omega^k\ r)\right| \geq C\,|\alpha\,\varphi_2\,(a_\omega^k)|.$$

Thus, by Van der Corput lemma in [29], we get

$$J_{j,\omega,\Phi}(\xi,u') \leqslant \left|\lambda\,\varphi_1(a_\omega^j)\,\xi\cdot u'\right|^{-\frac{1}{d+1}}$$

and

$$J_{k,\omega,\Psi}(\eta,v') \leqslant \left|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'\right|^{-\frac{1}{b+1}}. \tag{2.4}$$

Thus,

$$|A_{j,k,\omega}(\xi,u',\eta,v')| \leq \left|\lambda\,\varphi_1(a_\omega^j)\,\xi\cdot u'\right|^{-\frac{1}{d+1}}\ \left|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'\right|^{-\frac{1}{b+1}}. \tag{2.5}$$

Now, it is radially seen that

$$|A_{j,k,\omega}(\xi,u',\eta,v')| \leq C\ (\ln(a_\omega))^2 \leq C\,(\omega+1)^2. \tag{2.6}$$

Then, by interpolation between (2.5) and (2.6) with $\epsilon = \frac{1}{q'}$, we get

$$\begin{aligned}&|A_{j,k,\omega}(\xi,u',\eta,v')| \\ &\quad\leq C\,(\omega+1)^2\ \left|\lambda\,\varphi_1(a_\omega^j)\,\xi\cdot u'\right|^{-\frac{1}{q'(d+1)}}\ \left|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'\right|^{-\frac{1}{q'(b+1)}}.\end{aligned} \tag{2.7}$$

Now, let

$$\mathcal{F}_d = \sup_{\xi'\in\mathbb{S}^{n-1}}\left(\int_{\mathbb{S}^{n-1}}|\xi'\cdot u'|^{-\frac{1}{d+1}}\,d\sigma(u')\right)^{\frac{1}{q'}} \tag{2.8}$$

and

$$\mathcal{G}_b = \sup_{\eta'\in\mathbb{S}^{m-1}}\left(\int_{\mathbb{S}^{m-1}}|\eta'\cdot v'|^{-\frac{1}{b+1}}\,d\sigma(v')\right)^{\frac{1}{q'}}.$$

Since $\mathcal{F}_d$ and $\mathcal{G}_b < \infty$, we have

$$\begin{aligned}\mathcal{H}_{d,b} &= \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\xi'\cdot u'|^{-\frac{1}{d+1}}\ |\eta'\cdot v'|^{-\frac{1}{b+1}}\,d\sigma(u')d\sigma(v')\right)^{\frac{1}{q'}} \\ &\leq \mathcal{F}_d\,\mathcal{G}_b < \infty.\end{aligned} \tag{2.9}$$

Thus, by (2.3) and (2.7), we get

$$\begin{aligned}&\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)\right| \\ &\quad\leq C\,(\omega+1)^2\,\|\Omega\|_{L^q}\end{aligned} \tag{2.10}$$

$$\left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\lambda\,\varphi_1\,(a_\omega^j)\,\xi\cdot u'|^{-\frac{1}{d+1}}\,|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'|^{-\frac{1}{b+1}}\,d\sigma(u')\,d\sigma(v')\right)^{\frac{1}{q'}}$$
$$\le C\,(\omega+1)^2\,\|\Omega\|_{L^q}\,\mathcal{H}_{d,b}\,|\lambda\,\varphi_1\,(a_\omega^j)\,\xi\,|^{-\frac{1}{q'(d+1)}}\;|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\,|^{-\frac{1}{q'(b+1)}}$$
$$\le C\,(\omega+1)^2\|\Omega\|_{L^q}|\lambda\,\varphi_1\,(a_\omega^j)\,\xi\,|^{-\frac{1}{q'(d+1)}}\;|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\,|^{-\frac{1}{q'(b+1)}}.$$

It can be easily seen that $\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}$ satisfies

$$\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)\right| \leqslant (\ln(a_\omega))^2\,\|\Omega\|_{L^1} \leqslant C\,(\omega+1)^2\|\Omega\|_{L^1}. \tag{2.11}$$

Finally, by interpolation between (2.10) and (2.11) with $0<\epsilon=\frac{1}{\omega+1}<1$, we get

$$\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)\right| \le C\,B_{\Omega,\omega}\,|\lambda\,\varphi_1\,(a_\omega^j)\,\xi\,|^{-\frac{1}{q'(d+1)(\omega+1)}}\,|\alpha\,\varphi_2\,(a_\omega^k)\,\eta|^{-\frac{1}{q'(b+1)(\omega+1)}}.$$

To get (iii), we have

$$\begin{aligned}&\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)-\widehat{\sigma}_{\omega,j,k}^{(d,b+1)}(\xi,\eta)\right|\\ &=\left|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}}\Omega(u',v')\,e^{-i\Psi(r)\,\eta\cdot v'}\right.\\ &\qquad\left.\left(e^{-i\Phi(t)\,\xi\cdot u'}-e^{-iP(t)\,\xi\cdot u'}\right)\frac{dt}{t}\frac{dr}{r}d\sigma(u')\,d\sigma(v')\right|.\end{aligned}$$

Thus by Fubini's theorem, the fact that $a_\omega^j<t<a_\omega^{j+1}$ and φ_1 is increasing, we get

$$\begin{aligned}&\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)-\widehat{\sigma}_{\omega,j,k}^{(d,b+1)}(\xi,\eta)\right| \qquad (2.12)\\ &\le\left|\lambda\varphi_1(a_\omega^{j+1})\,\xi\right|\int_{\mathbb{S}^{n-1}}\int_{I_{j,\omega}}\left|\int_{\mathbb{S}^{m-1}}\int_{I_{k,\omega}}\Omega(u',v')\,e^{-i\Psi(r)\,\eta\cdot v'}d\sigma(v')\frac{dr}{r}\right|\frac{dt}{t}\,d\sigma(u')\\ &\le\ \ln(a_\omega)\left|\lambda\varphi_1(a_\omega^{j+1})\,\xi\right|\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{m-1}}|\Omega(u',v')|\;J_{k,\omega,\Psi}(\eta,v')\,d\sigma(u')\,d\sigma(v').\end{aligned}$$

As $J_{k,\omega,\Psi}(\eta,v')$ in (2.4), we have

$$J_{k,\omega,\Psi}(\eta,v')\leqslant\left|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'\right|^{-\frac{1}{b+1}}. \tag{2.13}$$

On the other hand, we have

$$J_{k,\omega,\Psi}(\eta,v')\le C\ (\omega+1). \tag{2.14}$$

Thus by interpolation between (2.13) and (2.14) with $0<\epsilon=\frac{1}{q'}<1$, we obtain

$$J_{k,\omega,\Psi}(\eta,v')\le C\,(\omega+1)\,\left|\alpha\,\varphi_2\,(a_\omega^k)\,\eta\cdot v'\right|^{-\frac{1}{q'(b+1)}}. \tag{2.15}$$

Thus, by (2.12), (2.15) and Hölder's inequality, we get

$$\left|\widehat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)-\widehat{\sigma}_{\omega,j,k}^{(d,b+1)}(\xi,\eta)\right| \tag{2.16}$$

$$\leq C\,(\omega+1)^2\|\Omega\|_{L^q}\left|\lambda\varphi_1(a_\omega^{j+1})\,\xi\right|\left|\alpha\varphi_2(a_\omega^{k})\,\eta\right|^{-\frac{1}{q'(b+1)}}.$$

Thus by (2.16) and the observation that

$$\begin{aligned}\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)\right| &\leq (\ln(a_\omega))^2\,\|\Omega\|_{L^1}\\ &\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1},\end{aligned}$$

we get

$$\begin{aligned}&\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq C\,B_{\Omega,\omega}\,\left|\lambda\varphi_1(a_\omega^{j+1})\,\xi\right|^{\frac{1}{\omega+1}}\,\left|\alpha\varphi_2(a_\omega^{k})\,\eta\right|^{-\frac{1}{q'(b+1)\,(\omega+1)}}.\end{aligned}$$

Similarly, we can obtain the estimates (iv), we omit details.

For the estimate (v), we have

$$\begin{aligned}(2.17)\qquad &\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}}|\Omega(u',v')|\,|e^{-i\lambda\varphi_1(t)\,\xi\cdot u'}-1|\\ &\qquad |e^{-i\alpha\varphi_2(r)\,\eta\cdot v'}-1|\frac{dt}{t}\,\frac{dr}{r}\,d\sigma(u')\,d\sigma(v')\\ &\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}\,|\lambda\varphi_1(a_\omega^{j+1})\,\xi|\,|\alpha\varphi_2(a_\omega^{k+1})\,\eta|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(2.18)\qquad &\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq (\ln(a_\omega))^2\,\|\Omega\|_{L^1}\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}.\end{aligned}$$

By interpolation between (2.17) and (2.18) with $0<\epsilon=\frac{1}{\omega+1}<1$, we obtain the estimate (v).

Now, to obtain the estimate (vi), we have

$$\begin{aligned}&\left|\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\\ &\leq \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}}|\Omega(u',v')|\,\left|e^{-i\lambda\varphi_1(t)\,\xi\cdot u'}-1\right|\frac{dt}{t}\,\frac{dr}{r}\,d\sigma(u')\,d\sigma(v').\end{aligned}$$

Now, since $a_\omega^j<t<a_\omega^{j+1}$ and φ_1 is increasing. Then by change of variables, we obtain

$$(2.19)\qquad \left|\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}\,|\lambda\varphi_1(a_\omega^{j+1})\,\xi|.$$

In addition, we have

$$(2.20)\qquad \left|\widehat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\right|\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}.$$

By interpolation between (2.19) and (2.20) with $0<\epsilon=\frac{1}{\omega+1}<1$, we get the estimate (vi). By same procedure, we obtain (vii). This ends the proof of Lemma 2.5. □

Now, we move to the estimates concerning the measures $\{\sigma_{\omega,j,k}^{(l,s)}\}$:

Lemma 2.6. *Let* $\{\sigma_{\omega,j,k}^{(l,s)} : 0 \le l \le d, 0 \le s \le b\}$ *be as above. Suppose that* $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *for some* $q > 1$. *Let* $B_{\Omega,\omega} = (\omega+1)^2 \left(\|\Omega\|_{L^1}\right)^{1-\frac{1}{\omega+1}} \left(\|\Omega\|_{L^q}\right)^{\frac{1}{\omega+1}}$. *Then*

(i) $\|\sigma_{\omega,j,k}^{(l,s)}\| \le C\ (\omega+1)^2 \|\Omega\|_{L^1}$;

(ii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\right|$
$\le C\, B_{\Omega,\omega} \, |c_{l,1}\,(a_\omega^j)^l\ l!\ \xi|^{-\frac{1}{q'\,l\,(\omega+1)}}\ |\,c_{s,2}\,(a_\omega^k)^s\ s!\ \eta|^{-\frac{1}{q'\,s\,(\omega+1)}}$;

(iii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta)\right|$
$\le C\, B_{\Omega,\omega}\, |c_{l,1}\,(a^{j+1})^l \xi|^{\frac{1}{\omega+1}} |c_{s,2}\,(a_\omega^k)^s\, s!\, \eta\,|^{-\frac{1}{q' s(\omega+1)}}$;

(iv) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta)\right|$
$\le C\, B_{\Omega,\omega}\ \left|c_{l,1}\,(a_\omega^j)^l\, l!\, \xi\,\right|^{-\frac{1}{q'\,l\,(\omega+1)}} \left|c_{s,2}(a_\omega^{k+1})^s\, \eta\right|^{\frac{1}{\omega+1}}$;

(v) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta) + \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right|$
$\le C\,(\omega+1)^2 \|\Omega\|_{L^1}\, |c_{l,1}(a_\omega^{j+1})^l\, \xi|^{\frac{1}{\omega+1}}\ |c_{s,2}\,(a_\omega^{k+1})^s\, \eta|^{\frac{1}{\omega+1}}$;

(vi) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right| \le C\ (\omega+1)^2 \|\Omega\|_{L^1} |c_{l,1}\,(a_\omega^{j+1})^l\, \xi|^{\frac{1}{\omega+1}}$;

(vii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right| \le C\ (\omega+1)^2 \|\Omega\|_{L^1} \left|c_{s,2}(a_\omega^{k+1})^s\, \eta\right|^{\frac{1}{\omega+1}}$,

where C *is independent of* ω *and* $k, j \in \mathbb{Z}, (\xi,\eta) \in (\mathbb{R}^n, \mathbb{R}^m)$.

Proof of Lemma 2.6. First, the estimate (i) is trivial. To prove (ii), by Hölder's inequality and polar coordinates, we get

$$\begin{aligned}
(2.21)\quad & \left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\right| \\
&\le \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega(u',v')|^q \, d\sigma(u')d\sigma(v')\right)^{\frac{1}{q}} \\
&\quad \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left|\int_{I_{j,\omega}\times I_{k,\omega}} e^{-i\left(P_l(t)\,\xi\cdot u' + Q_s(r)\,\eta\cdot v'\right)} \frac{dt}{t}\,\frac{dr}{r}\right|^{q'} d\sigma(u')d\sigma(v')\right)^{\frac{1}{q'}} \\
&\le \|\Omega\|_q \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |L_{j,k,\omega}(\xi,u',\eta,v')|^{q'}\ d\sigma(u')d\sigma(v')\right)^{\frac{1}{q'}},
\end{aligned}$$

where

$$L_{j,k,\omega}(\xi,u',\eta,v') = \int_{I_{j,\omega}\times I_{k,\omega}} e^{-i\left(P_l(t)\,\xi\cdot u' + Q_s(r)\,\eta\cdot v'\right)} \frac{dt}{t}\,\frac{dr}{r}.$$

Thus, by change of variables and triangle inequality, we get

$$|L_{j,k,\omega}(\xi,u',\eta,v')| \le U_{j,\omega,P_l}(\xi,u')\ U_{k,\omega,Q_s}(\eta,v'),$$

where

$$U_{j,\omega,P_l}(\xi,u') = \left|\int_1^{a_\omega} e^{-i(P_l(a_\omega^j\, t)\,\xi\cdot u')}\,\frac{dt}{t}\right|$$

and

$$U_{k,\omega,Q_s}(\eta,v') = \left|\int_1^{a_\omega} e^{-i(Q_s(a_\omega^k\, r)\,\eta\cdot v')}\,\frac{dr}{r}\right|.$$

Since,

$$\frac{d^l}{dt}P_l(a_\omega^j\, t) = c_{l,1}\,(a_\omega^j)^l\;l! \quad \text{and} \quad \frac{d^s}{dr}Q_s(a_\omega^k\, r) = c_{s,2}\,(a_\omega^k)^s\;s!.$$

Thus, by Van der Corput lemma in [29], we obtain

$$U_{j,\omega,P_l}(\xi,u') = \left|\int_1^{a_\omega} e^{-i(P_l(a_\omega^j\, t)\,\xi\cdot u')}\,\frac{dt}{t}\right| \le |\,c_{l,1}\,(a_\omega^j)^l\;l!\;\xi\cdot u'|^{-\frac{1}{l}},$$

and

$$U_{k,\omega,Q_s}(\eta,v') \le |\,c_{s,2}\,(a_\omega^k)^s\;s!\;\eta\cdot v'|^{-\frac{1}{s}}. \tag{2.22}$$

Thus,

$$|L_{j,k,\omega}(\xi,u',\eta,v')| \le\; |\,c_{l,1}\,(a_\omega^j)^l\;l!\;\xi\cdot u'|^{-\frac{1}{l}}\;|\,c_{s,2}\,(a_\omega^k)^s\;s!\;\eta\cdot v'|^{-\frac{1}{s}}. \tag{2.23}$$

It is easy to see that

$$|L_{j,k,\omega}(\xi,u',\eta,v')| \le\; C\,(\omega+1)^2. \tag{2.24}$$

Therefore, by interpolation between (2.23) and (2.24) with $\epsilon = \frac{1}{q'}$, we get

$$\begin{aligned} &|L_{j,k,\omega}(\xi,u',\eta,v')| \\ &\le C\,(\omega+1)^2\,|\,c_{l,1}\,(a_\omega^j)^l\;l!\;\xi\cdot u'|^{-\frac{1}{q'l}}\;|\,c_{s,2}\,(a_\omega^k)^s\;s!\;\eta\cdot v'|^{-\frac{1}{q's}}. \end{aligned} \tag{2.25}$$

Now, by the same argument as in (2.8)-(2.9), (2.21) and (2.25), we obtain

$$\begin{aligned} &\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)\right| \\ &\le C\,(\omega+1)^2\,\|\Omega\|_{L^q}\,|c_{l,1}\,(a_\omega^j)^l\;l!\;\xi|^{-\frac{1}{q'l}}\;|\,c_{s,2}\,(a_\omega^k)^s\;s!\;\eta|^{-\frac{1}{q's}}. \end{aligned} \tag{2.26}$$

On the other hand, we can obtain

$$\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)\right| \le C\,(\omega+1)^2\,\|\Omega\|_{L^1}. \tag{2.27}$$

Finally, by interpolation between (2.26) and (2.27) with $\epsilon = \frac{1}{\omega+1}$, we get

$$\begin{aligned} &\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)\right| \\ &\le C\,B_{\Omega,\omega}\,|c_{l,1}\,(a_\omega^j)^l\;l!\;\xi|^{-\frac{1}{q'\,l\,(\omega+1)}}\,|\,c_{s,2}\,(a_\omega^k)^s\;s!\,\eta|^{-\frac{1}{q'\,s\,(\omega+1)}}. \end{aligned}$$

For (iii), we have

$$\begin{aligned} &\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right| \\ = &\left|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}} e^{-i\left(P_l(t)\,\xi\cdot u' + Q_s(r)\,\eta\cdot v'\right)}\,\Omega(u',v')\,\frac{dt}{t}\,\frac{dr}{r}d\sigma(u')\,d\sigma(v')\right. \end{aligned}$$

$$- \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \int_{I_{j,\omega}\times I_{k,\omega}} e^{-i\left(P_{l-1}(t)\,\xi\cdot u'+Q_s(r)\,\eta\cdot v'\right)}\,\Omega(u',v')\,\frac{dt}{t}\,\frac{dr}{r}d\sigma(u')\,d\sigma(v')\Bigg|$$

$$= \Bigg|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \int_{I_{j,\omega}\times I_{k,\omega}} \Omega(u',v')\,e^{-i\,Q_s(r)\,\eta\cdot v'}$$

$$\left(e^{-i\,P_l(t)\,\xi\cdot u'} - e^{-i\,P_{l-1}(t)\,\xi\cdot u'}\right)\frac{dt}{t}\frac{dr}{r}d\sigma(u')\,d\sigma(v')\Bigg|.$$

By Fubini's theorem, we get

$$\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right|$$

$$\leq \int_{\mathbb{S}^{n-1}}\int_{I_{j,\omega}}\left|\int_{\mathbb{S}^{m-1}}\int_{I_{k,\omega}}\Omega(u',v')e^{-i\,Q_s(r)\,\eta\cdot v'}d\sigma(v')\frac{dr}{r}\right|$$

$$\left|e^{-iP_{l-1}(t)\,\xi\cdot u'}\left(e^{-i\,c_{l,1}\,t^l\,\xi.u'}-1\right)\right|\,\frac{dt}{t}\,d\sigma(u')$$

$$\leq \left|c_{l,1}\,(a^j_\omega)^l\,\xi\right|\int_{\mathbb{S}^{n-1}}\int_{I_{j,\omega}}\left|\int_{\mathbb{S}^{m-1}}\int_{I_{k,\omega}}\Omega(u',v')\,e^{-i\,Q_s(r)\,\eta\cdot v'}d\sigma(v')\frac{dr}{r}\right|\frac{dt}{t}\,d\sigma(u')$$

$$\leq \ln(a_\omega)\left|c_{l,1}\,(a^j_\omega)^l\xi\right|\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(u',v')|\;U_{k,\omega,Q_s}\,d\sigma(u')d\sigma(v'),$$

where

$$U_{k,\omega,Q_s}(\eta,v') = \left|\int_1^{a_\omega} e^{-i\,(Q_s(a^k_\omega r)\,\eta\cdot v')}\,\frac{dr}{r}\right|.$$

As (2.22), we have

$$U_{k,\omega,Q_s}(\eta,v') \leq |\,c_{s,2}\,(a^k_\omega)^s\;\; s!\;\eta\cdot v'|^{-\frac{1}{s}}. \tag{2.28}$$

On the other hand, we have

$$U_{k,\omega,Q_s}(\eta,v') \leq C\,(\omega+1). \tag{2.29}$$

By interpolation between (2.28) and (2.29) with $0<\epsilon=\frac{1}{q'}<1$, we obtain

$$U_{k,\omega,Q_s}(\eta,v') \leq C\,(\omega+1)|\,c_{s,2}\,(a^k_\omega)^s\;\; s!\;\eta\cdot v'|^{-\frac{1}{q's}}. \tag{2.30}$$

Thus, by (2.30) and Hölder's inequality, we get

$$\begin{aligned}
&\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right| \\
&\leq C\,(\omega+1)\,\ln(a_\omega)\,\|\Omega\|_{L^q}\,|c_{l,1}\,(a^{j+1})^l\xi|\,|c_{s,2}\,(a^k_\omega)^s\;\; s!\;\eta|^{-\frac{1}{q's}} \\
&\leq C\,(\omega+1)^2\;\|\Omega\|_{L^q}|c_{l,1}\,(a^{j+1})^l\xi|\,|c_{s,2}\,(a^k_\omega)^s\;\; s!\;\eta|^{-\frac{1}{q's}}.
\end{aligned} \tag{2.31}$$

By straightforward calculations, it is easy to obtain that

$$\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right| \leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}. \tag{2.32}$$

Finally, we combine (2.31) and (2.32), to get

$$\begin{aligned}&\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right|\\&\leq C\,B_{\Omega,\omega}\,|c_{l,1}\,(a^{j+1})^l\xi|^{\frac{1}{\omega+1}}\,|c_{s,2}(a^k_\omega)^s\ s!\ \eta|^{-\frac{1}{q'\ s(\omega+1)}}.\end{aligned}$$

Similarly, we can obtain the estimates (iv), we omit details.

For the estimate (v), we have

$$\begin{aligned}(2.33)\qquad&\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\\&\leq\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}}|\Omega(u',v')|\,|e^{-ic_{l,1}\,t^l\,\xi\cdot u'}-1|\\&\qquad|e^{-ic_{s,2}\,r^s\,\eta\cdot v'}-1|\frac{dt}{t}\,\frac{dr}{r}\,d\sigma(u')\,d\sigma(v')\\&\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}\,|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|\,|c_{s,2}\,(a^{k+1}_\omega)^s\,\eta|,\end{aligned}$$

where $a^j_\omega<t<a^{j+1}_\omega,\ a^k_\omega<r<a^{k+1}_\omega$ and $l,s>0$. Also, we have

$$\begin{aligned}(2.34)\qquad&\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\\&\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}.\end{aligned}$$

Thus, by interpolation between (2.33) and (2.34) with $0<\epsilon=\frac{1}{\omega+1}<1$, we get

$$\begin{aligned}&\left|\widehat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)+\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\\&\leq C\ (\omega+1)^2\,\|\Omega\|_{L^1}\,|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a^{k+1}_\omega)^s\,\eta|^{\frac{1}{\omega+1}}.\end{aligned}$$

On the other hand, we get (vi) as follows:

$$\begin{aligned}&\left|\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\\&\leq\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\int_{I_{j,\omega}\times I_{k,\omega}}|\Omega(u',v')|\ \left|e^{-ic_{l,1}\,t^l\,\xi\cdot u'}-1\right|\frac{dt}{t}\,\frac{dr}{r}\,d\sigma(u')\,d\sigma(v')\\&\leq|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|\,\|\Omega\|_{L^1}\int_{I_{j,\omega}\times I_{k,\omega}}\frac{dt}{t}\,\frac{dr}{r}.\end{aligned}$$

Then, by change of variables, we obtain

$$(2.35)\qquad\left|\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\leq C\ (\omega+1)^2\,\|\Omega\|_{L^1}\,|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|,$$

where $a^j_\omega<t<a^{j+1}_\omega$ and $l>0$. In addition, we have

$$(2.36)\qquad\left|\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\leq C\,(\omega+1)^2\,\|\Omega\|_{L^1}.$$

Finally, by interpolation between (2.35) and (2.36) with $0<\epsilon=\frac{1}{\omega+1}<1$, we get

$$\left|\widehat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right|\leq C\ (\omega+1)^2\,\|\Omega\|_{L^1}\,|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|^{\frac{1}{\omega+1}}.$$

By same procedure, we can obtain (vii). This ends the proof of Lemma 2.6. □

3. Proof of Theorem 1.2

Assume that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. By the same decomposition introduced in [11], we shall decompose the function Ω as follows: For $\omega \in \mathbb{N}$, let $\mathbb{E}_\omega$ be the set of points $(x', y') \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ satisfying $2^\omega \leq |\Omega(x', y')| < 2^{\omega+1}$. Also, we let $\mathbb{E}_0$ be the set of all those points $(x', y') \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ satisfying $|\Omega(x', y')| < 2$. For $\omega \in \mathbb{N} \cup \{0\}$, set $b_\omega = \Omega \chi_{\mathbb{E}_\omega}$ and $\theta_\omega = \|b_\omega\|_1$. Set $\mathbb{D} = \{\omega \in \mathbb{N} : \theta_\omega \geq 2^{-3\omega}\}$ and define the sequence of functions $\{\Omega_\omega\}_{\omega \in \mathbb{D} \cup \{0\}}$ by

$$\begin{aligned}\Omega_0(x, y) = {} & b_0(u', v') + \sum_{\omega \notin \mathbb{D}} b_\omega(x', y') \\ & - \int_{\mathbb{S}^{n-1}} b_0(u', y') d\sigma(u') - \int_{\mathbb{S}^{m-1}} b_0(x', v') d\sigma(v') \\ & - \sum_{\omega \notin \mathbb{D}} \left(\int_{\mathbb{S}^{n-1}} b_\omega(u', y') d\sigma(u') + \int_{\mathbb{S}^{m-1}} b_\omega(x', v') d\sigma(v') \right) \\ & + \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} \left(b_0(u', v') + \sum_{\omega \notin \mathbb{D}} b_\omega(u', v') \right) d\sigma(u')\, d\sigma(v')\end{aligned} \tag{3.1}$$

and for $\omega \in \mathbb{D}$,

$$\begin{aligned}\Omega_\omega(x, y) = (\theta_\omega)^{-1} \Bigg(& b_\omega(x', y') - \int_{\mathbb{S}^{n-1}} b_\omega(u', y') d\sigma(u') - \int_{\mathbb{S}^{m-1}} b_\omega(x', v') d\sigma(v') \\ & + \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} b_\omega(u', v') d\sigma(u')\, d\sigma(v') \Bigg).\end{aligned}$$

Thus, Ω_ω satisfies the following:

$$\int_{\mathbb{S}^{n-1}} \Omega_\omega(u', \cdot) d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega_\omega(\cdot, v') d\sigma(v') = 0,$$

$$\|\Omega_\omega\|_1 \leq 4, \quad \|\Omega_\omega\|_2 \leq 4(a_\omega)^2, \tag{3.2}$$

$$\Omega(x, y) = \sum_{\omega \in \mathbb{D} \cup 0} \theta_\omega \Omega_\omega(x, y), \tag{3.3}$$

$$\sum_{\omega \in \mathbb{D} \cup 0} (\omega + 1)^2 \, \theta_\omega \leq \|\Omega\|_{L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})}, \tag{3.4}$$

where $\theta_0 = 1$. By (3.3), we get

$$T_{\Omega,\, \Phi,\, \Psi}(f) = \sum_{\omega \in \mathbb{D} \cup 0} \theta_\omega \; T_{\Omega_\omega,\, \Phi,\, \Psi}(f)(x, y), \tag{3.5}$$

where

$$T_{\Omega_\omega,\,\Phi,\,\Psi} f = p.v. \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x - \Phi(|u|)u', y - \Psi(|v|)v') \,\frac{\Omega_\omega(u', v')}{|u|^n\,|v|^m}\, dudv.$$

Therefore, by (3.4) and (3.5), our main goal is to show that

$$\|T_{\Omega_\omega,\,\Phi,\,\Psi} f\|_p \le C_p(\omega+1)^2 \|f\|_p. \tag{3.6}$$

In fact, when Ω is replaced by Ω_ω, we can see

$$T_{\Omega_\omega,\,\Phi,\,\Psi} f(x,y) = \sum_{j,\,k\in\mathbb{Z}} \sigma_{\omega,j,k}^{\,(d+1,b+1)} * f, \tag{3.7}$$

where $\{\sigma_{\omega,j,k}^{\,(l,s)} : 0 \le l \le d+1, 0 \le s \le b+1\}$ is the sequence of measure defined in (2.1)-(2.2). Notice that

$$\|\Omega_\omega\|_2^{\frac{1}{\omega+1}} \le \left(4\,(a_\omega)^2\right)^{\frac{1}{\omega+1}} \le C$$

and

$$\|\Omega_\omega\|_1^{1-\frac{1}{\omega+1}} \le 4.$$

Thus by (3.2), Lemma 2.5 and Lemma 2.6 with $q=2$, $c_{d+1,1} = \lambda$ and $c_{b+1,2} = \alpha$, we have for $0 \le l \le d+1$ and $0 \le s \le b+1$ that

(i) $\|\sigma_{\omega,j,k}^{\,(l,s)}\| \le C\ (\omega+1)^2$;

(ii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\right|$
$\le C\,(\omega+1)^2\,|c_{l,1}\,(a_\omega^j)^l\ l!\ \xi|^{-\frac{1}{2\,l\,(\omega+1)}}\,|\,c_{s,2}\,(a_\omega^k)^s\ s!\ \eta|^{-\frac{1}{2\,s\,(\omega+1)}}$;

(iii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta)\right|$
$\le C\,(\omega+1)^2\,|c_{l,1}\,(a^{j+1})^l \xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a_\omega^k)^s\ s!\ \eta|^{-\frac{1}{2s(\omega+1)}}$;

(iv) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta)\right|$
$\le C\,(\omega+1)^2\ \left|c_{l,1}\,(a_\omega^j)^l\,l!\,\xi\right|^{-\frac{1}{2\,l\,(\omega+1)}}\ \left|c_{s,2}(a_\omega^{k+1})^s\,\eta\right|^{\frac{1}{\omega+1}}$;

(v) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta) + \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right|$
$\le C\ (\omega+1)^2\,|c_{l,1}(a_\omega^{j+1})^l\,\xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a_\omega^{k+1})^s\,\eta|^{\frac{1}{\omega+1}}$;

(vi) $\left|\widehat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right| \le C\ (\omega+1)^2\,|c_{l,1}\,(a_\omega^{j+1})^l\,\xi|^{\frac{1}{\omega+1}}$;

(vii) $\left|\widehat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta) - \widehat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\right| \le C\ (\omega+1)^2\ \left|c_{s,2}(a_\omega^{k+1})^s\,\eta\right|^{\frac{1}{\omega+1}}$.

We choose and fix a function $\phi(t) \in C_0^\infty(\mathbb{R})$ such that $\phi(t) = 1$ for $|t| \leqslant \frac{1}{2}$ and $\phi(t) = 0$ for $|t| \geqslant 1$. For $j \in \mathbb{Z}$ and $i \in \mathbb{N}$, let $\varphi_j^{(i)}$ be defined by

$$\psi_j^{(i)}(t) = \phi(((a_\omega)^{ij} t)^2).$$

For $1 \le l \le d+1$ and $1 \le s \le b+1$, we define the family of measures $\{\tau_{\omega,j,k}^{(l,s)} : j,\,k \in \mathbb{Z}\}$ by

$$\hat{\tau}_{\omega,j,k}^{(l,s)}(\xi,\eta) \tag{3.8}$$

$$
\begin{aligned}
&= \hat{\sigma}^{(l,s)}_{\omega,j,k}(\xi,\eta) \prod_{l<i\le d+1} \psi^{(i)}_j(|c_{i,1}\,\xi|) \prod_{s<r\le b+1} \psi^{(r)}_k(|c_{r,2}\eta|) \\
&\quad - \hat{\sigma}^{(l-1,s)}_{\omega,j,k}(\xi,\eta) \prod_{l-1<i\le d+1} \psi^{(i)}_j(|c_{i,1}\,\xi|) \prod_{s<r\le b+1} \psi^{(r)}_k(|c_{r,2}\eta|) \\
&\quad - \hat{\sigma}^{(l,s-1)}_{\omega,j,k}(\xi,\eta) \prod_{l<i\le d+1} \psi^{(i)}_j(|c_{i,1}\,\xi|) \prod_{s-1<r\le b+1} \psi^{(r)}_k(|c_{r,2}\eta|) \\
&\quad + \hat{\sigma}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta) \prod_{l-1<i\le d+1} \psi^{(i)}_j(|c_{i,1}\,\xi|) \prod_{s-1<r\le b+1} \psi^{(r)}_k(|c_{r,2}\eta|)
\end{aligned}
$$

with the convention that $\prod_{i\in\emptyset} E_i = 1$, where $\emptyset$ is the empty. It is worth noticing that

$$
\begin{aligned}
\hat{\tau}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta) &= \hat{\sigma}^{(d+1,b+1)}_{\omega,j,k}(\xi,\eta) - \hat{\sigma}^{(d,b+1)}_{\omega,j,k}(\xi,\eta)\psi^{(d+1)}_j(|c_{d+1,1}\,\xi|) \\
&\quad - \hat{\sigma}^{(d+1,b)}_{\omega,j,k}(\xi,\eta)\psi^{(b+1)}_k(|c_{b+1,2}\eta|) \\
&\quad + \hat{\sigma}^{(d,b)}_{\omega,j,k}(\xi,\eta)\psi^{(d+1)}_j(|c_{d+1,1}\,\xi|)\psi^{(b+1)}_k(|c_{b+1,2}\eta|)
\end{aligned}
$$

which implies that $\hat{\tau}^{(d+1,b+1)}_{\omega,j,k}$ and $\hat{\sigma}^{(d+1,b+1)}_{\omega,j,k}$ have the same support that is equal to $\mathbb{R}^n\times\mathbb{R}^m$.

It is clearly that the measures $\{\tau^{(l,s)}_{\omega,j,k}(\xi,\eta)\}$ satisfy:

$$
\|\tau^{(l,s)}_{\omega,j,k}\| \le C\ (\omega+1)^2, \tag{3.9}
$$

$$
\begin{aligned}
&\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta)\right| \\
&\le C\,(\omega+1)^2\,|c_{l,1}\,(a^j_\omega)^l\ l!\ \xi|^{-\frac{1}{2\,l\,(\omega+1)}}\,|\,c_{s,2}\,(a^k_\omega)^s\ s!\ \eta|^{-\frac{1}{2\,s\,(\omega+1)}},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l-1,s)}_{\omega,j,k}(\xi,\eta)\right| \\
&\le C\,(\omega+1)^2\,|c_{l,1}\,(a^{j+1})^l\ \xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a^k_\omega)^s\,s!\,\eta|^{-\frac{1}{2s(\omega+1)}},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l,s-1)}_{\omega,j,k}(\xi,\eta)\right| \\
&\le C\,(\omega+1)^2\ \left|c_{l,1}\,(a^j_\omega)^l\,l!\,\xi\right|^{-\frac{1}{2\,l\,(\omega+1)}}\ \left|c_{s,2}(a^{k+1}_\omega)^s\,\eta\right|^{\frac{1}{\omega+1}},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l-1,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l,s-1)}_{\omega,j,k}(\xi,\eta) + \widehat{\tau}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right| \\
&\le C\ (\omega+1)^2\,|c_{l,1}(a^{j+1}_\omega)^l\,\xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a^{k+1}_\omega)^s\,\eta|^{\frac{1}{\omega+1}},
\end{aligned}
$$

$$
\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right| \le C\ (\omega+1)^2\,|c_{l,1}\,(a^{j+1}_\omega)^l\,\xi|^{\frac{1}{\omega+1}},
$$

$$
\left|\widehat{\tau}^{(l,s)}_{\omega,j,k}(\xi,\eta) - \widehat{\tau}^{(l-1,s-1)}_{\omega,j,k}(\xi,\eta)\right| \le C\ (\omega+1)^2\ \left|c_{s,2}(a^{k+1}_\omega)^s\,\eta\right|^{\frac{1}{\omega+1}}. \tag{3.10}
$$

Also, we can see that

$$\sum_{l=0}^{d+1}\sum_{s=0}^{b+1}\tau_{\omega,j,k}^{(l,s)}=\sigma_{\omega,j,k}^{(d+1,b+1)}. \tag{3.11}$$

To see (3.11), we make use of the definition of $\tau_{\omega,j,k}^{(l,s)}$ in (3.8) and the fact that $\sigma_{\omega,j,k}^{(l,0)}=\sigma_{\omega,j,k}^{(0,s)}=0$ for all $0\le l\le d+1$ and $0\le s\le b+1$. In fact, for $1\le l\le d+1$ and $1\le s\le b+1$, let

$$\Delta^{(l,s)}(\xi,\eta)=\prod_{l<i\le d+1}\psi_j^{(i)}(|c_{i,1}\,\xi|)\prod_{s<r\le b+1}\psi_k^{(r)}(|c_{r,2}\eta|).$$

Then

$$\begin{aligned}
&\sum_{l=0}^{d+1}\sum_{s=0}^{b+1}\widehat{\tau}_{\omega,j,k}^{(l,s)}(\xi,\eta)\\
&=\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\widehat{\tau}_{\omega,j,k}^{(l,s)}(\xi,\eta)\\
&=\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\Delta^{(l,s)}(\xi,\eta)-\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l-1,s)}(\xi,\eta)\Delta^{(l-1,s)}(\xi,\eta)\\
&\quad-\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l,s-1)}(\xi,\eta)\Delta^{(l,s-1)}(\xi,\eta)+\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l-1,s-1)}(\xi,\eta)\Delta^{(l-1,s-1)}(\xi,\eta)\\
&=\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\Delta^{(l,s)}(\xi,\eta)-\sum_{l=0}^{d}\sum_{s=1}^{b+1}\hat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\Delta^{(l,s)}(\xi,\eta)\\
&\quad-\sum_{l=1}^{d+1}\sum_{s=0}^{b}\hat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\Delta^{(l,s)}(\xi,\eta)+\sum_{l=0}^{d}\sum_{s=0}^{b}\hat{\sigma}_{\omega,j,k}^{(l,s)}(\xi,\eta)\Delta^{(l,s)}(\xi,\eta)\\
&=\hat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)\Delta^{(d+1,b+1)}(\xi,\eta)\\
&=\hat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta)\prod_{i\in\emptyset}\psi_j^{(i)}(|c_{i,1}\,\xi|)\prod_{r\in\emptyset}\psi_k^{(r)}(|c_{r,2}\eta|)\\
&=\hat{\sigma}_{\omega,j,k}^{(d+1,b+1)}(\xi,\eta).
\end{aligned}$$

Thus,

$$\begin{aligned}
T_{\Phi,\Psi,\Omega_\omega}f(x,y)&=\sum_{j,k\in\mathbb{Z}}\sum_{l=0}^{d+1}\sum_{s=0}^{b+1}\tau_{\omega,j,k}^{(l,s)}*f\\
&=\sum_{l=0}^{d+1}\sum_{s=0}^{b+1}\left(\sum_{j,k\in\mathbb{Z}}\tau_{\omega,j,k}^{(l,s)}*f\right)
\end{aligned}$$

which implies that

$$\|T_{\Phi,\Psi,\Omega_\omega} f\|_p \leq \sum_{l=0}^{d+1}\sum_{s=0}^{b+1} \|T^{(l,s)}_{\omega,j,k} f\|_p, \tag{3.12}$$

where

$$T^{(l,s)}_{\omega,j,k} f(x,y) = \sum_{j,k\in\mathbb{Z}} \tau^{(l,s)}_{\omega,j,k} * f.$$

On the other hand, for $\Phi \in \mathscr{PC}_\lambda(d), \Psi\mathscr{PC}_\alpha(b)$ for some d, $b > 0$, let

$$\mathscr{M}_{\Psi,\Omega_\omega}(f)(x,y) = \sup_{k\in\mathbb{Z}} \left| \int_{I^{(m)}_{k,\omega}} f(\cdot\,, y - \Psi(|v|)v')\, \frac{\Omega_\omega(\cdot\,, v')}{|v|^m}\, d(v) \right|$$

and

$$\mathscr{M}_{\Phi,\Omega_\omega}(f)(x,y) = \sup_{j\in\mathbb{Z}} \left| \int_{I^{(n)}_{j,\omega}} f(x - \Phi(|u|)u'),\, \cdot)\, \frac{\Omega_\omega(u',\, \cdot)}{|u|^n}\, d(u) \right|.$$

Then

$$(\tau^{(l,s)}_{\omega,j,k})^*(f) = \sup_{k,j\in\mathbb{Z}} \left|\, |\tau^{(l,s)}_{\omega,j,k}| * f \right| = \mathscr{M}_{\Phi,\Psi,\Omega_\omega}(f)(x,y), \tag{3.13}$$

where

$$\begin{aligned} &\mathscr{M}_{\Phi,\Psi,\Omega_\omega}(f)(x,y) \\ &= \sup_{j,k\in\mathbb{Z}} \left| \int_{I^{(m)}_{j,\omega}} \int_{I^{(n)}_{j,\omega}} f(x - \Phi(|u|)u', y - \Psi(|v|)v'))\, \frac{\Omega_\omega(u',v')}{|u|^n|v|^m}\, d(u)\, d(v) \right|. \end{aligned}$$

Since, $\mathscr{M}_{\Phi,\Psi,\Omega_\omega}(f) \leq \mathscr{M}_{\Phi,\Omega_\omega}(f) \circ \mathscr{M}_{\Psi,\Omega_\omega}(f)$. Thus, by using Theorem 2.3, we get

$$\begin{aligned} \|(\tau^{(l,s)}_{\omega,j,k})^*(f)\|_p &\leq \|\mathscr{M}_{\Phi,\Omega_\omega}(f) \circ \mathscr{M}_{\Psi,\Omega_\omega}(f)\|_p \\ &\leq C\,(\omega+1)^2\ \|f\|_p. \end{aligned} \tag{3.14}$$

Hence, by (3.14) and an argument similar to that used in the proof of Lemma 8 in [2], we obtain

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} |\tau^{(l,s)}_{\omega,j,k} * g_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p \leq C(\omega+1)^2 \left\| \left(\sum_{j,k\in\mathbb{Z}} |g_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p \tag{3.15}$$

for arbitrary functions $\{g_{k,j}\}$ on $\mathbb{R}^N \times \mathbb{R}^M$. Thus, by (3.9)-(3.10), (3.15) and Theorem 2.4 with $L_l(\xi) = c_{k,1}\,\xi$ and $Q_s(\eta) = c_{k,2}\,\eta$, we obtain

$$\left\| T^{(l,s)}_{\omega,j,k} f \right\|_p = \left\| \sum_{j,k\in\mathbb{Z}} \tau^{(l,s)}_{\omega,j,k} * f \right\|_p \leq C_p\,(\omega+1)^2 \|f\|_p. \tag{3.16}$$

Finally, by (3.6), (3.7), (3.12) and (3.16), we get

$$\|T_{\Omega_\omega,\,\Phi,\,\Psi} f\|_p = \Big\| \sum_{j,\,k\in\mathbb{Z}} \sigma^{(d+1,b+1)}_{\omega,j,k} * f \Big\|_p \le C_p\,(\omega+1)^2 \|f\|_p. \tag{3.17}$$

By (3.17), (3.12) and (3.4), the proof is complete.

4. Preparation for Marcinkiewicz integral operators

In this section, we shall set up the needed estimates to prove the L^p boundedness of $\mathcal{M}_{\Omega,\Phi,\Psi}$. Let $\{\Omega_\omega\}$ be the decomposition given in (3.1)-(3.4) with (3.4) replaced by

$$\sum_{\omega\in\mathbb{D}\cup 0} (\omega+1)\,\theta_\omega \le \|\Omega\|_{L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}. \tag{4.1}$$

For $\Lambda_\omega(t',s') = \{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m : |u|\le a_\omega^{t'} \text{ and } |v|\le a_\omega^{s'}\}$, we have

$$\begin{aligned}\mathcal{M}_{\Omega,\Phi,\Psi} f(x,y)\\ = (\omega+1)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|F^{(\Phi,\Psi)}_{t',s'}(f)(x,y)\right|^2 a_\omega^{-2(t'+s')} dt'\,ds'\right)^{\frac{1}{2}},\end{aligned} \tag{4.2}$$

where

$$F^{(\Phi,\Psi)}_{t',s'}(f)(x,y) = \int\int_{\Lambda_\omega(t',s')} f(x-\Phi(|u|)u', y-\Psi(|v|)v')\,\frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,du\,dv.$$

Thus, by (3.3), we have

$$\mathcal{M}_{\Omega,\Phi,\Psi} f(x,y) \le C \sum_{\omega\in\mathbb{D}\cup 0} \theta_\omega\, \mathcal{M}_{\Omega_\omega,\Phi,\Psi} f(x,y), \tag{4.3}$$

where $\mathcal{M}_{\Omega_\omega,\Phi,\Psi}$ is given by (4.2) with Ω replaced by Ω_ω. Define the family of measures $\{\sigma^{(d+1,b+1)}_{\omega,t',s'} : t',\, s'\in\mathbb{R}\}$ by

$$\begin{aligned}\widehat{\sigma}^{(d+1,b+1)}_{\omega,t',s'}(\xi,\eta)\\ = a_\omega^{-(t'+s')} \int\int_{\Gamma(a_\omega^{t'},a_\omega^{s'})} e^{-i\left(\Phi(|u|)\,\xi\cdot u' + \Psi(|v|)\,\eta\cdot v'\right)} \frac{\Omega_\omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,du\,dv\end{aligned} \tag{4.4}$$

and define $\{\sigma^{(l,s)}_{\omega,t',s'} : 0\le l\le d,\ 0\le s\le b\}$ by

$$\begin{aligned}\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta)\\ = a_\omega^{-(t'+s')} \int\int_{\Gamma(a_\omega^{t'},a_\omega^{s'})} e^{-i\left(P_l(|u|)\,\xi\cdot u' + Q_s(|v|)\,\eta\cdot v'\right)} \frac{\Omega_\omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}\,du\,dv.\end{aligned} \tag{4.5}$$

Here, $\Gamma(a_\omega^{t'},a_\omega^{s'}) = \{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m : a_\omega^{t'-1} < |u| \le a_\omega^{t'} \text{ and } a_\omega^{s'-1} < |v| \le a_\omega^{s'}\}$. Notice that

$$\widehat{\sigma}^{(0,0)}_{\omega,t',s'} = \widehat{\sigma}^{(0,b+1)}_{\omega,t',s'} = \widehat{\sigma}^{(d+1,0)}_{\omega,t',s'} = 0.$$

Thus, by (4.3)-(4.5), we have

$$\mathcal{M}_{\Omega,\Phi,\Psi}(f)(x,y)$$
$$\leq C \sum_{\omega\in\mathbb{D}\cup\{0\}} (\omega+1)\,\theta_\omega \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\sigma^{(d+1,b+1)}_{\omega,t',s'} * (f)(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}.$$

Therefore, to prove Theorem 1.3 we need to obtain the L^p-norm of the operator

$$\left\|\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\sigma^{(d+1,b+1)}_{\omega,t',s'} * (f)(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}\right\|_p. \tag{4.6}$$

To this end, we have the following two lemmas:

Lemma 4.1. *Let* $\{\sigma^{(d+1,b+1)}_{\omega,t',s'} : t',\, s'\in\mathbb{R}\}$ *be the measures given in* (4.4). *Then*

(i) $\|\sigma^{(d+1,b+1)}_{\omega,t',s'}\| \leq C$;

(ii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,|\lambda\,\varphi_1\,(a_\omega^{t'-1})\,\xi\,|^{-\frac{1}{2\,(d+1)(\omega+1)}}\;|\alpha\,\varphi_2\,(a_\omega^{s'-1})\,\eta\,|^{-\frac{1}{2\,(b+1)(\omega+1)}}$;

(iii) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,\left|\lambda\varphi_1(a_\omega^{t'})\,\xi\right|^{\frac{1}{\omega+1}}\,\left|\alpha\varphi_2(a_\omega^{s'-1})\eta\right|^{-\frac{1}{2(b+1)\,(\omega+1)}}$;

(iv) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d+1,b)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,\left|\,\lambda\,\varphi_1(a_\omega^{t'-1})\,\xi\right|^{-\frac{1}{2\,(d+1)\,(\omega+1)}}\,\left|\,\alpha\,\varphi_2(a_\omega^{s'})\,\eta\right|^{\frac{1}{\omega+1}}$;

(v) $\left|\widehat{\sigma}^{(d+1,b+1)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d,b+1)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d+1,b)}_{\omega,t',s'}(\xi,\eta) + \widehat{\sigma}^{(d,b)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C|\lambda\varphi_1(a_\omega^{t'})\,\xi|^{\frac{1}{\omega+1}}\,|\alpha\varphi_2(a_\omega^{s'})\,\eta|^{\frac{1}{\omega+1}}$;

(vi) $\left|\widehat{\sigma}^{(d+1,b)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d,b)}_{\omega,t',s'}(\xi,\eta)\right| \leq C\,|\lambda\varphi_1(a_\omega^{t'})\,\xi|^{\frac{1}{\omega+1}}$;

(vii) $\left|\widehat{\sigma}^{(d,b+1)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(d,b)}_{\omega,t',s'}(\xi,\eta)\right| \leq C\,\left|\,\alpha\,\varphi_2(a_\omega^{s'})\,\eta\right|^{\frac{1}{\omega+1}}$,

where C *is independent of* ω *and* $(\xi,\eta)\in(\mathbb{R}^n,\mathbb{R}^m)$.

Lemma 4.2. *Let* $\{\sigma^{(l,s)}_{\omega,t',s'} : 0\leq l\leq d,\ 0\leq s\leq b\}$ *be the measure given in* (4.5). *Then*

(i) $\|\sigma^{(l,s)}_{\omega,t',s'}\| \leq C$;

(ii) $\left|\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,|c_{l,1}\,(a_\omega^{t'-1})^l\;l!\;\xi|^{-\frac{1}{2\,l\,(\omega+1)}}\;|\,c_{s,2}\,(a_\omega^{s'-1})^s\;s!\;\eta|^{-\frac{1}{2\,s\,(\omega+1)}}$;

(iii) $\left|\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta) - \widehat{\sigma}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,|c_{l,1}\,(a_\omega^{t'})^l\xi|^{\frac{1}{\omega+1}}\;|c_{s,2}\,(a_\omega^{s'-1})^s\,s!\;\eta|^{-\frac{1}{2\,s\,(\omega+1)}}$;

(iv) $\left|\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\left|c_{l,1}\,(a_\omega^{t'-1})^l\,l!\,\xi\right|^{-\frac{1}{2\,l\,(\omega+1)}}\left|c_{s,2}(a_\omega^{s'})^s\,\eta\right|^{\frac{1}{\omega+1}};$

(v) $\left|\widehat{\sigma}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)+\widehat{\sigma}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|$
$\leq C\,|c_{l,1}(a_\omega^{t'})^l\,\xi|^{\frac{1}{\omega+1}}\,|c_{s,2}\,(a_\omega^{s'})^s\,\eta|^{\frac{1}{\omega+1}};$

(vi) $\left|\widehat{\sigma}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\,|c_{l,1}\,(a_\omega^{t'})^l\,\xi|^{\frac{1}{\omega+1}};$

(vii) $\left|\widehat{\sigma}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\sigma}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\left|c_{s,2}(a_\omega^{s'})^s\,\eta\right|^{\frac{1}{\omega+1}},$

where C is independent of ω and $(\xi,\eta)\in(\mathbb{R}^n,\mathbb{R}^m)$.

Proof of Lemma 4.1 and Lemma 4.2. To prove Lemmas 4.1 and 4.2, we use similar ideas as in the proofs of Lemma 2.5 and Lemma 2.6, but here we use the property of Ω_ω with $q=2$. We omit details.

Now, we define the measures $\{\tau^{(l,s)}_{\omega,t',s'}:0\leq l\leq d+1, 0\leq s\leq b+1\}$ as in (3.8) with proper modifications of the involved measures. Therefore, the following estimates hold:

$$\|\tau^{(l,s)}_{\omega,t',s'}\|\leq C; \tag{4.7}$$

$$\left|\widehat{\tau}^{(l,s)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\,|a_{\omega,l,t'}\,L_l(\xi)|^{-\frac{1}{2\,\beta_l\,(\omega+1)}}\,|a_{\omega,s,s'}\,Q_s(\eta)|^{-\frac{1}{2\,\delta_s\,(\omega+1)}}; \tag{4.8}$$

$$\begin{aligned}&\left|\widehat{\tau}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)\right|\\ &\leq C\,|a_{\omega,l,t'}\,L_l(\xi)|^{\frac{1}{\omega+1}}\,|a_{\omega,s,s'}\,Q_s(\eta)|^{-\frac{1}{2\,\delta_s\,(\omega+1)}};\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\left|\widehat{\tau}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\\ &\leq C\,|a_{\omega,l,t'}\,L_l(\xi)|^{-\frac{1}{2\,\beta_l\,(\omega+1)}}\,|a_{\omega,s,s'}\,Q_s(\eta)|^{\frac{1}{\omega+1}};\end{aligned} \tag{4.10}$$

$$\begin{aligned}&\left|\widehat{\tau}^{(l,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)+\widehat{\tau}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\\ &\leq C\,|a_{\omega,l,t'}\,L_l(\xi)|^{\frac{1}{\omega+1}}\,|a_{\omega,s,s'}\,Q_s(\eta)|^{\frac{1}{\omega+1}};\\ &\left|\widehat{\tau}^{(l,s-1)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\,|a_{\omega,l,t'}\,L_l(\xi)|^{\frac{1}{\omega+1}};\\ &\left|\widehat{\tau}^{(l-1,s)}_{\omega,t',s'}(\xi,\eta)-\widehat{\tau}^{(l-1,s-1)}_{\omega,t',s'}(\xi,\eta)\right|\leq C\,|a_{\omega,s,s'}\,Q_s(\eta)|^{\frac{1}{\omega+1}}\end{aligned} \tag{4.11}$$

and

$$\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\tau^{(l,s)}_{\omega,t',s'}=\sigma^{(d+1,b+1)}_{\omega,t',s'}, \tag{4.12}$$

where

$$L_l(\xi)=\begin{cases}\lambda\,\xi, & l=d+1,\\ c_{l,1}\,\xi, & l\neq d+1,\end{cases}\quad Q_s(\eta)=\begin{cases}\alpha\,\eta, & s=b+1,\\ c_{s,2}\,\eta, & s\neq b+1,\end{cases}$$

$$a_{\omega,l,t'}=\begin{cases}\varphi_1(a_\omega^{t'-1}), & l=d+1,\\ (a_\omega^{t'-1})^l, & l=d-1,\\ (a_\omega^{t'-1})^l\,l!, & l\neq d+1,d-1,\end{cases}\qquad a_{\omega,s,s'}=\begin{cases}\varphi_2(a_\omega^{s'-1}), & s=b+1,\\ (a_\omega^{s'-1})^s, & s=b-1,\\ (a_\omega^{s'-1})^s\,s!, & s\neq b+1,b-1,\end{cases}$$

$$\beta_l=\begin{cases}d+1, & l=d+1,\\ l, & l\neq d+1,\end{cases}$$

and

$$\delta_s=\begin{cases}b+1, & s=b+1,\\ s, & s\neq b+1.\end{cases}$$

Thus, by (4.12), we obtain that

$$\begin{aligned}(4.13)\qquad &\mathcal{M}_{\Omega_\omega,\Phi,\Psi}(f)(x,y)\\ &\leq C\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\tau_{\omega,t',s'}^{(l,s)}*(f)(x,y)\right)\right|^2dt'\,ds'\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, to prove (4.6), we need to obtain the L^p boundedness of the operator

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\tau_{\omega,t',s'}^{(l,s)}*(f)(x,y)\right)\right|^2dt'\,ds'\right)^{\frac{1}{2}}\tag{4.14}$$

for all $1<p<\infty$. □

5. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{M}_{\Omega_\omega,\varsigma}^{(l,s)}$ be operators given by (4.14). By (4.13), Minkowski's inequality and (4.1), we need to prove that

$$\|\mathcal{M}_{\Omega_\omega,\varsigma}^{(l,s)}(f)\|_p\leq C_p\,\|f\|_p$$

for $1<p<\infty$ and C_p is independent of ω. Now, choose two collections of $\mathcal{C}^\infty$ functions $\{\varpi_k^{(l)}\}_{k\in\mathbb{Z}}$ and $\{\varpi_k^{(s)}\}_{k\in\mathbb{Z}}$ on $(0,\infty)$ satisfying the following properties:

$$(5.1)\quad supp(\varpi_k^{(l)})\subseteq\left[\frac{1}{a_{\omega,l,k+1}},\frac{1}{a_{\omega,l,k-1}}\right]\text{ and }supp(\varpi_k^{(s)})\subseteq\left[\frac{1}{a_{\omega,s,k+1}},\frac{1}{a_{\omega,s,k-1}}\right],$$

$$0\leq\varpi_k^{(l)},\varpi_k^{(s)}\leq 1;$$

$$\sum_{k\in\mathbb{Z}}\varpi_k^{(l)})(u)=\sum_{k\in\mathbb{Z}}\varpi_k^{(s)})(u)=1;\tag{5.2}$$

$$\left|\frac{d^r\varpi_k^{(l)}}{du^r}(u)\right|, \quad \left|\frac{d^r\varpi_k^{(s)}}{du^r}(u)\right| \le \frac{C_r}{u^r},$$

where C_r is independent of ω. Define the measures $\{\upsilon_k^{(l)} : k \in \mathbb{Z}\}$ on $\mathbb{R}^n$ and $\{\upsilon_k^{(s)} : k \in \mathbb{Z}\}$ on $\mathbb{R}^m$ by

$$\widehat{(}\upsilon_k^{(l)})(x) = \varpi_k^{(l)}(|x|^2) \text{ and } \widehat{(}\upsilon_k^{(s)})(y) = \varpi_k^{(s)}(|y|^2).$$

By (5.2), we immediately obtain

$$\begin{aligned}(5.3)\quad (\widehat{\tau_{\omega,t',s'}^{(l,s)} * f})(\xi,\eta) &= \widehat{\tau}_{\omega,t',s'}^{(l,s)}(\xi,\eta) \cdot \widehat{f}(\xi,\eta)\sum_{j\in\mathbb{Z}}\widehat{\upsilon}_j^{(l)}(\xi) \cdot \sum_{k\in\mathbb{Z}}\widehat{\upsilon}_k^{(s)}(\eta)\\ &= \widehat{\tau}_{\omega,t',s'}^{(l,s)}(\xi,\eta) \cdot \widehat{f}(\xi,\eta)\sum_{j\in\mathbb{Z}}\widehat{\upsilon}_{\lfloor t'\rfloor+j}^{(l)}(\xi) \cdot \sum_{k\in\mathbb{Z}}\widehat{\upsilon}_{\lfloor s'\rfloor+k}^{(s)}(\eta),\end{aligned}$$

where $\lfloor t'\rfloor$ is the greatest integer functions such that $t'-1 < \lfloor t'\rfloor < t'$, similarly for $\lfloor s'\rfloor$ (see Al-Salman in [7,8]). Hence, by taking the inverse Fourier transform for (5.3), we get

$$(5.4)\qquad (\tau_{\omega,t',s'}^{(l,s)} * f)(x,y) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * \tau_{\omega,t',s'}^{(l,s)} * f(x,y).$$

Thus, by (3.3), (4.13), (5.4) and by Minkowski's inequality, $\mathcal{M}_{\Omega_\omega,\varsigma}^{(l,s)}$ is dominated as follows:

$$(5.5)\qquad \mathcal{M}_{\Omega_\omega,\varsigma}^{(l,s)}(f)(x,y) \le C\sum_{\omega\in\mathbb{D}\cup 0}(\omega+1)\,\theta_\omega\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} I_{k,j,\omega}^{(l,s)}(f)(x,y),$$

where

$$\begin{aligned}&I_{k,j,\omega}^{(l,s)}(f)(x,y)\\ &= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * \tau_{\omega,t',s'}^{(l,s)} * f(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}.\end{aligned}$$

Now, let

$$S_{k,j}(f)(x,y) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * f(x,y)\right|^2 dt'\,ds'\right)^{\frac{1}{2}}.$$

Thus, by Littlewood-Paley theory in [28], we get

$$(5.6)\qquad \|S_{k,j}(f)\|_p \le C\,\|f\|_p$$

for $1 < p < \infty$ and C is independent of ω.

For $p > 2$, we take $q = \left(\frac{p}{2}\right)'$. Then there exists $g \in L^q(\mathbb{R}^n \times \mathbb{R}^m)$ with $\|g\|_q \le 1$ such that

$$\begin{aligned}\|I_{k,j,\omega}^{(l,s)}(f)\|_p^2 = \int_{\mathbb{R}^n\times\mathbb{R}^m}&\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon_{\lfloor t'\rfloor+j}^{(l)} \otimes \upsilon_{\lfloor s'\rfloor+k}^{(s)}\right) * \tau_{\omega,t',s'}^{(l,s)} * f(x,y)\right|^2 dt'\,ds'\right)\\ &|g(x,y)|\,dx\,dy.\end{aligned}$$

Now, by the Cauchy-Schwarz inequality and (4.7), we have

$$\begin{aligned}&\left|\left(\upsilon^{(l)}_{\lfloor t'\rfloor+j}\otimes\upsilon^{(s)}_{\lfloor s'\rfloor+k}\right)*\tau^{(l,s)}_{\omega,t',s'}*f(x,y)\right|^2\\ &\le|\tau^{(l,s)}_{\omega,t',s'}|*|\left(\upsilon^{(l)}_{\lfloor t'\rfloor+j}\otimes\upsilon^{(s)}_{\lfloor s'\rfloor+k}\right)|*|f(x,y)|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|I^{(l,s)}_{k,j,\omega}(f)\|_p^2\\ &\le\int_{\mathbb{R}^n\times\mathbb{R}^m}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\tau^{(l,s)}_{\omega,t',s'}|*|\left(\upsilon^{(l)}_{\lfloor t'\rfloor+j}\otimes\upsilon^{(s)}_{\lfloor s'\rfloor+k}\right)*f(x,y)|^2dt'\,ds'\right)\\ &\quad|g(x,y)|\,dx\,dy.\end{aligned}$$

By definition of convolution and change of variables, we obtain

$$\begin{aligned}(5.7)\quad&\|I^{(l,s)}_{k,j,\omega}(f)\|_p^2\\ &\le\int_{\mathbb{R}^n\times\mathbb{R}^m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon^{(l)}_{\lfloor t'\rfloor+j}\otimes\upsilon^{(s)}_{\lfloor s'\rfloor+k}\right)*f(x,y)\right|^2dt'\,ds'\,({\tau^{(l,s)}_{\omega,t',s'}}^*(g))\,dx\,dy,\end{aligned}$$

where $({\tau^{(l,s)}_{\omega,t',s'}}^*(g))$ has similar definition as in (3.13). Thus, by Hölder's inequality, we get

$$\begin{aligned}&\|I^{(l,s)}_{k,j,\omega}(f)\|_p^2\\ &\le\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\left(\upsilon^{(l)}_{\lfloor t'\rfloor+j}\otimes\upsilon^{(s)}_{\lfloor s'\rfloor+k}\right)*f(x,y)\right|^2dt'\,ds'\right\|_{\frac{p}{2}}\|{\tau^{(l,s)}_{\omega,t',s'}}^*(g)\|_q\\ &\le C\,\|g\|_q\,\|S_{k,j}(f)\|_{\frac{p}{2}}\le C_p\,\|f\|,\end{aligned}$$

where the last inequality obtained by (3.14) and (5.6).

Next, we compute the L^2-norm of $I^{(l,s)}_{k,j,\omega}(f)$. We shall apply the same method introduced by Al-Salman in [8]. By (5.1), we have

$$\frac{1}{\theta(a_\omega^{k+\lfloor t'\rfloor})}<|\xi|<\frac{1}{\theta(a_\omega^{k+\lfloor t'\rfloor-2})},\ \beta(a_\omega^{k+\lfloor s'\rfloor-2})<\frac{1}{|\eta|}<\beta(a_\omega^{k+\lfloor s'\rfloor}),$$

where

$$\theta(t)=\begin{cases}\varphi_1(t), & l=d+1,\\ t, & l\neq d+1,\end{cases}\qquad \beta(t)=\begin{cases}\varphi_2(t), & s=b+1,\\ t, & s\neq b+1.\end{cases}$$

Then, by applying θ^{-1} and make use of the fact that $t'-1<\lfloor t'\rfloor<t'$, we get

$$a_\omega^{t'-3}<a_\omega^{-k}\theta^{-1}(|\xi|^{-1})$$

which implies that

$$t'<\log_{a_\omega}(a_\omega^{-k+3}\theta^{-1}|\xi|^{-1}).$$

Similarly, for the lower bound, we obtain

$$\log_{a_\omega}(a_\omega^{-k}\beta^{-1}(|\eta|^{-1}))<t'.$$

Similar argument can be applied for η. Thus, define the intervals $E_k^{(l)}$ and $E_j^{(s)}$ in $\mathbb{R}$ by

$$E_k^{(l)}(\xi) = \left[\log_{a_\omega}(a_\omega^k \theta^{-1}(|\xi|^{-1})), \log_{a_\omega}(a_\omega^{k+3}\theta^{-1}(|\xi|^{-1}))\right];$$

$$E_j^{(s)}(\eta) = \left[\log_{a_\omega}(a_\omega^j \beta^{-1}(|\eta|^{-1})), \log_{a_\omega}(a_\omega^{j+3}\beta^{-1}(|\eta|^{-1}))\right].$$

By similar argument as in [7] and [8], it can be verified that $E_k^{(l)}(\xi)$ and $E_j^{(s)}(\eta)$ satisfy the following:

$$|E_k^{(l)}(\xi)| = |E_j^{(s)}(\eta)| = 3;$$

$$\theta(a_\omega^{-k}\theta^{-1}(|\xi|^{-1})) \le \theta(a_\omega^{t'}) \le \theta(a_\omega^{-k+3}\theta^{-1}(|\xi|^{-1}));$$

$$\beta(a_\omega^{-j}\beta^{-1}(|\eta|^{-1})) \le \beta(a_\omega^{s'}) \le \beta(a_\omega^{-j+3}\beta^{-1}(|\eta|^{-1}))$$

for non-zero $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, $(t', s') \in E_k^{(l)}(\xi) \times E_j^{(s)}(\eta)$ and $k, j \in \mathbb{Z}$.

Now, for $k \geqslant 3$, $t' \in E_k^{(l)}(\xi)$ and by (i) in Lemma 2.2, we get

$$\begin{aligned} \theta(a_\omega^{t'}) &\le \theta(a_\omega^{-k+3}\theta^{-1}(|\xi|^{-1})) \\ &\le a_\omega^{-k+3}\,\theta(\theta^{-1}(|\xi|^{-1})) \\ &\le a_\omega^{-k+3}\,|\xi|^{-1}. \end{aligned} \tag{5.8}$$

For $k \le -2$, $t' \in E_k^{(l)}(\xi)$, we get by (ii) in Lemma 2.2 that

$$\begin{aligned} \theta(a_\omega^{t'-1}) &\ge \theta(a_\omega^{-k-1}\theta^{-1}(|\xi|^{-1})) \\ &\ge a_\omega^{-k-1}\,\theta(\theta^{-1}(|\xi|^{-1})) \\ &\ge a_\omega^{-k-1}\,|\xi|^{-1}. \end{aligned} \tag{5.9}$$

Similarly, we can obtain

$$\beta(a_\omega^{s'}) \le a_\omega^{-j+3}\,|\eta|^{-1} \quad \text{for } j \ge 3 \text{ and } s' \in E_j^{(s)}(\eta); \tag{5.10}$$

$$\beta(a_\omega^{s'-1}) \ge a_\omega^{-j-1}\,|\eta|^{-1} \quad \text{for } j \le -2 \text{ and } s' \in E_j^{(s)}(\eta). \tag{5.11}$$

Thus, by Plancherel's theorem, we have

$$\begin{aligned} &\|I_{k,j,\omega}^{(l,s)}(f)\|_2^2 \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\varpi_k^{(l)}(|\xi|^2)\cdot\varpi_k^{(s)}(|\eta|^2)\right|^2 |\widehat{\tau}_{\omega,t',s'}^{(l,s)}(\xi,\eta)|^2 dt'\,ds'\right) d\xi\,d\eta \\ &\le \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 \left(\int_{E_k^{(l)}}\int_{E_j^{(s)}} |\widehat{\tau}_{\omega,t',s'}^{(l,s)}(\xi,\eta)|^2 dt'\,ds'\right) d\xi\,d\eta. \end{aligned}$$

Thus, if $k, j \le -2$, by (4.7), (4.8), (5.9) and (5.11) , we get

$$\begin{aligned} \|I_{k,j,\omega}^{(l,s)}(f)\|_2^2 &\le C\, a_\omega^{\frac{2\,(k+1)}{l\,(\omega+1)}} \times a_\omega^{\frac{2\,(j+1)}{s\,(\omega+1)}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\,d\eta \\ &\le C\; 2^{\frac{ks+jl}{l+s}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\,d\eta. \end{aligned} \tag{5.12}$$

Also, if $k, j \geq 3$, by (4.7), (4.11), (5.8) and (5.10), we get

$$\begin{aligned}\|I^{(l,s)}_{k,j,\omega}(f)\|_2^2 &\leq C\, a_\omega^{\frac{2(-k+3)}{\omega+1}}\, a_\omega^{\frac{2(-j+3)}{\omega+1}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta \\ &\leq C\ 2^{-k-j} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta.\end{aligned}$$

If $k \leq -2$, $j \geq 3$, by (5.9), (5.10) and (4.10), we get

$$\begin{aligned}\|I^{(l,s)}_{k,j,\omega}(f)\|_2^2 &\leq C\, a_\omega^{\frac{2(k+1)}{l\,(\omega+1)}}\, a_\omega^{\frac{2(-j+3)}{\omega+1}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta \\ &\leq C\ 2^{\frac{k-jl}{l}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta.\end{aligned}$$

Similarly, for $k \geq 3$, $j \leq -2$, by (5.8), (5.11) and (4.9), we obtain

$$\|I^{(l,s)}_{k,j,\omega}(f)\|_2 \leq C\ 2^{\frac{-ks+j}{s}} \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta.$$

If $-2 \leq k,\ j \leq 3$, we have

$$\|I^{(l,s)}_{k,j,\omega}(f)\|_2 \leq C \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta. \tag{5.13}$$

Hence, by combining all estimates in (5.12)-(5.13), we get

$$\|I^{(l,s)}_{k,j,\omega}(f)\|_2 \leq \Theta_{k,j}\, \|f\|_2, \tag{5.14}$$

where

$$\Theta_{k,j} = \begin{cases} 2^{\frac{ks+jl}{l+j}}, & \text{if } k,j \leq -2, \\ 2^{-k-j}, & \text{if } k,j \geq 3, \\ 2^{\frac{k-jl}{l}}, & \text{if } k \leq -2 \text{ and } j \geq 3, \\ 2^{\frac{-ks+j}{s}}, & \text{if } k \geq 3 \text{ and } j \leq -2, \\ 1, & \text{if } k \geq -2 \text{ and } j \leq 3. \end{cases}$$

Next, by interpolation between (5.7) and (5.14), we get

$$\|I^{(l,s)}_{k,j,\omega}(f)\|_p \leq C^{1-\gamma}\, \Theta^\gamma_{k,j}\ \|f\|_p \tag{5.15}$$

for $0 < \gamma < 1$ and $1 < p < \infty$. Finally, by (5.5) and (5.15), we have

$$\begin{aligned}\|\mathcal{M}^{(l,s)}_{\Omega_\omega,\varsigma}(f)\|_p &\leq C \sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} I^{(l,s)}_{k,j,\omega}(f) \\ &\leq C\, \|\Omega_\omega\|_{L(\log)L(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\, \|f\|_p.\end{aligned}$$

This completes the proof. □

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