Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 621–628
https://doi.org/10.4134/CKMS.c220208
pISSN: 1225-1763 / eISSN: 2234-3024

# A NOTE ON STATISTICAL MANIFOLDS WITH TORSION

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Abstract. Given a linear connection $\nabla$ and its dual connection $\nabla^*$, we discuss the situation where $\nabla + \nabla^* = 0$. We also discuss statistical manifolds with torsion and give new examples of some type for linear connections inducing the statistical manifolds with non-zero torsion.

## 1. Introduction

A metric connection $\nabla$ satisfies $\nabla g = 0$, where $g$ is a given metric tensor on a manifold $M$. This means

$$d\langle X, Y\rangle_g = \langle \nabla X, Y\rangle_g + \langle X, \nabla Y\rangle_g.$$

So, the metric $g$ is preserved by the parallel transport with respect to a metric connection $\nabla$. In this case we say that the metric $g$ is preserved by the connection $\nabla$.

We now consider another case where the metric $g$ is preserved by two given linear connections $\nabla$, $\nabla^*$, that is

$$d\langle X, Y\rangle_g = \langle \nabla X, Y\rangle_g + \langle X, \nabla^* Y\rangle_g.$$

Then $\nabla$, $\nabla^*$ are called dual connections with respect to the metric $g$. These dual connections are introduced by A. P. Norden (under the name "conjugate connections"), Nagaoka and Amari ([2, 10–12]). The geometrical methods including dual connections are first used to define a statistical structure ([8], 1987) and now also in other fields of science. In particular, dually flat manifold as a dualistic extension of the Euclidean structure is useful in application.

The torsion tensor $T^{\nabla}$ of a connection $\nabla$ is defined by

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where $[X, Y]$ is the Lie-bracket.

Given a metric $g$, there exists a unique metric connection with zero-torsion. This connection is the Levi-Civita connection denoted by $\nabla^g$. The difference

Received July 15, 2022; Revised October 24, 2022; Accepted November 25, 2022.
2020 *Mathematics Subject Classification.* Primary 53A35, 53B05, 53B15, 53C05, 53C80.
*Key words and phrases.* Metric connection, torsion, information geometry, dual connections, statistical manifold.
This work was supported by 2019 Hannam University Research Fund.

of a linear connection $\nabla$ with the Levi-civita connection $\nabla^g$ is a $(2,1)$-tensor field denoted by $A$, that is

$$\nabla_X Y = \nabla^g_X Y + A(X,Y). \tag{1}$$

Then some geometric properties of the connection $\nabla$ are induced from the symmetric or antisymmetric properties of $A$, for details see Section 2. This tensor $A(X,Y)$ is very useful for studying connections, for example, following Cartan, the types of the torsion tensors of metric connections are classified algebraically. For details, we refer to [1,5,13].

If $A(X,Y,Z)$, as a $(3,0)$-tensor, is totally symmetric with respect to $X,Y,Z$, then $(M,g,\nabla)$ is a statistical manifold whose torsion is necessarily zero, for details we refer to [4]. For torsion-free linear connections $\nabla$, $\nabla^*$, a family of connections $\nabla^{(\alpha)}$ is defined and it is known that $(M,g,\nabla^{(\alpha)})$ is a statistical manifold for all $\alpha$. In particular, for $\alpha = 0$, it holds that

$$\nabla^{(0)} = \frac{\nabla + \nabla^*}{2} = \nabla^g. \tag{2}$$

In this article, we consider a general linear connection $\nabla = \nabla^g + A$, where $A$ is an element of $\otimes^3 TM$. In Section 3.1, we will see that a linear connection is actually the sum of a metric connection and a connection which satisfies the property (2).

A notion of statistical manifolds, which allow non-zero torsion, is introduced in [7]. In Section 3.2, we discuss these generalized statistical manifolds using the tensor $A(X,Y,Z)$. Finally in Section 3.3, defining a vectorial tensor $A$, we give examples for linear connections of some type that induce statistical manifolds with non-zero torsion.

## 2. Preliminaries

### 2.1. Connections

Let $(M,g)$ be a Riemannian manifold and $\Gamma(M)$, $\Gamma^*(M)$ denote the set of sections of the tangent bundle $TM$, $T^*M$, respectively. Then a linear connection $\nabla$ can be considered as a map

$$\nabla : \Gamma(M) \otimes \Gamma(M) \to \Gamma(M).$$

A metric connection $\nabla$ is a linear connection, which gives isometries between tangent spaces by parallel transport, that is

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y). \tag{3}$$

The condition (3) is equivalent to $\nabla g = 0$, since for $(2,0)$-tensor field $g$

$$(\nabla_V g)(X,Y) = V(g(X,Y)) - g(\nabla_V X, Y) - g(X, \nabla_V Y).$$

The Levi-Civita connection, denoted by $\nabla^g$, is the unique metric connection with torsion $T = 0$. The difference of a linear connection $\nabla$ with the Levi-Civita connection $\nabla^g$ is a $(2,1)$-tensor field $A$, that is, for any tangent vector fields $X, Y \in \Gamma(M)$,

$$\nabla_X Y = \nabla^g_X Y + A(X, Y).$$

The conditions for a linear connection $\nabla$ to be metric or torsion-free or geodesics-preserving can be determined by the $(2,1)$-tensor $A$ as follows:

- $\nabla$ is torison-free if and only if $A(X, Y) = A(Y, X)$,
- $\nabla$ is metric if and only if $A(X, Y, Z) + A(X, Z, Y) = 0$,
- the geodesics with respect to $\nabla$ are the same ones with respect to $\nabla^g$ if and only if $A(X, Y) + A(Y, X) = 0$,

where the notation $A$ is also used for the $(3,0)$-tensor defined by

$$A(X, Y, Z) = \langle A(X, Y), Z\rangle.$$

### 2.2. Dual connections

Given a Riemannian manifold $(M, g)$, we now consider the case where the metric $g$ is preserved by two given linear connections $\nabla$, $\nabla^*$ as follows.

**Definition 2.1** (Dual Connections)**.** For a linear connection $\nabla$, the dual connection $\nabla^*$ of $\nabla$ with respect to $g$ is defined by

$$Z\langle X, Y\rangle_g = \langle \nabla_Z X, Y\rangle_g + \langle X, \nabla^*_Z Y\rangle_g.$$

We now consider the notation (1) and let

$$\nabla_X Y = \nabla^g + A(X, Y), \tag{4}$$

$$\nabla^*_X Y = \nabla^g + A^*(X, Y). \tag{5}$$

By computation we can easily check the following.

**Lemma 2.1.** *For a linear connection $\nabla$ and its dual connection $\nabla^*$ with tensors $A$ and $A^*$ respectively, as above* (4), (5)*, it holds that*

$$\langle A(Z, X), Y\rangle + \langle X, A^*(Z, Y)\rangle = A(Z, X, Y) + A^*(Z, Y, X) = 0. \tag{6}$$

*Remark* 2.2. For torsion-free dual connections $\nabla$, $\nabla^*$, it holds

$$\nabla = \nabla^g + A \quad \text{and} \quad \nabla^* = \nabla^g - A,$$

where $A(X, Y, Z)$ is totally symmetric with respect to $X, Y, Z$. This property is equivalent to the formula

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0. \tag{7}$$

In this case, the manifold $(M, g, \nabla)$ is a statistical manifold whose torsion is zero, for details we refer to [4].

### 2.3. $\alpha$-geometry

On a statistical model $\mathcal{S}$ endowed with the fisher metric, for two special linear connections $\nabla^{(1)}$ and $\nabla^{(-1)}$, a 1-parameter family $(\nabla^{(\alpha)})$ is defined by

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla^{(1)} + \frac{1-\alpha}{2}\nabla^{(-1)}.$$

Then the statistical model $\mathcal{S}$ is also $\nabla^{(\alpha)}$-flat and the connections $(\nabla^{(\alpha)}, \nabla^{(-\alpha)})$ are dually coupled for all $\alpha$ . In particular, $\nabla^{(0)} = \frac{\nabla^{(1)}+\nabla^{(-1)}}{2}$ is the Levi-Civita connection with respect to the fisher metric. For details we refer to [2] and [4]. Now for a linear connection $\nabla$ and its dual connection $\nabla^*$, consider the $\alpha$-connection:

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*.$$

Then the following results are known.

**Proposition 2.3** ([2, 4])**.** *If the connections $\nabla$ and $\nabla^*$ are torsion-free, the connection $\nabla^{(\alpha)}$ satisfies the formula* (7) *for all $\alpha$ and $\nabla^{(0)}$ coincides with $\nabla^g$.*

## 3. Dual connections with torsions

In this section, we will discuss the dual connections in a general situation. So, given a linear connection $\nabla = \nabla^g + A$, the difference tensor $A$ is an element of $\otimes^3 TM$.

### 3.1. The mean connection with torsion

We now consider the connections $\nabla$ and $\nabla^*$ without the torsion-free condition.

**Lemma 3.1.** *For dual connections $\nabla$ and $\nabla^*$ with difference* $(3,0)$*-tensors $A$ and $A^*$ respectively, the followings are equivalent*:

- $A + A^* = 0$.
- *$A$ is symmetric with respect to the second and the third variables.*
- *$A^*$ is symmetric with respect to the second and the third variables.*

*Proof.* The duality of $\nabla$, $\nabla^*$ implies that $A(X,Y,Z) + A^*(X,Z,Y) = 0$. So $A + A^* = 0$ if and only if

$$A(X,Y,Z) = -A^*(X,Y,Z) = A(X,Z,Y) = -A^*(X,Z,Y). \qquad \square$$

Now let $\mathcal{A}^M$ and $\mathcal{A}^S$ denote the set of $(3,0)$-tensors $A$ which is skew-symmetric and symmetric respectively, that is

$$\mathcal{A}^M = TM \otimes \Lambda^2 TM = \{A \in \otimes^3 TM \mid A(X,Y,Z) = -A(X,Z,Y)\}$$

with the dimension $\frac{n^2(n-1)}{2}$, where $n$ is the dimension of $M$ and

$$\mathcal{A}^S = TM \otimes S^2 TM = \{A \in \otimes^3 TM \mid A(X,Y,Z) = A(X,Z,Y)\}$$

with the dimension $\frac{n^2(n+1)}{2}$.

**Theorem 3.2.** (i) *Given a linear connection* $\nabla = \nabla^g + A$*, the connection* $\nabla^{(0)}$ *coincides with* $\nabla^g$ *if and only if* $A \in \mathcal{A}^S$*.*
(ii) *A linear connection* $\nabla$ *can be represented as a direct sum of* $\nabla_1$ *and* $\nabla_2$*, where* $\nabla_1$ *is a metric connection and* $\nabla_2^{(0)} = \nabla^g$*.*

*Proof.* (i) Since $\nabla^{(0)} = \nabla^g + \frac{A+A^*}{2}$, $\nabla^{(0)} = \nabla^g$ if and only if

$$A + A^* = 0,$$

which is equivalent to the condition that $A \in \mathcal{A}^S$, by Lemma 3.1.

(ii) Since $\otimes^2 TM = \Lambda^2 TM \oplus S^2 TM$, it holds that

$$\begin{aligned}\otimes^3 TM &= \{TM \otimes \Lambda^2 TM\} \oplus \{TM \otimes S^2 TM\}\\ &= \mathcal{A}^M \oplus \mathcal{A}^S.\end{aligned}$$

We now recall that $\nabla$ is a metric connection for $A \in \mathcal{A}^M$ and $\nabla^{(0)} = \nabla^g$ for $A \in \mathcal{A}^S$, respectively. □

*Remark* 3.3. In fact, it holds that

$$2\nabla = \nabla^g + A + A^* + \nabla^g + A - A^*,$$

where $A + A^* \in \mathcal{A}^M$ and $A - A^* \in \mathcal{A}^S$.

### 3.2. Statistical manifolds admitting torsion

In [7], the notion of a statistical manifold admitting torsion is introduced.

**Definition 3.1** ([3, 7, 9])**.** A Riemannian manifold $(M, g, \nabla)$ is called a statistical manifold admitting torsion if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = -g(T^\nabla(X, Y), Z) \tag{8}$$

for $X, Y, Z \in \Gamma(TM)$, where $T^\nabla$ is the torsion tensor of $\nabla$.

Using the notation (4), we obtain the following result.

**Proposition 3.4.** *For* $\nabla = \nabla^g + A$*, the formula* (8) *is equivalent to*

$$A(X, Y, Z) = A(Z, Y, X) \tag{9}$$

*for* $X, Y, Z \in \Gamma(TM)$*.*

*Proof.* Recall that

$$(\nabla_X g)(Y, Z) = X[g(Y, Z)] - g(\nabla_X Y, Z) - g(Y, \nabla_X Z), \quad X, Y, Z \in \Gamma(TM).$$

We can then compute

$$\begin{aligned}&(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T^\nabla(X, Y), Z)\\ &= g(T^\nabla(X, Y) - \nabla_X Y + \nabla_Y X, Z) - g(Y, \nabla_X Z) + g(X, \nabla_Y Z)\\ &\quad + X[g(Y, Z)] - Y[g(X, Z)]\\ &= g(-[X, Y], Z) - g(Y, \nabla^g_X Z + A(X, Z)) + g(X, \nabla^g_Y Z + A(Y, Z))\\ &\quad + X[g(Y, Z)] - Y[g(X, Z)]\end{aligned}$$

$$= g(-[X,Y],Z) + g(\nabla^g_X Y, Z) - A(X,Z,Y) - g(\nabla^g_Y X, Z) + A(Y,Z,X)$$
$$= -A(X,Z,Y) + A(Y,Z,X),$$

since $\nabla^g$ is a torsion-free connection. □

The following property of statistical manifolds admitting torsion is known.

**Proposition 3.5** ([9]). *A manifold $(M, g, \nabla)$ is a statistical manifold admitting torsion if and only if its dual connection $\nabla^*$ is a torsion-free connection.*

*Remark* 3.6. The formula (9) implies

$$A^*(X,Z,Y) = A^*(Z,X,Y), \quad X,Y,Z \in \Gamma(TM).$$

So, Proposition 3.5 follows immediately from Proposition 3.4.

### 3.3. Examples [A difference tensor $A(X,Y)$ of vectorial type]

Given a linear connection $\nabla = \nabla^g + A$, we know that $\nabla^{(0)} = \frac{\nabla+\nabla^*}{2}$ is a metric connection. So, generalizing the difference tensor of vectorial type for a metric connection we can define the vectorial tensor $A$ as follows.

**Definition 3.2.** For fixed vector fields $V_1, V_2$, a difference tensor $A(X,Y)$ of vectorial type is defined by

$$A(X,Y) = g(X,Y)V_1 - g(V_2,Y)X \tag{10}$$

for $X,Y,Z \in \Gamma(M)$.

We then obtain the following results.

**Proposition 3.7.** *Let $A$ be a $(2,1)$-tensor of vectorial type with vector fields $V_1, V_2$ as above* (10). *Then for a linear connection $\nabla = \nabla^g + A$, we have*

(i) $A^*(X,Y) = g(X,Y)V_2 - g(V_1,Y)X$,
(ii) $A \in \mathcal{A}^M$ *if and only if* $V_1 = V_2$,
(iii) $A \in \mathcal{A}^S$ *if and only if* $V_1 = -V_2$,
(iv) $(M, g, \nabla)$ *is a statistical manifold admitting torsion if and only if* $V_1 = 0$.

*In particular, for $V_1 = 0, V_2 \neq 0$, we obtain a statistical manifold with non-zero torsion.*

*Proof.* (i) By (6) and (10), we compute

$$\begin{aligned} A^*(X,Y,Z) &= -A(X,Z,Y) \\ &= -\langle g(X,Z)V_1 - g(V_2,Z)X, Y\rangle \\ &= \langle g(X,Y)V_2 - g(V_1,Y)X, Z\rangle. \end{aligned}$$

So, we conclude that $A^*(X,Y) = g(X,Y)V_2 - g(V_1,Y)X$.

(ii) We compute

$$\begin{aligned} A(X,Y,Z) + A(X,Z,Y) &= g(X,Y)g(V_1,Z) - g(V_2,Y)g(X,Z) \\ &\quad + g(X,Z)g(V_1,Y) - g(V_2,Z)g(X,Y) \end{aligned}$$

$$= g(X,Y)g(V_1 - V_2, Z) - g(V_2 - V_1, Y)g(X,Z).$$

So, $A \in \mathcal{A}^M$ if and only if $V_1 = V_2$.

(iii) By a similar computation,

$$A(X,Y,Z) - A(X,Z,Y) = g(X,Y)g(V_1 + V_2, Z) - g(V_2 + V_1, Y)g(X,Z).$$

So, $A \in \mathcal{A}^S$ if and only if $V_1 = -V_2$.

(iv) By Proposition 3.4 and Remark 3.6, the formula (8) is satisfied if and only if

$$A^*(X,Y) = A^*(Y,X).$$

From Proposition 3.7(i) we compute then

$$A^*(X,Y) - A^*(Y,X) = -g(V_1, Y)X + g(V_1, X)Y.$$

So, $(M, g, \nabla)$ is a statistical manifold admitting torsion if and only if $V_1 = 0$. In particular, since

$$T^\nabla(X,Y) = A(X,Y) - A(Y,X) = -g(V_2, Y)X + g(V_2, X)Y, \tag{11}$$

we obtain a statistical manifold with non-zero torsion for $V_2 \neq 0$. □

Finally we recall that a linear connection $\nabla$, whose torsion satisfies the formula

$$T^\nabla(X,Y) = g(Y,V)X - g(X,V)Y \quad \text{for some}\, V \in \Gamma(TM),$$

is called a semi-symmetric connection ([6]). So, from the computation (11) we have the following remark.

*Remark* 3.8. A linear connection with a difference tensor of vectorial type is also a semi-symmetric connection.

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