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WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS OF A KENMOTSU MANIFOLD

Mohammad Shuaib

Abstract. For a pseudo-slant submanifold of a Kenmotsu manifold, we have worked out conditions in terms its canonical structure tensors, T and F, and its shape operator so that it reduces to a warped product submanifold.

1. Introduction

Initially R. L. Bishop and B. O'Neill [1] introduced the notion warped product manifolds in order to construct examples of manifolds of negative sectional curvatures. B. Y. Chen initiated the study of warped product spaces with extrinsic geometric point of view when he considered CR-submanifold of a Kaehler manifold as warped product and described many extrinsic geometric properties of the submanifolds in terms of tensor field (c.f. [6, 7]). Moreover Chen [4] has proved that, among CR-submanifolds of a Kaehler manifold, the CR products are characterized by the condition on T to be parallel. As a step forward V. A. Khan et al. [13, 14] worked out a characterization involving $\bar{\nabla}T$ and $\bar{\nabla}F$ under which a CR submanifolds of a Kaehler manifold and submanifold of Kenmotsu manifold reduces to a warped product submanifolds.

Slant immersion in complex geometry were defined by B. Y. Chen as a natural generalization of both holomorphic and totally real immersions [5]. In [15], A. Lotta has introduced the notion of slant immersions of Riemannian manifold into an almost contact metric manifold and he has proved some properties of such immersions.

The notion of semi-slant submanifold of almost Hermitian manifold was introduced by N. Papaghiuc [16], after that Cabrerizo et al. [2] defined and studied semi-slant submanifolds in the setting of almost contact metric manifold. A step forward. S. K. Hui et al. studied new class of warped product submanifolds as skew-CR submanifolds [9], Pointwise bi-slant submanifolds [10] of Kenmotsu manifold and also obtained some interesting results of submanifold of a kenmotsu manifold [11].

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In this present article, we study semi-slant submanifold and pseudo-slant submanifold of a Kenmotsu manifold. Section 2 deals with basic concepts formulas and even some known result that are relevant for the subsequent sections. In Section 3, we have explored conditions for the integrability of the distributions on a semi-slant submanifold of Kenmotsu manifold. These result will helped to develop the characterization in Section 4 under which a semi-slant submanifold of a Kenmotsu manifold reduces to warped product submanifolds. In Section 5, we studied the integrability conditions for the distributions and also indicate some geometric properties of the leaves of the distributions. finally, last section contained some result on pseudo-slant warped product submanifolds of a Kenmotsu manifold in terms of $\bar{\nabla}T$, $\bar{\nabla}F$ and shape operator.

2. Preliminaries

An almost contact structure on a $(2n+1)$-dimensional manifold $\bar{M}$ is defined by a $(1,1)$ tensor field ϕ, a vector field ξ and the dual 1-form η of ξ satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1.$$

There always exists a Riemannian metric g on $\bar{M}$ satisfying

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{2.1}$$

for any $U, V \in \bar{M}$. It is easy to observe that

$$g(\phi U, V) + g(U, \phi V) = 0,\ g(U, \xi) = \eta(U).$$

If $\bar{\nabla}$ is the Levi-Civita connection on $(\bar{M}, g)$, then the covariant derivative of ϕ is defined as

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi \bar{\nabla}_U V. \tag{2.2}$$

Let Ω be the fundamental 2-form on $\bar{M}$, i.e., $\Omega(U, V) = g(U, \phi V)$. If $\Omega = d\eta$, $\bar{M}$ is said to be a contact manifold. If ξ is a killing vector field with respect to g, the contact metric structure is called a k-contact structure. It is easy to show that a contact metric manifold is k-contact if $\bar{\nabla}_U \xi = -\phi U$ for each vector field U on $\bar{M}$. The almost contact structure on $\bar{M}$ is said to be normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. A Sasakian manifold is a normal contact metric manifold. It is known that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\bar{\nabla}_U \phi)V = g(U, V)\xi - \eta(V)U.$$

S. Tanno [17] classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of a plane section containing ξ is a constant c. He showed that they can be divided into three classes: (i) Homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) Global Riemannian product of a line and a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$

and (iii) A warped product space $R \times_f C^n$ if $c < 0$. It is known that manifolds of class (i) are characterized some tensorial equations. In fact, these manifolds admit Sasakian structure. The manifolds of class (ii) are characterized by a tensorial equation and admit cosymplectic structure. Kenmotsu [12] characterized differential geometric properties of manifolds of class (iii). The structure obtained on this class of manifolds is known as Kenmotsu structure. These structures are not Sasakian. In fact, Kenmotsu manifolds are characterized by the following tensorial equation:

$$(\bar{\nabla}_U \phi)V = g(\phi U, V) - \eta(V)\phi U. \tag{2.3}$$

It can also be seen that on a Kenmotsu manifold,

$$\bar{\nabla}_U \xi = -\phi^2 U = U - \eta(U)\xi \tag{2.4}$$

for all vector fields U, V on $\bar{M}$.

Throughout we denote M a submanifold of an almost contact metric manifold $\bar{M}$ with TM and $T^{\perp}M$ as the tangent and normal bundles on M, respectively. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on TM and $T^{\perp}M$, then Gauss and Weingarton formulae are

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.6}$$

for any $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. A_N and h, respectively, denote the shape operator (corresponding to the normal vector field N) and the second fundamental form of the immersion of M into $\bar{M}$. The two are related as

$$g(A_N U, V) = g(h(U, V), N), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced metric on M.

For any $U \in \Gamma(TM)$, we write

$$TU = tan(\phi U) \text{ and } FU = nor(\phi U) \tag{2.8}$$

similarly, for $N \in TM^{\perp}$, we write

$$tN = tan(\phi N) \text{ and } fN = nor(\phi N), \tag{2.9}$$

where '*tan*' and '*nor*' are the natural projections associated with the direct decomposition:

$$T_x\bar{M} = T_x M \oplus T_x^{\perp} M, \ x \in M.$$

The tensor fields on M determined by the endomorphism T and the normal valued 1-form F are denoted by the same letters T and F, respectively. Similarly, t and f are tangential and normal valued (1,1)-tensor fields on the normal bundle of M. The covariant differentiations of the tensor fields P, and F, are defined, respectively, as:

$$(\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V. \tag{2.11}$$

On a submanifold of a Kenmotsu manifold, by equations (2.4) and (2.5), we get

$$\nabla_X \xi = X - \eta(X)\xi,\ h(X, \xi) = 0$$

for each $U \in TM$. Also from equations (2.3), (2.5), (2.6) and (2.7) to (2.11), we obtain

$$(\bar{\nabla}_X T)Y = A_{FY}X + \mathrm{th}(X, Y) - g(X, TY) - \eta(Y)TX, \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = fh(X, Y) + h(X, TY) + \eta(Y)FX. \tag{2.13}$$

For any $x \in M$ and $X \in T_xM$, if the vectors X and ξ are linearly independent, then angle $\theta(X) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is well defined. If $\theta(X)$ doses not defined on the choice of $x \in M$ and $X \in T_xM$, we say that M is slant in $\bar{M}$. The contact angle θ is called the slant angle of M in $\bar{M}$. The anti-invariant submanifolds of an almost contact metric manifold are slant submanifolds with slant angle $\frac{\pi}{2}$ and invariant submanifolds are slant submanifolds with slant angle 0. If slant angle $\theta \neq 0, \frac{\pi}{2}$, the slant submanifold is called a proper slant submanifold of an almost contact metric manifold. If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM of M is decompose as

$$TM = D \oplus \{\xi\}$$

when $\{\xi\}$ denotes the distribution spanned by the structure vector field ξ and D is a complementary distribution $\{\xi\}$ in TM, known as slant distribution.

For a slant submanifold of a contact manifold Cabrerizo [3] proved the following theorem:

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi).$$

Furthermore, in such a case, if θ is the slant angle of M, then it verifies that $\lambda = \cos^2 \theta$. Thus one has the following consequence of the above formulae:

$$g(TX, TY) = \cos^2 \theta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.14}$$

$$g(FX, FY) = \sin^2 \theta[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.15}$$

The notion of warped (or more generally warped bundle) was introduce by R. L. Bishop and B. O'Neill [1] in order to construct a large variety of manifolds of negative sectional curvature and are generalized version of Riemannian product of two manifolds. We recall in the following paragraphes the notion of warped product manifolds and some intrinsic geometric properties of these manifolds.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f be a positive differentiable function on

M_1. Then the warped product $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with the Riemannian metric g given by

$$g = \pi^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where $\pi = (1, 2)$ are the projection maps of M on to M_1 and M_2, respectively. Then the function f, in this case known as the warping function. If the warping function is just a constant, the warped product is simply a Riemannian product, known as trivial warped product.

A warped product manifold is isometrically immersed into Riemannian manifold is known as warped product submanifold.

Few important observation and formulae revealing some geometric aspects of warped product manifold are obtained by R. L. Bishop and B. O'Neill [1]:

Theorem 2.2. *Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then*

(i) $\nabla_X Y \in TM_1$,
(ii) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$,
(iii) $nor(\nabla_Z W) = -g(Z, W)\nabla \ln f$,

where $nor(\nabla_Z W)$ denotes the component of $\nabla_Z W$ in TM_1 and ∇f is the gradient of f defined as

$$g(\nabla f, U) = Uf \tag{2.16}$$

for any $U \in TM$.

A couple of important consequences of the above theorem can be stated as:

Corollary 2.3. *On a warped product manifold $M = M_1 \times_f M_2$;*

(i) *M_1 is totally geodesic.*
(ii) *M_2 is totally umbilical.*

3. Pseudo-slant submanifold of a Kenmotsu manifold

A submanifold M of $\bar{M}$ is said to be a pseudo-slant submanifold of an almost contact metric manifold $\bar{M}$ if there exit two orthogonal complementary distributions $D^\perp$ and D^θ on M such that

(1) $TM = D^\perp \oplus D^\theta \oplus \{\xi\}$,
(2) the distribution $D^\perp$ is anti-invariant under ϕ, i.e., $\phi D^\perp \subset TM^\perp$,
(3) the distribution D^θ is slant with slant angle $\theta \neq 0$.

Throughout this section we studied the pseudo-slant warped product submanifold of the type $M_\perp \times_f M_\theta$ isometrically immersed into a Kenmotsu manifold $\bar{M}$ with structure vector field ξ tangential to $M_\perp$, where $M_\perp$ and M_θ are, respectively, ϕ-anti-invariant submanifold and proper slant submanifold of $\bar{M}$. In this setting the formula of Theorem 2.2(ii) can be written as

$$\nabla_X Z = \nabla_Z X = Z \ln f X \tag{3.1}$$

for each $X \in D^\theta$ and $Z \in D^\perp$. In this case the complement of $FD^\perp$ and FD^θ is an invariant subbundle of $TM^\perp$ dented by μ. Thus

$$TM^\perp = FD^\perp \oplus FD^\theta \oplus \mu.$$

Let M be a pseudo-slant submanifold and for any $X \in TM$, we can write as

$$X = BX + CX + \eta(X)\xi, \tag{3.2}$$

where $BX \in D^\perp$ and $CX \in D^\theta$. Now by using equations (2.8) and (3.2), we have

$$\phi X = \phi BX + TCX + FCX.$$

As $D^\perp$ is anti-invariant under ϕ we obtain

$$\phi BX = FBX, \ TBX = 0.$$

Thus

$$TX = TCX, \ FX = FBX + FCX.$$

Lemma 3.1. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then*

$$\phi D^\perp \perp FD^\theta$$

for each $X \in D^\theta$ and $Z \in D^\perp$.

Proof. From (2.8), we obtain $g(\phi Z, FCX) = g(\phi Z, \phi CX - TCX)$ for $Z \in D^\perp$ and $X \in D^\theta$, hence $g(\phi Z, FCX) = g(\phi Z, \phi X)$. Thus from (2.1), we have $g(\phi Z, FCX) = 0$ as $D^\perp$ and D^θ are orthogonal. □

If M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$, the applying ϕ to (2.8), we obtain

$$-X + \eta(X)\xi = T^2X + FTX + tFX + fFX$$

for $X \in TM$. Comparing the tangential and normal components, we derive

$$-X + \eta(X)\xi = T^2X + tFX, \ FTX + fFX = 0.$$

From Theorem 2.1, above equation reduces to

$$\sin^2\theta(-X + \eta(X)\xi) = tFX, \ FTX + fFX = 0.$$

From above we have the following result.

Corollary 3.2. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then*

$$tFX = \sin^2\theta(-X + \eta(X)\xi), \ fFX = -FTX$$

for any $X \in D^\theta$.

Since we want to study the pseudo-slant submanifold as a warped product submanifold, we need to ensure the existence of ϕ-anti-invariant and slant factor of the submanifolds. To this end we have:

Theorem 3.3. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then the anti-invariant distribution $D^\perp$ is always integrable.*

Proof. For $Z, W \in D^{\perp}$ and $X \in D^{\theta}$ and from (2.10), we have

$$g([Z,W],TX) = g((\nabla_Z T)W, X) - g((\nabla_W T)Z, X).$$

On using formula (2.12), we have

$$g([Z,W],TX) = 0.$$

Hence, $D^{\perp}$ is integrable. □

Theorem 3.4. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then the slant distribution D^{θ} is integrable if and only if*

$$t\{\nabla^{\perp}_X FY + h(X,TY) - \nabla^{\perp}_Y FX - h(Y,TX)\}$$

lies in D^{θ} for each $X, Y \in D^{\theta}$.

Proof. By using equations (2.5), (2.6), (2.8) and (2.12), we have

$$g([X,Y],Z) = g(\nabla^{\perp}_X FY + h(X,TY) - \nabla^{\perp}_Y FX - h(Y,TX), \phi Z)$$

for each $X, Y \in D^{\theta}$ and $Z \in D^{\perp}$. This proves the theorem completely. □

In this section we are going to study the problem when a pseudo-slant submanifold of a Kenmotsu manifold is a Riemannian product manifold of anti-invariant submanifold and slant submanifold.

Theorem 3.5. *Let M be a proper pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then the distribution D^{θ} defined a totally geodesic foliation if and only if*

$$g(A_{\phi Z}TY, X) = g(A_{FTY}Z, X)$$

for all $X, Y \in D^{\theta}$ and $Z \in D^{\perp}$.

Proof. From (2.5), we have $g(\nabla_X Y, Z) = g(\bar{\nabla}_X Y, Z)$ for $X, Y \in D^{\theta}$ and $Z \in D^{\perp}$. Then from (2.1) and (2.3), we get

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X \phi Y, \phi Z).$$

Using (2.8), we obtain

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X TY, \phi Z) + g(\bar{\nabla}_X FY, \phi Z).$$

Hence

$$g(\nabla_X Y, Z) = -g(\bar{\nabla}_X \phi Z, TY) - g(\bar{\nabla}_X \phi Z, FY).$$

Now, using (2.6) and (2.2), we obtain

$$g(\nabla_X Y, Z) = g(A_{\phi Z}X, TY) + g(\phi FY, \bar{\nabla}_X Z).$$

Then from (2.9), we get

$$g(\nabla_X Y, Z) = g(A_{\phi Z}X, TY) + g(\nabla_X Z, tFY) + g(h(X,Z), fFY).$$

Then from Corollary 3.2, we arrive at

$$g(\nabla_X Y, Z) = g(A_{\phi Z}X, TY) - \sin^2\theta g(\nabla_X Z, Y) - g(h(X,Z), FTY).$$

Then from (2.7), we get

$$\cos^2\theta\ g(\nabla_X Y, Z) = g(A_{\phi Z}TY, X) - g(A_{FTY}Z, X).$$

This proves the assertion of the theorem. □

Theorem 3.6. *Let M be a proper pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then the distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if*

$$g(A_{\phi W}TX, Z) = g(A_{FTX}W, Z)$$

for $Z, W \in D^{\perp}$ and $X \in D^{\theta}$.

Proof. From (2.1), (2.2), (2.3), (2.8) and (2.6), we obtain

$$g(\nabla_Z W, X) = -g(A_{FW}Z, TX) + g(\phi\bar{\nabla}_Z W, FX)$$

for $Z, W \in D^{\perp}$ and $X \in D^{\theta}$. Using (2.5), we get

$$g(\nabla_Z W, X) = -g(A_{FW}Z, TX) + g(FB\nabla_Z W, FX) + g(h(Z, W), \phi FX).$$

Thus using (2.9), (2.15) and Corollary 3.2, we derive

$$g(\nabla_Z W, X) = -g(A_{FW}, TX) + \sin^2\theta\ g(B\nabla_Z W, X) + g(h(Z, W), FTX).$$

Hence, we arrive at

$$\cos^2\theta\ g(\nabla_Z W, X) = -g(A_{FW}TX, Z) + g(A_{FTX}W, Z),$$

which proves the assertion. □

Thus from Theorems 3.5 and 3.6 we have the following result:

Corollary 3.7. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then M is locally a Riemannian product manifold $M = M_{\perp} \times M_{\theta}$ if and only if*

$$A_{\phi W}TX = A_{FTX}W$$

for each $X \in D^{\theta}$ and $Z, W \in D^{\perp}$, where $M_{\perp}$ is an anti-invariant submanifold and M_{θ} is a slant submanifold of $\bar{M}$.

4. Pseudo-slant warped product submanifold of a Kenmotsu manifold

In this section we consider the warped product pseudo-slant submanifold of the form $M_{\perp} \times_f M_{\theta}$, where $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$ and M_{θ} is a proper slant submanifold.

Lemma 4.1. *Let $M = M_{\perp} \times_f M_{\theta}$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$. Then*

$$g(h(U, V), FZ) = g(h(U, Z), FV) + Z \ln f g(CU, TV) + \eta(Z)g(\phi U, \phi V)$$

for each $U, V \in TM$ and $Z \in D^{\perp} \oplus \{\xi\}$.

Proof. By using Gauss formula (2.2) and (2.3), we have

$$g(h(U,V),FZ) = g(\phi U,V)\eta(Z) - g(\bar{\nabla}_U \phi V, Z).$$

Using (2.9), we get

$$g(h(U,V),FZ) = g(\phi U,V)\eta(Z) - g(\bar{\nabla}_U TV, Z) - g(\bar{\nabla}_U FV, Z).$$

Again using (2.5), (2.6), (3.1) and from the fact that $TV \in D^\theta$ for each $V \in TM$, we get

$$g(h(U,V),FZ) = g(h(U,Z),FV) + Z\ln f g(CU,TV) + \eta(Z)g(\phi U,\phi V),$$

which proves the lemma. □

Lemma 4.2. *Let $M = M_\perp \times_f M_\theta$ be a pseudo-slant warped product submanifold of a kenmotsu manifold $\bar{M}$. Then for each $U, V \in TM$, $X \in D^\theta$ and $Z \in D^\perp$,*

(i) $(\nabla_X T)Z = -(Z\ln f)TX$,
(ii) $(\nabla_U T)X = A_{FX}U + \mathrm{th}(U,X) + g(TU,X)\xi$,
(iii) $(\nabla_U T)\xi = -\eta(V)TU$.

Proof. In view of the formula (2.10), Theorem 2.2 and the fact that $TZ = 0$, we obtain

$$(\nabla_X T)Z = -(Z\ln f)TX.$$

This proves part (i). For part (ii), making use (2.10), (2.12) and Theorem 2.2, we get

$$(\nabla_U T)X = A_{FX}U + \mathrm{th}(U,X) + g(TU,X)$$

this verifies part (ii). Further by formula (2.6) and the fact that $TZ = 0$, we get part (iii). □

Lemma 4.3. *Let $M = M_\perp \times_f M_\theta$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$. Then*

(i) $g(h(X,Y),FZ) - g(h(Y,Z),FX) = g(TX,Y)\{-\eta(Z) + Z\ln f\}$,
(ii) $g(h(X,Z),FW) = g(h(Z,W),FX)$

for each $X, Y \in D^\theta$ and $Z, W \in D^\perp$.

Proof. On using (2.5), (2.2) and (2.3), we get

$$g(h(X,Y),FZ) = g(TY,X)\eta(Z) - g(\bar{\nabla}_Y \phi X, Z).$$

Then from (2.6), (2.8) and from Theorem 2.2, we obtain

$$g(h(X,Y),FZ) = g(TY,X)\eta(Z) + Z\ln f g(TX,Y) + g(A_{FX}Y,Z).$$

Hence, we arrive at

$$g(h(X,Y),FZ) - g(h(Y,Z),FX) = g(TX,Y)\{-\eta(Z) + Z\ln f\}.$$

For part (ii), using (2.5) and (2.3), we get

$$g(h(X,Z),FW) = g(\phi X, \bar{\nabla}_Z W).$$

Then from (2.8) and from Theorem 2.2, we arrive at

$$g(h(X,Z),FW) = g(h(Z,W),FX). \qquad \square$$

Now we have the following result in terms of shape operator:

Theorem 4.4. *A pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ is locally a pseudo-slant warped product if and only if there exists a C^{∞}-function μ on M such that $X\mu = 0$ satisfying*

$$A_{FZ}X - A_{FX}Z = \{\eta(Z) - Z(\ln f)\}TX \tag{4.1}$$

for all $X \in D^{\theta}$ and $Z \in D^{\perp}$.

Proof. From (2.12) and (2.2) we have

$$A_{FX}Z + \mathrm{th}(X,Z) = 0. \tag{4.2}$$

Similarly again from (2.12) and from Theorem 2.2, we get

$$\eta(Z)TX - A_{FZ}X - Z(\ln f)TX = \mathrm{th}(X,Z). \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$A_{FZ}X - A_{FX}Z = \{\eta(Z) - Z(\ln f)\}TX.$$

Conversely suppose that M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ satisfies (2.11), then for any $Z, W \in D^{\perp}$ and $X \in D^{\theta}$, from (2.12) and hypothesis of the theorem we have $g((\nabla_W T)Z, X) = 0$. On using (2.2) we get $g(\nabla_Z W, X) = 0$ which implies that leaf of the $D^{\perp} \oplus \{\xi\}$ is totally geodesic in M. Further from (4.1) and (2.2), we have

$$g(\nabla_X TY, Z) = -Z(\mu)g(TX,Y).$$

Let M_{θ} be a leaf of D^{θ} and $h^{'}$ be the second fundamental form of the immersion of M_{θ} in M. Then for any $X, Y \in D^{\theta}$ and $Z \in D^{\perp}$ the left hand side reduces to $g(\nabla_X TY, Z) = g^{'}(h(X,TY))$ from which we have

$$g(h^{'}(X,Y),Z) = Z\mu g(X,Y),$$

which implies that M_{θ} is totally umbilical in M with $\nabla\mu$ as the mean curvature vector with respect to the immersion ofM_{θ} in to M. Further as $X\mu = 0$ for each $X \in D^{\theta}$, $\nabla\mu$ is parallel, that is, leaves of D^{θ} are extrinsic spheres. Hence, by virtue of theorem of S. Hiepko [8], $M_{\perp} \oplus_f M_{\theta}$ is a warped product submanifold of $\bar{M}$. $\square$

Now we may establish the following characterization.

Theorem 4.5. *A pseudo-slant submanifold M of a Kenmotsu manifold $\bar{M}$ is a pseudo-slant warped product submanifold $M_{\perp} \times_f M_{\theta}$ if and only if there exists a smooth function μ on M with $X\mu = 0$ for each $X \in D^{\perp}$ satisfies the following*

$$\begin{aligned}(\nabla_U T)V = {} & A_{FCV}CU + \mathrm{th}(CU,CV) - BV(\ln f)TCU \\ & - \eta(V)TU + g(TCU,CV)\xi\end{aligned} \tag{4.4}$$

for each $U, V \in TM$.

Proof. Let $M_\perp \times_f M_\theta$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$. Then on using from Lemma 4.2(iii), we may write

$$(\nabla_U T)V = (\nabla_{BU} T)BV + (\nabla_{CU} T)BV + (\nabla_U T)CV - \eta(V)TU + \eta(U)(\nabla_\xi T)BV. \tag{4.5}$$

First term of the right hand side from (2.6) and the fact that $TZ = 0$ reduces to $(\nabla_{BU} T)BV = -T(\nabla_{BU})BV$. Further as $M_\perp$ is totally geodesic in M, we have $(\nabla_{BU} T)BV = 0$. Similarly $(\nabla_\xi T)BV = 0$.

On the other hand on making use of Theorem 2.2, equation (4.5) takes the form

$$(\nabla_U T)V = A_{FCV}CU + \mathrm{th}(CU, CV) - BV \ln fTCU - \eta(V)TU + g(TCU, CV)\xi.$$

Conversely suppose that M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that for each $U, V \in TM$ and for a smooth function μ on M satisfying $X\mu = 0$ for each $X \in D^\theta$, (4.4) holds. Now from (4.4), $(\nabla_Z T)W = 0$ for each $Z, W \in D^\perp$. Making use of (2.6) and the fact $TW = 0$ for each $W \in D^\perp$, we get $T\nabla_Z W = 0$, which implies that $\nabla_Z W \in D^\perp \oplus \{\xi\}$. Therefore $D^\perp \oplus \{\xi\}$ is parallel on M. That is $D^\perp \oplus \{\xi\}$ is integrable and its leaves $M_\perp$ are totally geodesic in M. For each $X \in D^\theta$ and $Z \in D^\perp \oplus \{\xi\}$, (4.4) reduces to

$$(\nabla_X T)Z = -BZ(\mu)TX - \eta(Z)TX.$$

Using (3.1) and $\xi \ln f = 1$, we obtain

$$(\nabla_X T)Z = -Z(\mu)TX.$$

From (2.10) and the fact that $TZ = 0$, after that taking inner product on both side with TY for each $Y \in D^\theta$, we get

$$g(T\nabla_X Z, TY) = g(TX, TY)Z(\mu).$$

On using (2.14) and (2.16), we deduce that

$$g(\nabla_X Y, Z) = -g(X, Y)g(\nabla\mu, Z).$$

Let us assume that M_θ is a leaf of D^θ and $h^{'}$ is the second fundamental form of the immersion of M_θ in to M. Then

$$g(h^{'}(X, Y), Z) = -g(X, Y)g(\nabla\mu, Z).$$

As $h^{'}(X, Y)$ lies in $D^\perp \oplus \{\xi\}$, it follows from above equation that

$$h^{'}(X, Y) = -g(X, Y)\nabla\mu.$$

That means M_θ is totally umbilical in M with $\nabla\mu$ as the mean curvature vector with respect to the immersion of M_θ in to M. Further as $X\mu = 0$ for each $X \in D^\theta$, $\nabla\mu$ is parallel, that is, leaves of D^θ are extrinsic spheres. Hence, by virtue of theorem of S. Hiepko [8], $M_\perp \oplus_f M_\theta$ is a warped product submanifold of $\bar{M}$.

If M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$, then from (2.10) and (2.12), we have

$$A_{FZ}X = \mathrm{th}(X,Z) + \eta(Z)TX + (\nabla_X T)Z \tag{4.6}$$

for each $X \in D^\theta$ and $Z \in D^\perp \oplus \{\xi\}$. Again from (2.10) and (2.12), we get

$$\mathrm{th}(X,Z) = (\nabla_Z T)X - A_{FX}Z. \tag{4.7}$$

Since M is a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$, the from (4.4) and $\xi \ln f = 1$, we obtain

$$(\nabla_X T)Z = -(Z \ln f)TX, \text{ and } (\nabla_Z T)X = 0.$$

From (4.6) and (4.7) and above equation, we arrive at

$$A_{FZ}X - A_{FX}Z = \{\eta(Z) - Z(\ln f)\}TX.$$

Hence, we conclude that. □

Corollary 4.6. *A pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ is a pseudo-slant warped product if and only if there exists a function μ on M with $X\mu = 0$ for each $X \in D^\theta$ such that*

$$A_{FZ}X - A_{FX}Z = \{\eta(Z) - Z(\ln f)\}TX$$

for each $X \in D^\theta$ and $Z \in D^\perp \oplus \{\xi\}$.

A characterization in terms of the canonical structure F is obtained in the following theorem:

Theorem 4.7. *A pseudo-slant submanifold M of a Kenmotsu manifold $\bar{M}$ is a pseudo-slant warped product submanifold $M_\perp \times_f M_\theta$ if and only if there exists a smooth function μ on M with $X\mu = 0$ for each $X \in D^\perp$ satisfying the following*:

$$\begin{aligned} g((\nabla_U F)V, FZ) = {} & g(CU,V)\cos^2\theta Z(\ln f) - g(h(U,Z),FTV) \\ & - \eta(V)g(BU,Z) - g(TU,TV)\eta(Z) \end{aligned} \tag{4.8}$$

for each $U, V \in TM$ and $Z \in D^\perp$.

Proof. Let $M = M_\perp \times_f M_\theta$ be a pseudo-slant warped product submanifold of a Kenmotsu manifold $\bar{M}$. Then for any $U, V \in TM$ and $W \in D^\perp \oplus \{\xi\}$, from (2.13), we have

$$g(h(U,V),FZ) = g(fh(U,V)) - g(h(U,TV)) - \eta(V)g(FU,FZ). \tag{4.9}$$

The first term in the right hand side of the above equation will be zero as $g(\phi h(U,V)\phi Z) = g(h(U,V),Z) = 0$. For the last term using (3.2) and (2.15) reduces to $-\eta(V)g(BU,Z)$. On applying Lemma 4.1 the middle term takes the form

$$g(h(U,TV),FZ) = g(h(U,Z),FTV) + g(T^2CU,V)Z(\ln f) + \eta(Z)g(TU,TV).$$

From (2.14) and using the fact $\eta(CU) = 0$, we have

$$g(h(U, TV), FZ) = g(h(U, Z), FTV) - g(CU, V) \cos^2\theta\, Z(\ln f) + \eta(Z)g(TU, TV).$$

Using above in (4.9), we arrive at (4.8).

Conversely suppose that M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that (4.8) holds for a C^∞-function μ on M with $X\mu = 0$ for each $X \in D^\theta$. Then for $X \in D^\theta$ and $Z \in D^\perp \oplus \{\xi\}$, we obtain

$$g((\nabla_W F)X, FZ) = -g(h(Z, W), FTX).$$

Making use of formula (2.13) while taking account of the fact that

$$g(fh(X, W)FZ) = 0 \text{ and } \eta(X) = 0 \text{ for each } X \in D^\theta,$$

the equation reduces to

$$g(h(W, TX), FZ) = g(h(Z, W), FTX).$$

On replacing X by TX and from (2.3) and (2.8), we get

$$g(h(W, X), FZ) = -g(\bar{\nabla}_Z \phi W, X) + g(T\nabla_Z W, X).$$

Using Weingarton formula on the right hand side and (2.7) on the left hand side, we deduce that

$$g(\nabla_Z W, TX) = 0,$$

from which we have $\nabla_Z W \in D^\perp \oplus \{\xi\}$ for $Z \in D^\perp \oplus \{\xi\}$. Hence $D^\perp \oplus \{\xi\}$ is parallel, i.e., $D^\perp \oplus \{\xi\}$ is integrable and its leaves are totally geodesic in M. Again by (4.9), we obtain

$$g((\nabla_X F)Y, FZ) = Z\mu\ \cos^2\theta\, g(CX, Y) - g(h(X, Z), FTY) - \eta(Z)g(TX, TY). \tag{4.10}$$

Using (2.13) and from the fact that $g(fh(X, Y), FZ) = 0$, the left hand side of the above equation reduces to $-g(h(X, TY), FZ)$, whereas the second term of the right hand side simplified as

$$g(h(X, Z), FTY) = g(\bar{\nabla}_X Z, \phi TY) - g(\bar{\nabla}_X Z, T^2 Y).$$

From Theorem 2.1, (2.6) and (2.3), we get

$$g(h(X, Z), FTY) = -\eta(Z)g(TX, TY) + g(h(X, TY), FZ) - \cos^2\theta\, g(\nabla_X Y, Z).$$

Substituting above equation and value of $g((\bar{\nabla}_X F), FZ)$ in (4.10), we have

$$g(\nabla_X Y, Z) = -Z(\mu)g(X, Y).$$

Let M_θ be a leaf of D^θ and h' be the second fundamental form of the immersion of M_θ into M. Then by Gauss formula and from (2.16), we arrive at

$$h'(X, Y) = g(X, Y)\nabla\mu,$$

which shows that M_θ is totally umbilical in M with $\nabla\mu$ as the mean curvature vector with respect to the immersion of M_θ in to M. Further as $X\mu = 0$ for each $X \in D^\theta$, $\nabla\mu$ is parallel, that is, leaves of D^θ are extrinsic spheres.

Hence, by virtue of theorem of S. Hiepko [8], $M_\perp \oplus_f M_\theta$ is a warped product submanifold of $\bar{M}$. □

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Mohammad Shuaib
Department of Basic Sciences
SRMS College of Engineering Technology and Research
Bareily, India
Email address: shuaibyousuf6@gmail.com