Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 525–545
https://doi.org/10.4134/CKMS.c220168
pISSN: 1225-1763 / eISSN: 2234-3024

UNIQUENESS OF MEROMORPHIC FUNCTION WITH ITS k-TH DERIVATIVE SHARING TWO SMALL FUNCTIONS UNDER DIFFERENT WEIGHTS

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Abstract. In the paper, we have exhaustively studied about the uniqueness of meromorphic function sharing two small functions with its k-th derivative as these types of results have never been studied earlier. We have obtained a series of results which will improve and extend some recent results of Banerjee-Maity [1].

1. Introduction and definitions

Let us denote by $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}$, the set of all complex numbers, the set of all natural numbers and the set of all integers, respectively. For the sake of convenience, we also put $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$. By $M(\mathbb{C})$ ($\mathcal{E}(\mathbb{C})$) we mean the field of meromorphic (entire) functions where a meromorphic function f is defined on $\mathbb{C}$.

We assume that the readers are familiar with the basics of Nevanlinna's value distribution theory of meromorphic functions in $\mathbb{C}$, such as the first and second main theorems, the standard notations such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, the counting functions $N(r, \infty; f)$ (counting multiplicity) and $\overline{N}(r, \infty; f)$ (ignoring multiplicity). For any non-constant meromorphic function f, by $S(r, f)$ we define any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \to \infty,\ r \notin E)$, where E denotes any set of positive real numbers having finite linear measure.

Definition 1.1. For a non constant meromorpic function f, the set of all small functions of f is denoted by $S(f)$, i.e., $S(f) = \{a \in M(\mathbb{C}) : T(r, a) = S(r, f)$ as $r \to \infty\}$. Clearly $S(f) \subset M(\mathbb{C})$.

Let f and g be two non-constant meromorphic functions and $a = a(z) \in S(f)$. We say that f and g share the small function a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. On the other hand, they said to

Received June 7, 2022; Accepted December 22, 2022.
2010 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Meromorphic function, uniqueness, shared sets, small function.

share the small function a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities.

Now we give the following definitions which actually introduced by Lahiri ([5], [6]), but for small functions, the same was modified by Zhang [11] in the following manner:

Definition 1.2 ([11])**.** Let k be a positive integer and $a \in S(f)$ and by $E_k(0; f-a)$ the set of all zeros of $f-a$, where a zero of multiplicity p is counted p times if $p \leq k$, and $k+1$ times if $p > k$. If $E_k(0; f-a) = E_k(0; g-a)$, we say that $f-a$, $g-a$ share the 0 with weight k and we write it as $f-a$ and $g-a$ share $(0, k)$.

Clearly if $f-a$, $g-a$ share $(0, k)$, then $f-a$, $g-a$ share $(0, p)$ for any $0 \leq p < k$. Also we note that $f-a$, $g-a$ share a value 0 IM or CM if and only if $f-a$, $g-a$ share $(0, 0)$ or $(0, \infty)$, respectively.

For the sake of convenience when a is chosen from $\mathbb{C}$ then $f-a$ and $g-a$ share $(0, k)$ is denoted as f, g share (a, k).

Definition 1.3 ([6])**.** Let $a \in S(f)$, by $N\left(r, \frac{1}{f-a} \mid\geq s\right)$ $\left(N\left(r, \frac{1}{f-a} \mid\leq s\right)\right)$ we denote the counting function of those zeros of $f-a$ of multiplicity $\geq s$ ($\leq s$).

Also $\overline{N}\left(r, \frac{1}{f-a} \mid\geq s\right)$ $\left(\overline{N}\left(r, \frac{1}{f-a} \mid\leq s\right)\right)$ are defined analogously.

Definition 1.4 ([5])**.** Let f and g be two non-constant meromorphic functions such that $f-a$ and $g-a$ share $(0, 0)$. Let z_0 be a zero of $f-a$ with multiplicity p, an zero of $g-a$ with multiplicity q. We denote by $\overline{N}_L\left(r, \frac{1}{f-a}\right)$ the reduced counting function of those zeros of $f-a$ and $g-a$, where $p > q$. Similarly by $\overline{N}_L\left(r, \frac{1}{g-a}\right)$ the reduced counting function of those zeros of $f-a$ and $g-a$, where $p < q$.

Also by $\overline{N}_*\left(r, \frac{1}{f-a}; \frac{1}{g-a}\right)$ the reduced counting function of those zeros of $f-a$ and $g-a$, where $p \neq q$ and clearly $\overline{N}_*\left(r, \frac{1}{f-a}; \frac{1}{g-a}\right) = \overline{N}_L\left(r, \frac{1}{f-a}\right) + \overline{N}_L\left(r, \frac{1}{g-a}\right)$.

Let us consider $f-a$ and $g-a$ share 0. Then by

$$N_{(m,n)}\left(r, \frac{1}{f-a}\right)\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f-a}\right)\right)$$

we denote the counting (reduced counting) function of those zeros of $f-a$ and $g-a$ of multiplicity m and n respectively.

2. Background

In the beginning of nineteenth century R. Nevanlinna inaugurated the value distribution theory with his famous Five value and Four value theorems which can be regarded as the bases of uniqueness theory. Later in [9], Rubel-Yang first investigated about the uniqueness of non-constant entire function f and

f' sharing two values. Rubel-Yang [9] exhibited that in the uniqueness theory of the special class of functions, the number of sharing values can be reduced from 5 to 2. Their [9] result was as follows:

Theorem A ([9])**.** *Let f be a non-constant entire function. If f and f' share two distinct values (a,∞) and (b,∞), then $f \equiv f'$.*

In 2000, Li-Yang [7] obtained the following result which settled the conjecture of Frank [4] as well.

Theorem B ([7])**.** *Let f be a non-constant meromorphic function and a, b be two small functions. If $f-a$ and $f^{(k)}-a$ share $(0,\infty)$, $f-b$ and $f^{(k)}-b$ share $(0,0)$ and $N(r,f)=S(r,f)$, then $f \equiv f^{(k)}$.*

Since then, a number of papers appeared in the literature (see [2, 3, 10, 12]) to make the same affluent. Recently in case of meromorphic functions, manipulating the notion of weighted sharing Banerjee-Maity [1] improved some previous results to obtain a series of results. Some of their results are given below.

Theorem C ([1])**.** *Let f be a non-constant meromorphic function such that f and $f^{(k)}$ share two distinct non-zero values (a,k), (b,k) and satisfies $\overline{N}(r,f) < \lambda T(r,f)$. Then $f \equiv f^{(k)}$, where $\lambda \in \left[0, \frac{1}{k+2}\right]$.*

Theorem D ([1])**.** *Let f be a non-constant meromorphic function satisfying $\overline{N}(r,f) < \lambda T(r,f)$. If f and $f^{(k)}$ share two distinct non-zero values $(a,k-1)$, $(b,k-1)$ and $\lambda \in \left[0, \frac{1}{3k+1}\right)$, then $f \equiv f^{(k)}$.*

Theorem E ([1])**.** *Let f be a non-constant meromorphic function such that f and $f^{(k)}$ share two distinct non-zero values $(0,\chi_k-1)$ and a non-zero value $(b,k-1)$ and satisfies $\overline{N}(r,f) < \lambda T(r,f)$, $\lambda \in \left[0, \frac{1}{3k^2+4k+2}\right)$. Then $f \equiv f^{(k)}$.*

Theorem F ([1])**.** *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share the value $(0,k)$ and a non-zero value (b,k) and satisfies $\overline{N}(r,f) < \lambda T(r,f)$ and $\lambda \in \left[0, \frac{k}{(k+1)^2}\right)$. Then $f \equiv f^{(k)}$.*

Theorem G ([1])**.** *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share the values $(0,\infty)$, $(b,k-1)$ and satisfies $\overline{N}(r,f) < \lambda T(r,f)$, where $\lambda \in \left[0, \frac{1}{k+1}\right)$. Then $f \equiv f^{(k)}$.*

From the definition of weighted sharing it is easy to verify that when f and $f^{(k)}$ share (a,k); then actually it shares (a,∞). But if we choose $a=a(z) \in S(f)$, then $f-a$ and $f^{(k)}-a$ share $(0,k)$ does not always imply that $f-a$ and $f^{(k)}-a$ share $(0,\infty)$. So it will be interesting to re-investigate Theorems C-G considering $a,b \in S(f)$.

In this paper, we have considered $a,b \in S(f)$ and diminished the weight of b to zero to obtain some results. Also considering the weighted sharing with various weights we obtain the following results which improve and extend those of Banerjee-Maity [1].

Theorem 2.1. *Let f be a non-constant transcendental meromorphic function, and a, b be two distinct small functions, k be a positive integer. If $f-a$ and $f^{(k)}-a$ share $(0,\infty)$ and $f-b$ and $f^{(k)}-b$ share $(0,0)$ and $\overline{N}(r,f)<\lambda T(r,f)$ and $\lambda\in[0,\frac{1}{22k+20}]$, then $f\equiv f^{(k)}$.*

Theorem 2.2. *Let f be a non-constant meromorphic function, and a, b be two distinct small functions, k be a positive integer. If $f-a$ and $f^{(k)}-a$ share $(0,1)$ and $f-b$ and $f^{(k)}-b$ share $(0,1)$ and $\overline{N}(r,f)<\lambda T(r,f)$, where $\lambda\in[0,\frac{1}{4k+5}]$, then $f\equiv f^{(k)}$.*

Theorem 2.3. *Let f be a non-constant meromorphic function and a, b be two distinct small functions, k be a positive integer. If $f-a$ and $f^{(k)}-a$ share $(0,1)$ and $f-b$ and $f^{(k)}-b$ share $(0,0)$ and $\overline{N}(r,f)<\lambda T(r,f)$ and $\lambda\in[0,\frac{1}{8k^2+36k+31}]$, then $f\equiv f^{(k)}$.*

Theorem 2.4. *Let f be a non-constant meromorphic function and b be a small function of f. If f and $f^{(k)}$ share $(0,\infty)$ and $f-b$ and $f^{(k)}-b$ share $(0,0)$ and satisfies $\overline{N}(r,f)<\lambda T(r,f)$, where $\lambda\in[0,\frac{1}{k+1}]$, then $f\equiv f^{(k)}$.*

Theorem 2.5. *Let f be a non-constant meromorphic function, b be a small function of f and $k(>1)$ be a positive integer. If f, $f^{(k)}$ share $(0,k)$, $f-b$ and $f^{(k)}-b$ share $(0,1)$ and satisfies $\overline{N}(r,f)<\lambda T(r,f)$, where $\lambda\in\left[0,\frac{k-1}{(k+1)^2}\right]$, then $f\equiv f^{(k)}$.*

Theorem 2.6. *Let f be a non-constant meromorphic function, b be a small function of f. If f, $f^{(k)}$ share $(0,k)$, $f-b$ and $f^{(k)}-b$ share $(0,0)$ and satisfies $\overline{N}(r,f)<\lambda T(r,f)$, where $\lambda\in\left[0,\frac{k}{2k^2+3k+1}\right]$, then $f\equiv f^{(k)}$.*

Theorem 2.7. *Let f be a non-constant meromorphic function a, b be two distinct non-zero rational small functions of f. If $f-a$ and $f^{(k)}-a$ share $(0,k)$ and $f-b$ and $f^{(k)}-b$ share $(0,k)$ and satisfies $\overline{N}(r,f)<\lambda T(r,f)$, where $\lambda\in[0,\frac{1}{2k+3}]$, then $f\equiv f^{(k)}$.*

Clearly here Theorems 2.2, 2.7 improve Theorem C; Theorem 2.4 improves Theorem G and Theorems 2.6, 2.5 improve Theorem F. Also Theorems 2.1, 2.3 improves Theorem B.

The following examples shows the sharpness of above Theorems 2.1, 2.3, 2.4 and 2.6.

Example 2.1. Let $f=\frac{2b}{1-Ae^{-2z}}$. Then it is easy to check that f and f' share $(0,\infty)$ and $(b,0)$ and $T(r,f)=\overline{N}(r,f)$ and $f\neq f'$.

Example 2.2. Let $f=\frac{2e^{2az}-8e^{az}+2}{(e^{az}+1)^2}$, where $a^2=-2$. It is easy to check that f and f'' share $(0,\infty)$ and $(1,0)$. Here $T(r,f)=2\overline{N}(r,f)$ and $f\not\equiv f''$.

3. Lemma

Lemma 3.1 ([8]). *Let f be a non-constant meromorphic function, and let a and b be two distinct small functions of f with $a \neq \infty$ and $b \neq \infty$. Set*

$$\Delta(f) = \begin{vmatrix} f-a & b-a \\ f'-a' & b'-a' \end{vmatrix} \quad \textit{and} \quad \Delta(f^{(k)}) = \begin{vmatrix} f^{(k)}-a & b-a \\ f^{(k+1)}-a' & b'-a' \end{vmatrix}.$$

Then we can have,

(i) $\Delta(f) \neq 0$ *and* $\Delta(f^k) \neq 0$,

(ii) $m\left(r, \frac{\Delta(f)}{(f-a)(f-b)}\right) = S(r,f) = m\left(r, \frac{\Delta(f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}\right)$,

(iii) $m\left(r, \frac{\Delta(f)}{(f-a)}\right) = S(r,f) = m\left(r, \frac{\Delta(f)}{(f-b)}\right)$,

(iv) $m\left(r, \frac{\Delta(f^{(k)})}{(f^{(k)}-a)}\right) = S(r,f) = m\left(r, \frac{\Delta(f^{(k)})}{(f^{(k)}-b)}\right)$.

Lemma 3.2. *Let f be a non-constant transcendental meromorphic function and a be a small function of f. If $f-a$, $f^{(k)}-a$ share $(0,\infty)$, $N(r, \frac{1}{f^{(k)}-a^{(k)}}) \leq (4k+3)\overline{N}(r,f)$ and $a \not\equiv a^{(k)}$, then*

$$\textit{either} \quad T(r,f^{(k)}) \leq (9k+7)\overline{N}(r,f) \quad \textit{or} \quad (f-a) = h_e(f^{(k)}-a),$$

where $T(r,h_e) = S(r,f)$.

Proof. Since f and $f^{(k)}$ share a CM, there is an entire function h_e such that

$$h_e = \frac{f-a}{f^{(k)}-a}. \tag{3.1}$$

Clearly,

$$f-a = h_e(f^{(k)}-a) = h_e(f^{(k)}-a^{(k)}) - h_e(a-a^{(k)}). \tag{3.2}$$

Differentiating (3.2) k time, putting $g = f^{(k)}-a^{(k)}$ and taking $b = a-a^{(k)}$ we have,

$$g = (h_e g)^{(k)} - (h_e b)^{(k)}. \tag{3.3}$$

Let us set the functions as $G = g$, $H = \frac{(h_e g)^{(k)}}{h_e g}$ and $C = \frac{(h_e b)^{(k)}}{h_e}$. Then from (3.3) we have

$$G = h_e(GH - C). \tag{3.4}$$

It is given that $G \not\equiv 0$. If $C \equiv 0$, then we have $h_e b = p_{k-1}$, where p_{k-1} ($\not\equiv 0$) is a polynomial of degree $k-1$ and therefore $T(r, \frac{1}{h_e}) = S(r,f)$. If $H \equiv 0$, then we have $h_e g = p_{k-1}$ and this implies $N(r,g) = S(r,f)$. Using this from (3.3) we get

$$\frac{1}{h_e^2} = \frac{-1}{p_{k-1}} \cdot \frac{(h_e b)^{(k)}}{h_e}.$$

This implies $m(r, \frac{1}{h_e}) = S(r,f)$ and hence $T(r, \frac{1}{h_e}) = S(r,f)$.

Next let us consider none of G, H and C is zero function and write (3.4) as

$$\frac{1}{h_e}=H-\frac{C}{G}. \tag{3.5}$$

Differentiating (3.5) we have,

$$-\frac{h_e'}{h_e^2}=H'-\frac{C'}{G}+\frac{CG'}{G^2}. \tag{3.6}$$

Using (3.5) in (3.6) we get

$$\frac{I}{G}=\frac{Hh_e'}{h_e}+H', \tag{3.7}$$

where

$$I=\frac{Ch_e'}{h_e}+C'-\frac{CG'}{G}. \tag{3.8}$$

Case-1. $I\not\equiv 0$. From (3.7) and (3.8) we get

$$\begin{aligned} m\left(r,\frac{1}{G}\right) &\le m\left(r,\frac{H}{I}\left(\frac{h_e'}{h_e}+\frac{H'}{H}\right)\right) \\ &\le m(r,H)+m(r,\frac{1}{I})+S(r,f) \\ &\le T(r,I)+S(r,f)\le N(r,I)+S(r,f). \end{aligned} \tag{3.9}$$

In view of (3.9) we have

$$\begin{aligned} T(r,G)=T\left(r,\frac{1}{G}\right) &= m\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G}\right) \\ &\le N(r,I)+N\left(r,\frac{1}{G}\right)\le (9k+7)\overline{N}(r,f). \end{aligned} \tag{3.10}$$

That is,

$$T(r,f^{(k)})\le (9k+7)\overline{N}(r,f).$$

Case-2. If $I\equiv 0$, then from (3.8) we have

$$\frac{h_e'}{h_e}+\frac{C'}{C}-\frac{G'}{G}=0,$$

which on integration yields $(h_eb)^{(k)}=cg$ and this implies $N(r,g)=S(r,f)$ and hence $N\left(r,\frac{1}{h_e}\right)=S(r,f)$. Now using $(h_eb)^{(k)}=cg$ in (3.3) we get

$$\frac{1}{h_e}=\frac{1}{1+c}\cdot\frac{(h_eg)^{(k)}}{h_eg}. \tag{3.11}$$

From (3.11) we can get $m\left(r,\frac{1}{h_e}\right)=S(r,f)$ and hence $T\left(r,\frac{1}{h_e}\right)=S(r,f)$. Also we note that $1+c\neq 0$, otherwise it implies $H\equiv 0$. □

Lemma 3.3. *Let* $f = \sum_{i=1}^{m} a_i h^i$, *where* $T(r, a_i) = S(r, f)$ *and* h *is a meromorphic function such that* $h' = \eta h$, *where* $T(r, \eta) = S(r, f)$. *Then* $T(r, f^{(k)}) = T(r, f) + S(r, f)$.

Proof. Let us assume $a_m \neq 0$. It is given that $f = a_0 + a_1 h + \cdots + a_m h^m$, differentiating we get $f' = m a_m h^{m-1} h' + \cdots + a_1 h' = m a_m \eta h^m + \cdots + a_1 \eta h$. Here $T(r, f) = mT(r, h) + S(r, f) = T(r, f') + S(r, f)$ holds. Clearly differentiating f, k times we will have $f^{(k)} = m^k a_m \eta^k h^m + \cdots + a_1 \eta^k h$, and hence $T(r, f^{(k)}) = mT(r, h) + S(r, f) = T(r, f)$. □

4. Proofs of the theorems

Proof of Theorem 2.1. At first let us suppose $f \not\equiv f^{(k)}$. It is given that $f - a$ and $f^{(k)} - a$ share $(0, \infty)$ and $f - b$ and $f^{(k)} - b$ share $(0, 0)$. Let us denote the function

$$\kappa = \frac{f^{(k)} - a}{f - a}. \tag{4.1}$$

Clearly κ has no zero and the only poles of κ come from the pole of f. Using the second main theorem for small functions we get

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - b}\right) + \overline{N}(r, f) + S(r, f) \tag{4.2}$$

and hence,

$$\begin{aligned} T(r, f) &\leq N\left(r, \frac{1}{f^{(k)} - a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + \overline{N}(r, f^{(k)}) + S(r, f) \\ &\leq 3T(r, f^{(k)}) + S(r, f). \end{aligned} \tag{4.3}$$

Also,

$$\begin{aligned} T(r, f^{(k)}) &\leq T(r, f) + k\overline{N}(r, f) + S(r, f) \\ &\leq (k+1)T(r, f) + S(r, f). \end{aligned} \tag{4.4}$$

So from (4.3), (4.4) we get $S(r, f^{(k)}) = S(r, f)$. Next, introduce the following auxiliary function,

$$\phi = \frac{\Delta(f)(f - f^{(k)})}{(f - a)(f - b)}, \tag{4.5}$$

and

$$\chi = \frac{\Delta(f^{(k)})(f - f^{(k)})}{(f^{(k)} - a)(f^{(k)} - b)}. \tag{4.6}$$

Clearly from (4.5) and (4.6) we have

$$\begin{aligned} T(r, \phi - \chi) &= m(r, \phi - \chi) + N(r, \phi - \chi) \\ &\leq m(r, \phi) + m(r, \chi) + (k+1)\overline{N}(r, f) + S(r, f) \\ &\leq m(r, \chi) + (k+1)\overline{N}(r, f) + S(r, f), \end{aligned} \tag{4.7}$$

and here,

$$
\begin{aligned}
(4.8)\qquad & m(r,\chi)\\
&\leq m\left(r,\frac{f-f^{(k)}}{f^{(k)}-a}\right)+S(r,f)\\
&\leq m\left(r,\frac{f-a}{f^{(k)}-a}\right)+N\left(r,\frac{f-a}{f^{(k)}-a}\right)-N\left(r,\frac{f-a}{f^{(k)}-a}\right)+S(r,f)\\
&\leq T\left(r,\frac{f^{(k)}-a}{f-a}\right)+S(r,f)\\
&\leq T(r,\kappa)+S(r,f)\\
&\leq m\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{f^{(k)}-a}{f-a}\right)+S(r,f)\\
&\leq m\left(r,\frac{1}{f-a}\right)+k\overline{N}(r,f).
\end{aligned}
$$

Case-1. Let $\phi-\chi\equiv 0$ and $f\not\equiv f^{(k)}$. Then from $\phi-\chi\equiv 0$ we get

$$\frac{\Delta(f)}{(f-a)(f-b)}=\frac{\Delta(f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}.$$

Integrating we have,

$$\frac{f-a}{f-b}=c_1\frac{f^{(k)}-a}{f^{(k)}-b}, \tag{4.9}$$

where c_1 is a constant. From (4.9) we have $T(r,f)=T(r,f^{(k)})+S(r,f)$. Clearly $c_1=1$ implies $f=f^{(k)}$. Now if $c_1\not\equiv 1$, then from (4.9) we have

$$(1-c_1)ff^{(k)}+ab(1-c_1)=(a-c_1b)f^{(k)}+(b-c_1a)f, \tag{4.10}$$

which implies $N(r,f)=S(r,f)$. Also from (4.9) we have

$$\frac{2f-a-b}{b-a}=\frac{(1+c_1)f^{(k)}-c_1a-b}{(c_1-1)f^{(k)}-c_1a+b}. \tag{4.11}$$

Since $N(r,f)=S(r,f)$, then from (4.11) we must have

$$N\left(r,\frac{1}{f^{(k)}-\frac{c_1a-b}{c_1-1}}\right)=S(r,f).$$

Using the second fundamental theorem for small function we have

$$
\begin{aligned}
(4.12)\qquad & 2T(r,f^{(k)})\\
&\leq \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-\frac{c_1a-b}{c_1-1}}\right)\\
&\quad+\overline{N}(r,f)+S(r,f)
\end{aligned}
$$

$$\leq \overline{N}\left(r, \frac{1}{f - f^{(k)}}\right) + S(r, f)$$
$$\leq T(r, f - f^{(k)}) + S(r, f) \leq T(r, f) + S(r, f).$$

Now from (4.12) and $T(r, f) = T(r, f^{(k)}) + S(r, f)$ implies $T(r, f) = S(r, f)$, a contradiction.

Case-2. $\phi - \chi \not\equiv 0$. It is easy to check

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq \overline{N}\left(r, \frac{1}{\phi - \chi}\right) \leq T(r, \phi - \chi) \tag{4.13}$$

and

$$\overline{N}\left(r, \frac{1}{f-b}\right) \leq \overline{N}\left(r, \frac{1}{\kappa - 1}\right) \leq T(r, \kappa) + S(r, f). \tag{4.14}$$

Using (4.13), (4.14), (4.7) and (4.8) from (4.2) we get

$$T(r, f) \leq 2m\left(r, \frac{1}{f-a}\right) + (3k+2)\overline{N}(r, f) + S(r, f). \tag{4.15}$$

Next we will discuss three cases, which are given below.

Case-2.1. If $a^{(k)} \not\equiv a, b$, then from (4.15) we have

$$\begin{aligned}
(4.16)\quad & T(r, f) \\
&\leq 2m\left(r, \frac{f^{(k)} - a^{(k)}}{f - a}\right) + 2m\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) + (3k+2)\overline{N}(r, f) \\
&\quad + S(r, f) \\
&\leq 2T(r, f^{(k)}) - 2N\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) + (3k+2)\overline{N}(r, f) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) \\
&\quad + (3k+3)\overline{N}(r, f) - 2N\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{f - f^{(k)}}\right) + (3k+3)\overline{N}(r, f) - N\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) + S(r, f) \\
&\leq T(r, f) + (4k+3)\overline{N}(r, f) - N\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) + S(r, f).
\end{aligned}$$

(4.16) implies

$$N\left(r, \frac{1}{f^{(k)} - a^{(k)}}\right) \leq (4k+3)\overline{N}(r, f) + S(r, f). \tag{4.17}$$

Again from (4.17) and Lemma 3.2 we must have $T(r, \frac{1}{\kappa}) = T(r, \kappa) + O(1) = S(r, f)$, otherwise from the given condition we will have

$$T(r, f) \leq 2T(r, f^{(k)}) + \overline{N}(r, f) \leq (18k+15)\overline{N}(r, f) < \frac{18k+15}{22k+20} T(r, f),$$

a contradiction. Then from (4.14), we have $\overline{N}(r, \frac{1}{f-b}) \leq S(r,f)$. Hence from $T(r,\kappa) = S(r,f)$ we have $N(r,\kappa) = k\overline{N}(r,f) = S(r,f)$. Dealing in the same way as in (4.8) and (4.7) we get

$$T(r, \phi - \chi) = S(r,f). \tag{4.18}$$

Using (4.18), from (4.13), (4.14) and (4.2) we get $T(r,f) = S(r,f)$, a contradiction.

Case-2.2. If $a^{(k)} \equiv a$, then we have $m(r,\kappa) = m\left(r, \frac{f^{(k)}-a^{(k)}}{f-a}\right) = S(r,f)$. Next dealing in the same way as done in (4.8) we have

$$m(r,\chi) \leq k\overline{N}(r,f). \tag{4.19}$$

Using (4.19), (4.14), (4.13), (4.7) and (4.2) we get

$$T(r,f) \leq (3k+2)\overline{N}(r,f),$$

a contradiction.

Case-2.3. If $a^{(k)} \equiv b$, from (4.2), (4.13) and (4.7), (4.8) we have

$$\begin{aligned} T(r,f) & \\ &\leq m\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + (2k+2)\overline{N}(r,f) + S(r,f) \\ &\leq m\left(r, \frac{1}{f^{(k)}-a^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) + (2k+2)\overline{N}(r,f) + S(r,f) \\ &\leq T(r, f^{(k)}) + (2k+2)\overline{N}(r,f) + S(r,f). \end{aligned} \tag{4.20}$$

(4.20) yields

$$\begin{aligned} T(r,f) + T\left(r, f^{(k)}\right) & \\ &\leq 2T\left(r, f^{(k)}\right) + (2k+2)\overline{N}(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f^{(k)}-c}\right) \\ &\quad + (2k+2)\overline{N}(r,f) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f-f^{(k)}}\right) + T\left(r, \frac{1}{f^{(k)}-c}\right) + (2k+3)\overline{N}(r,f) \\ &\quad - m\left(r, \frac{1}{f^{(k)}-c}\right) + S(r,f) \\ &\leq T(r,f) + T\left(r, f^{(k)}\right) + (3k+3)\overline{N}(r,f) - m\left(r, \frac{1}{f^{(k)}-c}\right) + S(r,f), \end{aligned} \tag{4.21}$$

where $c \neq a, b$. Immediately from (4.21), we have

$$m\left(r, \frac{1}{f^{(k)}-c}\right) \leq (3k+3)\overline{N}(r,f) + S(r,f). \tag{4.22}$$

Next in view of (4.20) and (4.22) we have

$$\begin{aligned}(4.23)\quad & m\left(r,\frac{f-c}{f^{(k)}-c}\right)\\ &\leq m\left(r,\frac{f}{f^{(k)}-c}\right)+(3k+3)\overline{N}(r,f)\\ &\leq T\left(r,\frac{f^{(k)}-c}{f}\right)-N\left(r,\frac{f}{f^{(k)}-c}\right)+(3k+3)\overline{N}(r,f)+S(r,f)\\ &\leq m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f^{(k)}-c}\right)+(4k+3)\overline{N}(r,f)+S(r,f)\\ &\leq T\left(r,\frac{1}{f}\right)-T\left(r,\frac{1}{f^{(k)}-c}\right)+(7k+6)\overline{N}(r,f)+S(r,f)\\ &\leq (9k+8)\overline{N}(r,f)+S(r,f).\end{aligned}$$

We can write (4.6) as

$$(4.24)\qquad \chi=\left(\frac{a-c}{a-b}.\frac{\Delta(f^{(k)})}{f^{(k)}-a}-\frac{b-c}{a-b}.\frac{\Delta(f^{(k)})}{f^{(k)}-b}\right)\left(\frac{f-c}{f^{(k)}-c}-1\right).$$

Clearly from (4.23) and (4.24) we have

$$(4.25)\qquad m(r,\chi)\leq (9k+8)\overline{N}(r,f)+S(r,f).$$

So using (4.25), (4.7), (4.13) and (4.2), (4.20) we have

$$\begin{aligned}(4.26)\quad & T(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f-b}\right)+(10k+10)\overline{N}(r,f)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+(10k+10)\overline{N}(r,f)+S(r,f)\\ &\leq T\left(r,\frac{1}{f^{(k)}-a^{(k)}}\right)-m\left(r,\frac{1}{f^{(k)}-a^{(k)}}\right)+(10k+10)\overline{N}(r,f)\\ &\quad +S(r,f)\\ &\leq T(r,f)-m\left(r,\frac{1}{f^{(k)}-a^{(k)}}\right)+(11k+10)\overline{N}(r,f)+S(r,f).\end{aligned}$$

Hence we have

$$(4.27)\qquad m\left(r,\frac{1}{f^{(k)}-a^{(k)}}\right)\leq (11k+10)\overline{N}(r,f)+S(r,f).$$

Again from (4.26) and (4.27) we can have

$$\begin{aligned}(4.28)\qquad T(r,f) &\leq \overline{N}\left(r,\frac{1}{f-b}\right)+(10k+10)\overline{N}(r,f)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{\kappa-1}\right)+10(k+1)\overline{N}(r,f)+S(r,f)\end{aligned}$$

$$\leq m\left(r,\frac{1}{f-a}\right)+(11k+10)\overline{N}(r,f)+S(r,f)$$
$$\leq m\left(r,\frac{1}{f^{(k)}-a^{(k)}}\right)+(11k+10)\overline{N}(r,f)+S(r,f)$$
$$\leq (22k+20)\overline{N}(r,f)+S(r,f).$$

As (4.28) gives a contradiction, therefore we must have $f\equiv f^{(k)}$. $\square$

Proof of Theorem 2.2. Here it is given that $f-a$ and $f^{(k)}-a$ share $(0,1)$ and $f-b$ and $f^{(k)}-b$ share $(0,1)$. First let us assume $f\not\equiv f^{(k)}$. Then clearly from **Case-1** of Theorem 2.1, we have $\phi-\chi\not\equiv 0$, where ϕ, χ are given by (4.5), (4.6). Now using the second fundamental theorem for small functions we have

$$\begin{aligned}
(4.29)\quad & T(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+\overline{N}(r,f)+S(r,f)\\
&\leq \overline{N}_{(1,1)}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(1,1)}\left(r,\frac{1}{f-b}\right)\\
&\quad+\sum_{p\geq 2,q\geq 2}\overline{N}_{(p,q)}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(p,q)}\left(r,\frac{1}{f-b}\right)\\
&\quad+\overline{N}(r,f)+S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{\phi-\chi}\right)+\overline{N}(r,f)+S(r,f)\\
&\leq T(r,\phi-\chi)+\overline{N}(r,f)+S(r,f)\\
&\leq m(r,\phi)+m(r,\chi)+(k+2)\overline{N}(r,f)\\
&\leq m\left(r,\frac{f-f^{(k)}}{f^{(k)}-a}\right)+(k+2)\overline{N}(r,f)+S(r,f)\\
&\leq m\left(r,\frac{f-a}{f^{(k)}-a}\right)+N\left(r,\frac{f-a}{f^{(k)}-a}\right)-N\left(r,\frac{f-a}{f^{(k)}-a}\right)\\
&\quad+(k+2)\overline{N}(r,f)+S(r,f)\\
&\leq T\left(r,\frac{f^{(k)}-a}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}-a}\right)+(k+2)\overline{N}(r,f)+S(r,f)\\
&\leq m\left(r,\frac{1}{f-a}\right)+N\left(\frac{f^{(k)}-a}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}-a}\right)\\
&\quad+(k+2)\overline{N}(r,f)+S(r,f)\\
&\leq m\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-a}\right)+N\left(r,f^{(k)}-a\right)-N\left(r,\frac{1}{f^{(k)}-a}\right)\\
&\quad-N(r,f-a)+(k+2)\overline{N}(r,f)+S(r,f)
\end{aligned}$$

$$\leq T\left(r, \frac{1}{f-a}\right) - \overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) + (2k+2)\overline{N}(r,f).$$

Hence from (4.29) we have

$$\overline{N}\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) \leq (2k+2)\overline{N}(r,f) + S(r,f). \tag{4.30}$$

Proceeding exactly in the same way as done in (4.29) we will get

$$\overline{N}\left(r, \frac{1}{f-b}\right) = \overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) \leq (2k+2)\overline{N}(r,f) + S(r,f). \tag{4.31}$$

In view of the second fundamental theorem for small functions and (4.30), (4.31) we get

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + \overline{N}(r,f) + S(r,f) \\ &\leq (4k+5)\overline{N}(r,f) + S(r,f) < T(r,f) + S(r,f). \end{aligned} \tag{4.32}$$

As (4.32) contradicts the given condition. Therefore $f \equiv f^{(k)}$. □

Proof of Theorem 2.3. First let us assume $f \not\equiv f^{(k)}$. Here it is given that $f-a$, $f^{(k)}-a$ share $(0,1)$ and $f-b$, $f^{(k)}-b$ share $(0,0)$. Now let us again choose the same function κ, as in the proof of Theorem 2.1. As $f-a$ and $f^{(k)}-a$ share $(0,1)$ it follows that if z_0 is a zero of $f-a$ of multiplicity >1, then it is a zero of $f^{(k)}-a$ of multiplicity >1.
Again,

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f-a} \mid \geq 2\right) &= \overline{N}\left(r, \frac{1}{f^{(k)}-a} \mid \geq 2\right) \\ &\leq \overline{N}\left(r, \frac{1}{\phi}\right) \leq (k+1)\overline{N}(r,f) + S(r,f). \end{aligned} \tag{4.33}$$

It follows immediately,

$$\overline{N}(r,\kappa) + \overline{N}\left(r, \frac{1}{\kappa}\right) \leq (k+1)\overline{N}(r,f) + k\overline{N}(r,f) + S(r,f). \tag{4.34}$$

If z_0 is a zero of $f-b$ of multiplicity $>k$, then it becomes a zero of $b^{(k)}-b$. Here if $b^{(k)}-b \not\equiv 0$, then we have $\overline{N}\left(r, \frac{1}{f-b} \mid \geq k+1\right) = S(r,f)$. If $b^{(k)}-b \equiv 0$, then clearly,

$$\overline{N}\left(r, \frac{1}{f-b} \mid \geq k+2\right) \leq \overline{N}\left(r, \frac{1}{\phi}\right) \leq (k+1)\overline{N}(r,f) + S(r,f).$$

First let us consider the following function

$$\sigma = \frac{f^{(k+1)}-a'}{f^{(k)}-a} - \frac{f'-a'}{f-a}. \tag{4.35}$$

From (4.35) clearly we have

$$\begin{aligned}T(r,\sigma) &= m(r,\sigma)+N(r,\sigma)\\ &\le \overline{N}_*\left(r,\frac{1}{f-a};\frac{1}{f^{(k)}-a}\right)+\overline{N}(r,f)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f-a}\mid\ge 2\right)+\overline{N}(r,f)\\ &\le (k+2)\overline{N}(r,f)+S(r,f).\end{aligned} \tag{4.36}$$

From (4.5) we have

$$\phi=\frac{\Delta(f)}{f-b}\left(1-\frac{f^{(k)}-a}{f-a}\right),$$

i.e.,

$$\phi\frac{f-b}{\Delta(f)}=1-\kappa. \tag{4.37}$$

Differentiating (4.37) we get

$$\phi'\frac{f-b}{\Delta(f)}+\phi\frac{(f'-b')}{\Delta(f)}-\phi\frac{f-b}{\Delta(f)}\frac{\Delta(f)'}{\Delta(f)}=-\kappa\sigma=\left(\phi\frac{f-b}{\Delta(f)}-1\right)\sigma, \tag{4.38}$$

i.e.,

$$\begin{aligned}&\phi'+\phi\frac{f'-b'}{f-b}-\phi\frac{\Delta(f)'}{\Delta(f)}-\phi\sigma+\frac{\Delta(f)}{f-b}\sigma=0,\\ &\phi'-\phi\frac{\Delta(f)'}{\Delta(f)}-\phi\sigma+\frac{f'-b'}{f-b}(\phi-(b-a)\sigma)+(b'-a')\sigma=0.\end{aligned} \tag{4.39}$$

Next we are going to discuss the following two cases, which are explained below.

Case-1. $\phi-(b-a)\sigma\not\equiv 0$. Clearly from the above discussion we have

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{f-b}\right)\\ &=\sum_{p=1}^{k+2}\overline{N}_{(p,1)}\left(r,\frac{1}{f-b}\right)+\sum_{q=2}^{\infty}\overline{N}_{(1,q)}\left(r,\frac{1}{f-b}\right)\\ &\quad+\sum_{p,q\ge 2}\overline{N}_{(p,q)}\left(r,\frac{1}{f-b}\right)\\ &\le\sum_{p=0}^{k+2}\overline{N}\left(r,\frac{1}{(\phi-p(b-a)\sigma)}\right)\\ &\le\sum_{p=0}^{k+2}T(r,(\phi-p(b-a)\sigma))\\ &\le 2(k+1)(k+3)\overline{N}(r,f)+S(r,f).\end{aligned} \tag{4.40}$$

In view of (4.40) and second fundamental theorem for small functions we have

$$
\begin{aligned}
(4.41)\qquad & T(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+\overline{N}(r,f)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right)+2(k+1)(k+3)\overline{N}(r,f)+\overline{N}(r,f)+S(r,f)\\
&\le T(r,f^{(k)})+(2(k+1)(k+3)+1)\overline{N}(r,f)+S(r,f).
\end{aligned}
$$

Using (4.41), and proceeding exactly in the same way as done in (4.21)-(4.25), we can get $m(r,\chi)\le(6k^2+27k+23)\overline{N}(r,f)+S(r,f)$. Then using (4.13), (4.40) we can obtain from (4.2), $T(r,f)\le(8k^2+36k+31)\overline{N}(r,f)+S(r,f)$, a contradiction. Clearly $\phi\not\equiv 0$ and from the assumption $\phi-(b-a)\sigma\not\equiv 0$, we can conclude that at least one of $\phi-p(b-a)\sigma\equiv 0$ for some $1<p\le k+2$. With the help of $\phi-p(b-a)\sigma\equiv 0$ from (4.39) we have

$$
(4.42)\qquad \frac{\phi'}{\phi}-\frac{\Delta(f)'}{\Delta(f)}-\sigma+\left(1-\frac{1}{p}\right)\frac{f'-b'}{f-b}+\frac{1}{p}\cdot\frac{b'-a'}{b-a}\equiv 0.
$$

Integrating (4.42) we have

$$
\begin{aligned}
(4.43)\qquad (f-b)^{p-1}(b-a)&=c\frac{(f^{(k)}-a)^p}{(f-a)^p}\cdot\left(\frac{\Delta(f)}{\phi}\right)^p\\
&=c\left(\frac{(f-b)(f^{(k)}-a)}{f-f^{(k)}}\right)^p.
\end{aligned}
$$

Hence we get

$$
(4.44)\qquad f-b=\frac{b-a}{c}\left(\frac{f-f^{(k)}}{f^{(k)}-a}\right)^p=\frac{b-a}{c}\left(\frac{1-\kappa}{\kappa}\right)^p=\frac{b-a}{c}\left(\frac{1}{\kappa}-1\right)^p,
$$

where c is integrating constant. By (4.1) we immediately get

$$
\begin{aligned}
(4.45)\qquad f^{(k)}-b&=(b-a)\kappa\left(\frac{1}{\kappa}-1\right)\left(\frac{1}{c}\left(\frac{1}{\kappa}-1\right)^{p-1}-1\right)\\
&=(b-a)\kappa\left(\frac{1}{\kappa}-1\right)\left(\frac{(-1)^{p-1}}{c}\left(1-\frac{1}{\kappa}\right)^{p-1}-1\right).
\end{aligned}
$$

<u>Subcase-1.1.</u> If $c=(-1)^{p-1}$, then from (4.44), (4.45) we get

$$
f-b=-(b-a)\left(1-\frac{1}{\kappa}\right)^p,
$$

$$
f^{(k)}-b=(b-a)\kappa\left(\frac{1}{\kappa}-1\right)\left(\left(1-\frac{1}{\kappa}\right)^{p-1}-1\right).
$$

Since $f-b$ and $f^{(k)}-b$ share 0, then clearly $\left(\left(1-\frac{1}{\kappa}\right)^{p-1}-1\right)$ has all exceptional values that is κ must have $p-2$ exceptional values. Here $p<5$, since otherwise

f reduces to a constant. Now for $p>2$ we will get $\overline{N}\left(r,\frac{1}{\kappa\left(\frac{1}{\kappa}-1\right)^{p-1}-\kappa}\right)$ $=S(r,f)$, which implies $T(r,\kappa)\leq(2k+1)\overline{N}(r,f)+S(r,f)$ and hence from (4.44) we will have

$$T(r,f)\leq pT(r,\kappa)+S(r,f)\leq 4(2k+1)\overline{N}(r,f))+S(r,f),$$

a contradiction. For the case $p=2$, we get $f-b=-(b-a)\left(1-\frac{1}{\kappa}\right)^2$ and $f^{(k)}-b=(b-a)\left(1-\frac{1}{\kappa}\right)$. Hence we get $T(r,f)=2T(r,\kappa)+S(r,f)$ and $T(r,f^{(k)})=T(r,\kappa)+S(r,f)$. From the construction of κ it is easy to verify

$$s\overline{N}(r,f)+\overline{N}_L\left(r,\frac{1}{f-a}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-a}\right)\leq\overline{N}(r,\kappa)+\overline{N}\left(r,\frac{1}{\kappa}\right).$$

Let z_0 be a pole of f and hence a pole of κ. Then from $f-b=-(b-a)\left(1-\frac{1}{\kappa}\right)^2$ we have z_0 is a pole of $b-a$. Again let z_1 be a zero of $f-a$ $(f^{(k)}-a)$ of multiplicity p (q). Now if $p>q$, then z_1 is a pole of κ and from $f^{(k)}-b=(b-a)\left(1-\frac{1}{\kappa}\right)$ we have z_1 is a zero of $a-b$. Also if $p<q$, then z_1 is a zero of κ and from the same relation we can get z_1 is either a zero of $b-a$ or $\frac{1}{b-a}$. Now from the above discussion, since z_0,z_1 are arbitrary then we will must have

$$\begin{aligned}\overline{N}(r,f)+\overline{N}_*\left(r,\frac{1}{f-a};\frac{1}{f^{(k)}-a}\right)&\\ \leq\overline{N}(r,\kappa)+\overline{N}\left(r,\frac{1}{\kappa}\right)&\\ \leq\overline{N}(r,b-a)+\overline{N}\left(r,\frac{1}{b-a}\right)&=S(r,f).\end{aligned}\tag{4.46}$$

So from (4.46), (4.36) we have $T(r,\sigma)=S(r,f)$ and $\frac{k'}{k}=\sigma$, implies κ is a transcendental function.
Now from Lemma 3.3 we must have

$$T(r,\kappa)+S(r,f)=T(r,f^{(k)})=T(r,f)+S(r,f)=2T(r,\kappa)+S(r,f),$$

this implies $T(r,\kappa)=S(r,f)$, again a contradiction.
Subcase-1.2. If $c\neq(-1)^{p-1}$, then from (4.44) and (4.45) we must have $\overline{N}\left(r,\frac{1}{\kappa\left(\frac{1}{c}\left(\frac{1}{\kappa}-1\right)^{p-1}-1\right)}\right)=S(r,f)$ and $p<3$.
Then finally we have $T(r,\kappa)\leq(2k+1)\overline{N}(r,f)+S(r,f)$ and we will have $T(r,f)\leq 3(2k+1)\overline{N}(r,f)+S(r,f)$, again a contradiction.
Case-2. Let $\phi-(b-a)\sigma\equiv 0$. Then from (4.39) we have

$$\frac{\phi'}{\phi}-\frac{\Delta'(f)}{\Delta(f)}-\sigma-\frac{b'-a'}{b-a}=0,\tag{4.47}$$

which on integration yields

$$\frac{f-f^{(k)}}{(f^{(k)}-a)(f-b)}=\frac{C}{(b-a)}.\tag{4.48}$$

Therefore clearly we can get

$$\overline{N}(r,f)=S(r,f), \tag{4.49}$$
$$\overline{N}_*\left(r,\frac{1}{f-a};\frac{1}{f^{(k)}-a}\right)=\overline{N}_*\left(r,\frac{1}{f-b};\frac{1}{f^{(k)}-b}\right)=S(r,f).$$

Using (4.49) we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)\\
&\le \overline{N}_{(1,1)}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(1,1)}\left(r,\frac{1}{f-b}\right)+\sum_{p\ge 2}\overline{N}_{(p,p)}\left(r,\frac{1}{f-a}\right)\\
&\quad+\sum_{p\ge 2}\overline{N}_{(p,p)}\left(r,\frac{1}{f-b}\right)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{\phi-\chi}\right)+S(r,f).
\end{aligned}$$

Next proceeding exactly in the same manner as done in (4.29)–(4.32) of Theorem 2.2, we can get a contradiction again. □

Proof of Theorem 2.4. It is given that f and $f^{(k)}$ share $(0,\infty)$, therefore 0 must be an exceptional value of both f and $f^{(k)}$. Next consider the function

$$\mu=\frac{f^{(k)}}{f}. \tag{4.50}$$

Using the second fundamental theorem for small function we have

$$\begin{aligned}
T(r,f)&\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+\overline{N}(r,f)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{\mu-1}\right)+\overline{N}(r,f)\le T(r,\mu)+\overline{N}(r,f)+S(r,f)\\
&\le (k+1)\overline{N}(r,f)+S(r,f)<T(r,f),
\end{aligned}$$

a contradiction. Hence the result follows. □

Proof of Theorem 2.6. First let us assume that $f\not\equiv f^{(k)}$. It is given that f and $f^{(k)}$ share $(0,k)$ and $f-b$, $f^{(k)}-b$ share $(0,0)$. Now clearly from the property of weighted sharing, the zeros of f must have multiplicity $\ge 2k+1$ and the zeros of $f^{(k)}$ have multiplicity $k+1\ge 2$. Considering the function μ as defined in (4.50) we can construct the following function

$$\psi=\frac{(f'(f-b)-f(f'-b'))(f-f^{(k)})}{f(f-b)}.$$

It is easy to verify that $k\overline{N}(r,\frac{1}{f})\le N(r,\frac{1}{\psi})\le (k+1)\overline{N}(r,f)+S(r,f)$.

Using the second fundamental theorem for small functions we have

$$\begin{aligned}
T(r,f) &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+\overline{N}(r,f)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\mu-1}\right)+\overline{N}(r,f)\\
&\le T(r,\mu)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&\le \frac{(k+1)^2}{k}\overline{N}\left(r,f\right)+(k+1)\overline{N}(r,f)\\
&\le \frac{(k+1)(2k+1)}{k}\overline{N}(r,f)+S(r,f)<T(r,f),
\end{aligned}$$

a contradiction. Again if $\psi\equiv 0$, then we will $T(r,f)=S(r,f)$ which leads a contradiction again. Hence $f\equiv f^{(k)}$. □

Proof of Theorem 2.5. It is given that f and $f^{(k)}$ share $(0,k)$ and $f-b$, $f^{(k)}-b$ share $(0,1)$. It is given that $k\ge 2$. As f and $f^{(k)}$ share the values $(0,k)$, it follows that the zeros of f are of multiplicity $\ge 2k+1$. Now let us denote the function,

$$\gamma=\frac{f^{(k+1)}}{f^{(k)}}-\frac{f'}{f},$$

$$\delta=\frac{f^{(k+1)}-b'}{f^{(k)}-b}-\frac{f'-b'}{f-b}.$$

If $\delta\equiv 0$, then we have $f^{(k)}-b=c(f-b)$ for some constant c. This implies $f-b$ and $f^{(k)}-b$ share $(0,\infty)$ and $N(r,f)=S(r,f)$ then clearly from Theorem B we will get $f\equiv f^{(k)}$. Similarly if $\gamma\equiv 0$, then we will also get $f\equiv f^{(k)}$.
So we can assume $\delta\not\equiv 0$ and $\gamma\not\equiv 0$. Since f and $f^{(k)}$ share $(0,k)$, it follows that the zeros of f are of multiplicities $2k+1$ and that of $f^{(k)}$ are of multiplicity k (≥ 2). Therefore we must have

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{f}\mid\ge k+1\right) &\le \frac{1}{k}N\left(r,\frac{1}{\delta}\right)+S(r,f)\\
&\le \frac{1}{k}\left\{\overline{N}\left(r,\frac{1}{f-b}\mid\ge 2\right)+\overline{N}(r,f)\right\}\\
&\le \frac{1}{k}N\left(r,\frac{1}{\gamma}\right)+\frac{1}{k}\overline{N}(r,f)+S(r,f)\\
&\le \frac{1}{k}\overline{N}\left(r,\frac{1}{f}\mid\ge k+1\right)+\frac{2}{k}\overline{N}(r,f)+S(r,f),
\end{aligned}$$

i.e.,

$$\overline{N}\left(r,\frac{1}{f}\mid\ge k+1\right)\le\frac{2}{k-1}\overline{N}(r,f)+S(r,f). \tag{4.51}$$

From (4.50) it is clear that,

$$\begin{aligned}(4.52)\qquad & \overline{N}\left(r,\frac{1}{f-b}\right)\\ & \leq \overline{N}\left(r,\frac{1}{\mu-1}\right) \leq k\overline{N}\left(r,\frac{1}{f}\mid\geq k+1\right)+k\overline{N}(r,f)+S(r,f).\end{aligned}$$

Next from the second fundamental theorem and (4.51), (4.52) for small functions we get

$$\begin{aligned}(4.53)\qquad & T(r,f)\\ & \leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+\overline{N}(r,f)+S(r,f)\\ & \leq \frac{2+2k}{k-1}\overline{N}(r,f)+(k+1)\overline{N}(r,f)\\ & = \frac{(k+1)^2}{k-1}\overline{N}(r,f)\\ & < T(r,f)+S(r,f).\end{aligned}$$

As (4.53) gives a contradiction, we have $f \equiv f^{(k)}$. $\square$

Proof of Theorem 2.7. It is given that $f-a$, $f^{(k)}-a$ share $(0,k)$ and $f-b$, $f^{(k)}-b$ share $(0,k)$. Since a, b be two non-zero rationals, clearly we have

$$\overline{N}\left(r,\frac{1}{f-a}\mid\geq k+1\right)=\overline{N}\left(r,\frac{1}{f^{(k)}-a}\mid\geq k+1\right)=S(r,f),$$
$$\overline{N}\left(r,\frac{1}{f-b}\mid\geq k+1\right)=\overline{N}\left(r,\frac{1}{f^{(k)}-b}\mid\geq k+1\right)=S(r,f).$$

Let us consider the following functions

$$\alpha=\frac{f-a}{f^{(k)}-a}\ ,\ \beta=\frac{f-b}{f^{(k)}-b}.$$

Here we have,

$$\Gamma=\frac{\alpha}{\beta}=\frac{(f-a)(f^{(k)}-b)}{(f^{(k)}-a)(f-b)}.$$

It is easy to check,

$$\begin{aligned}(4.54)\qquad & \overline{N}\left(r,\frac{\alpha}{\beta}\right)+\overline{N}\left(r,\frac{\beta}{\alpha}\right)\\ & \leq \overline{N}\left(r,\frac{1}{f-a}\mid\geq k+1\right)+\overline{N}\left(r,\frac{1}{f-b}\mid\geq k+1\right)=S(r,f).\end{aligned}$$

Now using (4.54), we have

$$\begin{aligned}(4.55)\qquad & T(r,\Gamma)\\ & \leq \overline{N}(r,\Gamma)+\overline{N}\left(r,\frac{1}{\Gamma}\right)+\overline{N}\left(r,\frac{1}{\Gamma-1}\right)\leq \overline{N}\left(r,\frac{1}{\Gamma-1}\right)+S(r,f)\end{aligned}$$

$$\leq N\left(r, \frac{1}{\Gamma - 1}\right) + S(r,f) \leq T(r,\Gamma) + S(r,f).$$

Clearly from (4.55) we have $T(r,\Gamma) = N\left(r, \frac{1}{\Gamma-1}\right) + S(r,f) = \overline{N}\left(r, \frac{1}{\Gamma-1}\right) + S(r,f)$ and hence $m\left(r, \frac{1}{\Gamma-1}\right) = S(r,f)$.
Again we have

$$\frac{ab}{f^2} = \frac{a-b}{\Gamma-1}\left(1 - \frac{f^{(k)}}{f}\right)\frac{1}{f} + \frac{a}{f} + \frac{f^{(k)}}{f}\left(\frac{b}{f} - 1\right) \tag{4.56}$$
$$= \frac{1}{f}\left(\frac{a-b}{\Gamma-1}\left(1 - \frac{f^{(k)}}{f}\right) + a + b\frac{f^{(k)}}{f}\right) - \frac{f^{(k)}}{f}.$$

From (4.56) we have $m\left(r, \frac{1}{f}\right) = S(r,f)$ and hence $m(r, f^{(k)}) = S(r,f)$. Now

$$\Delta = \frac{\Gamma'}{\Gamma} = \frac{f'-a'}{f-a} - \frac{f'-b'}{f-b} + \frac{f^{(k+1)}-b'}{f^{(k)}-b} - \frac{f^{(k+1)}-a'}{f^{(k)}-a}. \tag{4.57}$$

It is easy to verify

$$N(r,f) - \overline{N}(r,f) \leq N\left(r, \frac{1}{\Delta}\right) \leq T(r,\Delta) + S(r,f) = S(r,f),$$

and hence $N(r,f) \leq \overline{N}(r,f) + S(r,f)$ and hence $N(r,f) = \overline{N}(r,f)$. From above we have $T(r,f^{(k)}) = m(r,f^{(k)}) + N(r,f^{(k)}) = (k+1)\overline{N}(r,f)$. Using this and the second main theorem for small functions we get

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) + \overline{N}(r,f) + S(r,f) \\ &\leq 2T(r,f^{(k)}) + \overline{N}(r,f) + S(r,f) \\ &\leq (2k+3)\overline{N}(r,f) + S(r,f), \end{aligned}$$

a contradiction. If $\Delta \equiv 0$, then integrating from (4.57) we have $\frac{f-a}{f-b} = c_1\frac{f^{(k)}-a}{f^{(k)}-b}$, where c_1 is a constant. If $c_1 \neq 1$, then clearly from this relation we get $N(r,f) = S(r,f)$. Now we have $T(r,f^{(k)}) = N(r,f^{(k)}) + S(r,f) = S(r,f)$ and this implies $T(r,f) = S(r,f)$, a contradiction. □

Acknowledgements. The authors wish to thank the referee for his/her valuable suggestions towards the improvement of this paper. The second author is thankful to Council of Scientific and Industrial Research (India) for their financial support under File No: 09/106(0200)/2019-EMR-I.

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