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VANISHING THEOREMS FOR WEIGHTED HARMONIC 1-FORMS ON SMOOTH METRIC MEASURE SPACES

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Abstract. In this paper, we prove some vanishing theorems under the assumptions of weighted BiRic curvature or m-Bakry-Émery-Ricci curvature bounded from below.

1. Introduction

It is well known that the space of harmonic 1-forms is isomorphic to its first de Rham cohomology group for compact manifolds. In particular, if the space of harmonic 1-forms is trivial, then the first de Rham cohomology group is trivial. It is an interesting problem in geometry and topology to find sufficient conditions for the space of harmonic 1-forms to be trivial.

Li and Wang [18] proved that if M is a complete Riemannian manifold with $\lambda_1(M) > 0$ and $Ric_M \geq -\frac{n\lambda_1(M)}{n-1} + \epsilon$ for some $\epsilon > 0$, then $H^1(L^2(M)) = 0$. Later, Lam [17] generalized the above result to manifolds satisfying a weighted Poincaré inequality, although adding assumption of growth rate of the weight function. Vieira [25] improves Lam's theorem by removing the assumptions of sign and growth rate of the weight function. Other related vanishing results have been obtained (see [4–7, 9, 12, 13, 21, 26] for details).

In [19], Lott discussed the topology of compact smooth metric measure spaces with non-negative *Bakry-Émery-Ricci* curvature using harmonic forms. Recently, there are some interesting vanishing type theorems on smooth metric measure spaces or gradient Ricci solitons. For example, Munteanu and Wang [20] considered a smooth metric measure space with $Ric_f \geq 0$. Vieira [24] obtained a vanishing result for L^2 weighted harmonic 1-forms if the underlying manifold M satisfies $Ric_f \geq 0$ and the bottom of the weighted spectrum $\lambda_1(\Delta_f) > 0$. For more vanishing results about harmonic forms or harmonic maps on smooth metric measure spaces, we will refer the reader to

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[1, 2, 10, 11, 14–16, 23, 27–29] for recent progress on this topic and references therein. In particular, Dung-Duc-Pyo ([10]) considered immersed f-minimal hypersurfaces in a weighted Riemannian manifold and prove that if such a hypersurface is weighted stable, then the space of L^2 weighted harmonic 1-forms is trivial.

Theorem 1.1 ([10])**.** *Let M be an f-minimal hypersurface immersed in a smooth metric measure space $(\overline{M}, g, e^{-f}dv)$. If M is f-stable and $\overline{M}$ has $\overline{BiRic}_f \geq 0$, and there exists at least a point $x_0 \in M$ such that $\overline{BiRic}_f(x_0) > 0$, then there is no f-harmonic 1-form on M with finite weighted L^2 energy, or equivalently*

$$\mathcal{H}_f^1\left(L^2(M)\right) = 0,$$

where

$$\mathcal{H}_f^1\left(L^2(M)\right) := \left\{\omega \in \Lambda^1(M) : d\omega = \delta_f\omega = 0, \int_M |\omega|^2 e^{-f} < +\infty\right\}.$$

In the first part of this paper, we investigate harmonic forms on non-compact smooth metric measure spaces with weighted BiRicci curvature $\overline{BiRic}_f^a$ bounded from below.

Definition 1.2. For orthonormal vector fields X and Y on $\overline{M}^{n+1}$, the weighted BiRicci curvature $\overline{BiRic}_f^a$ is defined by

$$\overline{BiRic}_f^a(X,Y) = \overline{Ric_f}(X,X) + a\overline{Ric_f}(Y,Y) - \overline{K}(X,Y),$$

where a is a constant and $\overline{K}$ is the sectional curvature of $\overline{M}^{n+1}$.

In particular, when $a = 1$, then $\overline{BiRic}_f^a = \overline{BiRic}_f$. If we assume further that f is a constant function, then it is the *BiRic* curvature defined by Shen and Ye ([23]).

Theorem 1.3. *Let M^n be an f-minimal hypersurface immersed in a smooth metric measure space $(\overline{M}, g, e^{-f}dv)$. If M is f-stable and $\overline{M}$ has nonnegative weighted BiRicci curvature $\overline{BiRic}_f^a$ for $1 \leq a < \frac{n}{n-1}$, and there exists at least a point $x_0 \in M$ such that $\overline{BiRic}_f^a(x_0) > 0$, then $\mathcal{H}_f^1\left(L^2(M)\right) = 0$.*

Theorem 1.4. *Let M be an f-minimal hypersurface immersed in a smooth metric measure space $(\overline{M}, g, e^{-f}dv)$. Assume that $\overline{M}$ has*

$$\overline{BiRic}_f^a \geq -k^2$$

for $1 \leq a < \frac{n}{n-1}$, and there exists at least a point $x_0 \in M$ such that $\overline{BiRic}_f^a(x_0) > -k^2$. If M is f-stable and $\lambda_{1,f}(M) > \frac{(n-1)k^2}{n-(n-1)a}$, then

$$\mathcal{H}_f^1\left(L^2(M)\right) = 0.$$

In the second part, we will consider the vanishing theorems of L^p weighted harmonic 1-forms when m-Bakry-Émery-Ricci curvature bounded from below. Similarly, we denote

$$\mathcal{H}_f^1\left(L^p(M)\right) := \left\{\omega \in \Lambda^1(M) : d\omega = \delta_f\omega = 0, \int_M |\omega|^p e^{-f} < +\infty\right\}.$$

Theorem 1.5. *Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space. Suppose*

$$Ric_f^m \geq -\delta$$

and $\lambda_{1,f}(M) > \frac{(m-1)\delta\sigma^2}{2(m-1)\sigma-(m-2)}$ with two constants $\delta \geq 0$ and $\sigma > \frac{m-2}{2(m-1)}$. Then every f-harmonic 1-form ω on M with $\lim_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^{2\sigma} e^{-f}dv = 0$ vanishes identically. In particular, $\mathcal{H}_f^1(L^{2\sigma}(M)) = 0$.

Remark 1.6. Choosing $\delta = 0$ and $m \geq n$ in Theorem 1.5, we have Theorem 1.3 in [16].

Moreover, if the weighted volume is polynomial growth at most, we have:

Theorem 1.7. *Let $(M^n, g, e^{-f}dv)$, $n \geq 3$ be a complete smooth metric measure space. Assume that*

$$Ric_f^m \geq -\delta,$$

where $0 \leq \delta < \frac{(m-1)\lambda_{1,f}(M)}{m-2}$. If the weighted volume of M satisfies

$$\mathrm{Vol}_f(B(r)) \leq Cr^{\frac{2q(m-1)}{m(q-2)-q+4}}$$

for some $C > 0$ and $q > \frac{2(m-2)}{m-1}$, then $\mathcal{H}_f^1(L^q(M)) = 0$.

2. Preliminaries

A smooth metric measure space $(M^n, g, e^{-f}\mathrm{d}v)$ is an n-dimensional Riemannian manifold (M, g) together with a weighted volume form $e^{-f}\mathrm{d}v$, where f is a smooth function on M and $\mathrm{d}v$ is the volume element induced by the Riemannian metric g. Recall that the formal adjoint of the exterior derivative d with respect to the measure $e^{-f}\mathrm{d}v$ is given by $\delta_f = \delta + \iota_{\nabla f}$. The associated weighted Laplacian $\triangle_f$ on smooth metric measure spaces is given by

$$\triangle_f(\cdot) = -(d\delta_f + \delta_f d)(\cdot) = \triangle(\cdot) - \langle\nabla f, \nabla(\cdot)\rangle.$$

A differential form ω on M is called f-harmonic if ω satisfies

$$d\omega = 0 \quad \text{and} \quad \delta_f\omega = 0.$$

The *Bakry-Émery-Ricci* curvature and *m-Bakry-Émery-Ricci* tensor are defined by the formula

$$Ric_f = Ric + Hess f,$$

and

$$Ric_f^m = Ric + Hess(f) - \frac{\nabla f \otimes \nabla f}{m-n},$$

respectively, where *Ric* denotes the *Ricci tensor* of (M,g) and $Hess(f)$ denotes the Hessian of f. Note that in the definition of Ric_f^m, we assume $m \geq n$ and $m = n$ if and only if f is constant.

A differential form ω is called an L_f^2 differential form if

$$\int_M |\omega|^2 e^{-f} \mathrm{d}v < \infty.$$

A function h is said to be f-harmonic if $\triangle_f h = 0$. It is clear that f-harmonic functions are characterized as the critical points of the weighted *Dirichlet* energy $\int_M |\nabla h|^2 e^{-f} \mathrm{d}v$.

Now let $i : M \hookrightarrow (\overline{M}^{n+1}, \bar{g}, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth immersion. Then i induces a metric $g = i^*\overline{g}$ on M so that $i : (M,g) \to (\overline{M}^{n+1}, \overline{g})$ is an isometric immersion. The restriction of f on M, still denoted by f, yields a weighted measure $e^{-f}\mathrm{d}v$ on M, and hence an induced smooth metric measure space $(M^n, g, e^{-f}\mathrm{d}v)$.

Definition 2.1. The weighted mean curvature H_f of the hypersurface M is defined by

$$H_f = H - \langle \overline{\nabla} f, \nu \rangle.$$

M is called an *f-minimal* hypersurface if it satisfies $H_f = 0$, i.e., $H = \langle \overline{\nabla} f, \nu \rangle$.

Definition 2.2. An f-minimal hypersurface M is said to be *f-stable* if the following stability inequality

$$\int_M \Big(|\nabla \eta|^2 - (|A|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu))\eta^2 \Big) e^{-f} \mathrm{d}v \geq 0 \tag{2.1}$$

holds true for any compactly supported smooth function $\eta \in C_0^\infty(M)$.

Lemma 2.3 ([24])**.** *Let ω be an f-harmonic 1-form on a smooth metric measure space $(M, g, e^{-f}\mathrm{d}v)$. Then there is weighted Bochner-Weitzenböck formula*

$$\frac{1}{2}\triangle_f |\omega|^2 = |\nabla \omega|^2 + \langle \triangle_f \omega, \omega \rangle + Ric_f(\omega^\sharp, \omega^\sharp). \tag{2.2}$$

Lemma 2.4 ([3,8])**.** *For a closed and co-closed k-form ω on M^n the following inequality holds*

$$|\nabla \omega|^2 \geq C_{n,k} \Big| \nabla |\omega| \Big|^2 \tag{2.3}$$

with

$$C_{n,k} = \begin{cases} 1 + \frac{1}{n-k}, & 1 \leq k \leq \frac{n}{2}; \\ 1 + \frac{1}{k}, & \frac{n}{2} \leq k \leq n-1. \end{cases}$$

3. Smooth metric measure spaces with weighted BiRicci curvature bounded from below

In this section, we investigate harmonic forms on non-compact smooth metric measure spaces with $\overline{BiRic}_f^a$ bounded from below. Firstly, we have:

Theorem 3.1. *Let M^n be an f-minimal hypersurface immersed in a smooth metric measure space $(\overline{M}, \bar{g}, e^{-f}\mathrm{d}v)$. If M is f-stable and $\overline{M}$ has nonnegative weighted BiRicci curvature $\overline{BiRic}_f^a$ for $1 \leq a < \frac{n}{n-1}$, and there exists at least a point $x_0 \in M$ such that $\overline{BiRic}_f^a(x_0) > 0$, then $\mathcal{H}_f^1\left(L^2(M)\right) = 0$.*

Proof. Assume ω is an f-harmonic 1-form on M^n and $\omega^\sharp$ is its dual vector field. We argue by contradiction, assuming that $\omega \not\equiv 0$. By weighted Bochner-Weitzenböck formula (2.2) and *Kato* inequality (2.3), we have

$$\begin{aligned} |\omega|\triangle_f|\omega| &= |\nabla\omega|^2 - |\nabla|\omega||^2 + Ric_f(\omega^\sharp, \omega^\sharp) \\ &\geq \frac{1}{n-1}|\nabla|\omega||^2 + Ric_f(\omega^\sharp, \omega^\sharp). \end{aligned} \tag{3.1}$$

We observe that for any smooth unit vector fields X, Y on M,

$$\begin{aligned} \text{Hess } f(X,Y) &= \nabla_X\nabla_Y f - (\nabla_X Y)\, f \\ &= \bar\nabla_X\bar\nabla_Y f - \left\{\bar\nabla_X Y - \left(\bar\nabla_X Y\right)^\perp\right\} f \\ &= \overline{\text{Hess}}\, f(X,Y) - h(X,Y)\frac{\partial f}{\partial \nu} \\ &= \overline{\text{Hess}}\, f(X,Y) - h(X,Y)H. \end{aligned} \tag{3.2}$$

Here, $h(X,Y)$ is the second fundamental form with respect to X, Y. Note also that the Gaussian equation implies that

$$\text{Ric}(X,X) = \sum_i \bar{R}\left(X, e_i, X, e_i\right) + h(X,X)H - \sum_i h\left(e_i, X\right)^2. \tag{3.3}$$

Then we conclude from (3.2) and (3.3), for unit vector field X,

$$\begin{aligned} \text{Ric}_f(X,X) = &\ \text{Ric}(X,X) + \text{Hess}\, f(X,X) \\ = &\ \overline{\text{Ric}}(X,X) + h(X,X)H - \sum_i h\left(e_i, X\right)^2 - \overline{K}(X,\nu) \\ &+ \overline{\text{Hess}} f(X,X) - h(X,X)H \\ \geq &\ \overline{\text{BiRic}}_f^a(X,\nu) - a\overline{\text{Ric}}_f(\nu,\nu) - |A|^2 \\ \geq &\ -a\overline{\text{Ric}}_f(\nu,\nu) - |A|^2. \end{aligned} \tag{3.4}$$

Thus (3.1) and (3.4) infers that

$$\begin{aligned} |\omega|\triangle_f|\omega| &\geq \frac{1}{n-1}|\nabla|\omega||^2 + (-a\overline{\text{Ric}}_f(\nu,\nu) - |A|^2)|\omega|^2 \\ &\geq \frac{1}{n-1}|\nabla|\omega||^2 - a\big(\overline{\text{Ric}}_f(\nu,\nu) + |A|^2\big)|\omega|^2 \end{aligned} \tag{3.5}$$

for any $a \geq 1$. Multiplying both sides of (3.5) by η^2, where η is a smooth function on M with compact support, and then integrating the obtained result over M, we obtain that

$$\begin{aligned} &\int_M \eta^2|\omega|\triangle_f|\omega|e^{-f} \\ &\geq \frac{1}{n-1}\int_M \eta^2|\nabla|\omega||^2e^{-f} - a\int_M \big(\overline{\mathrm{Ric}}_f(\nu,\nu)+|A|^2\big)\eta^2|\omega|^2e^{-f}. \end{aligned} \tag{3.6}$$

The stability condition (2.1) implies that

$$B(\eta) := \int_M |\nabla(\eta|\omega|)|^2e^{-f} - \int_M \big(|A|^2+\overline{\mathrm{Ric}}_f(\nu,\nu)\big)\eta^2|\omega|^2e^{-f} \geq 0. \tag{3.7}$$

Combining (3.6) and (3.7) and using Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} 0 &\leq aB(\eta) \\ &\leq a\int_M |\nabla(\eta|\omega|)|^2e^{-f} - \frac{1}{n-1}\int_M \eta^2|\nabla|\omega||^2e^{-f} \\ &\quad + \int_M \eta^2|\omega|\triangle_f|\omega|e^{-f} \\ &= a\int_M |\nabla(\eta|\omega|)|^2e^{-f} - \frac{1}{n-1}\int_M \eta^2|\nabla|\omega||^2e^{-f} \\ &\quad - \int_M \langle\nabla(\eta^2|\omega|),\nabla|\omega|\rangle e^{-f} \\ &= a\int_M |\omega|^2|\nabla\eta|^2e^{-f} + \Big(a-1-\frac{1}{n-1}\Big)\int_M \eta^2|\nabla|\omega||^2e^{-f} \\ &\quad + 2(a-1)\int_M \eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle e^{-f} \\ &\leq a\int_M |\omega|^2|\nabla\eta|^2e^{-f} + \Big(a-\frac{n}{n-1}\Big)\int_M \eta^2|\nabla|\omega||^2e^{-f} \\ &\quad + (a-1)\frac{1}{\epsilon}\int_M |\omega|^2|\nabla\eta|^2e^{-f} + (a-1)\epsilon\int_M \eta^2|\nabla|\omega||^2e^{-f} \end{aligned} \tag{3.8}$$

for any positive constant ϵ. Consequently,

$$0 \leq aB(\eta) \leq A\int_M |\omega|^2|\nabla\eta|^2e^{-f} - B\int_M \eta^2|\nabla|\omega||^2e^{-f}, \tag{3.9}$$

where

$$A = a+(a-1)\cdot\frac{1}{\epsilon},\ \ B = \frac{n}{n-1} - a - (a-1)\epsilon.$$

From the assumption, we know that $A, B > 0$ for sufficiently small $\epsilon > 0$.

For each $r > 0$, let B_r denote the geodesic ball of radius r on M centered at some fixed point o and let $\eta \in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases} \eta = 1 & \text{on } B_{\frac{r}{2}}(o), \\ \eta = 0 & \text{on } M \backslash B_r(o) \end{cases}$$

and $|\nabla \eta| \le \frac{c}{r}$ for some positive constant c. Applying this test function η to (3.9), we get that

$$0 \le aB(\eta) \le \frac{Ac^2}{r^2} \int_M |\omega|^2 e^{-f}.$$

Letting $r \to \infty$ and using the fact $\omega \in L_f^2(M)$, this implies that $B(\eta) \equiv 0$. Moreover, all inequalities used to verify this identity must be equalities. In particular, from (3.4), we have $\overline{\mathrm{BiRic}}_f^a \equiv 0$ which contradict with the assumption $\overline{\mathrm{BiRic}}_f^a(x_0) > 0$. Thus we complete the proof. $\square$

In what follow, under the assumption on the bottom spectrum $\lambda_{1,f}(M)$ of weighted Laplacian Δ_f, we will have the following vanishing theorem.

Theorem 3.2. *Let M be an f-minimal hypersurface immersed in a smooth metric measure space $(\bar{M}, g, e^{-f} \mathrm{d}v)$. Assume that $\bar{M}$ has*

$$\overline{BiRic}_f^a \ge -k^2$$

for $1 \le a < \frac{n}{n-1}$, and there exists at least a point $x_0 \in M$ such that $\overline{BiRic}_f^a(x_0) > -k^2$. If M is f-stable and $\lambda_{1,f}(M) > \frac{(n-1)k^2}{n-(n-1)a}$, then

$$\mathcal{H}_f^1\left(L^2(M)\right) = 0.$$

Proof. In this case, (3.4) and (3.8) are rewritten as

$$\mathrm{Ric}_f(X, X) \ge -k^2 - a\overline{\mathrm{Ric}}_f(\nu, \nu) - |A|^2$$

and

$$\begin{aligned} 0 \le aB(\eta) \le a \int_M |\omega|^2 |\nabla \eta|^2 e^{-f} + \Big(a - 1 - \frac{1}{n-1}\Big) \int_M \eta^2 |\nabla |\omega||^2 e^{-f} \\ + k^2 \int_M \eta^2 |\omega|^2 e^{-f} + 2(a-1) \int_M \eta |\omega| \langle \nabla \eta, \nabla |\omega| \rangle e^{-f}. \end{aligned} \tag{3.10}$$

By the definition of eigenvalues, we have that

$$\lambda_{1,f}(M) \le \frac{\int_M \left|\nabla\left(\eta|\omega|\right)\right|^2 e^{-f}}{\int_M \eta^2 |\omega|^2 e^{-f}}.$$

Then

$$\int_M \eta^2 |\omega|^2 e^{-f} \le \frac{1}{\lambda_{1,f}(M)} \int_M \left|\nabla\left(\eta|\omega|\right)\right|^2 e^{-f}.$$

Inserting this inequality into (3.10) and taking the same arguments, we have

$$0 \le aB(\eta) \le C \int_M |\omega|^2 |\nabla \eta|^2 e^{-f} - D \int_M \eta^2 |\nabla |\omega||^2 e^{-f},$$

where

$$C = a + \frac{k^2}{\lambda_{1,f}(M)} + \Big(a + \frac{k^2}{\lambda_{1,f}(M)} - 1\Big) \cdot \frac{1}{\epsilon},$$
$$D = \frac{n}{n-1} - a - \frac{k^2}{\lambda_{1,f}(M)} - \Big(a + \frac{k^2}{\lambda_{1,f}(M)} - 1\Big)\epsilon.$$

From the assumption, we know that $1 \leq a + \frac{k^2}{\lambda_{1,f}(M)} < \frac{n}{n-1}$, so $C, D > 0$ for sufficiently small $\epsilon > 0$. Then the proof follows the same argument as before. □

4. Smooth metric measure spaces with m-Bakry-Émery-Ricci curvature bounded from below

Let ω be any f-harmonic 1-form on M and let $\omega^{\#}$ be its dual vector field. By weighted Bochner-Weitzenböck formula (2.2), we have

$$\frac{1}{2}\Delta_f|\omega|^2 = |\nabla\omega|^2 + Ric_f(\omega^{\#}, \omega^{\#}).$$

By [24], we have $|\nabla\omega|^2 \geq \frac{1}{n-1}(|\nabla|\omega|| - |df(\omega^{\#})|)^2 + |\nabla|\omega||^2$. Hence, by the formula $(a-b)^2 \geq \frac{1}{1+\alpha}a^2 - \frac{1}{\alpha}b^2$, we get

$$\begin{aligned}\frac{1}{2}\Delta_f|\omega|^2 &\geq \frac{1}{n+1}(|\nabla|\omega|| - |df(\omega^{\#})|)^2 + |\nabla|\omega||^2 + Ric_f(\omega^{\#}, \omega^{\#}) \\ &\geq \Big(1 + \frac{1}{(n-1)(1+\alpha)}\Big)|\nabla|\omega||^2 - \frac{1}{(n-1)\alpha}|df(\omega^{\#})|^2 + Ric_f(\omega^{\#}, \omega^{\#}) \\ &= \Big(1 + \frac{1}{(n-1)(1+\alpha)}\Big)|\nabla|\omega||^2 + \Big(Ric_f - \frac{df \otimes df}{(n-1)\alpha}\Big)(\omega^{\#}, \omega^{\#})\end{aligned}$$

for any $\alpha > 0$. Let $(n-1)\alpha = m - n$, then $1 + \frac{1}{(n-1)(1+\alpha)} = \frac{m}{m-1}$. This yields

$$\frac{1}{2}\Delta_f|\omega|^2 \geq \frac{m}{m-1}|\nabla|\omega||^2 + Ric_f^m(\omega^{\#}, \omega^{\#}). \tag{4.1}$$

Lemma 4.1. *Let $(M, g, e^{-f}\mathrm{dv})$ be a complete smooth metric measure space with $\lambda_{1,f}(M) > 0$. Suppose h is a non-negative function satisfying the differential inequality*

$$h\Delta_f h \geq A|\nabla h|^2 - Bh^2 \tag{4.2}$$

in the weak sense, for some nonnegative constants A and B. Assume that

$$\int_{B_{x_0}(r)} h^{2\sigma} e^{-f} dv = o(r^2) \quad as\ r \to \infty \tag{4.3}$$

for some constant σ satisfying

$$2\sigma - 1 + A - \frac{B\sigma^2}{\lambda_{1,f}(M)} > 0. \tag{4.4}$$

Then h is identically zero.

Proof. Using (4.2), we compute

$$
\begin{aligned}
h^\alpha \Delta_f h^\alpha &= h^\alpha[\alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + \alpha h^{\alpha-1}\Delta_f h] \\
&\geq (\alpha - 1 + A)\alpha h^{2\alpha-2}|\nabla h|^2 - \alpha B h^{2\alpha} \\
&= \Big(1 - \frac{1-A}{\alpha}\Big)|\nabla h^\alpha|^2 - \alpha B h^{2\alpha} \qquad (4.5)
\end{aligned}
$$

for any $\alpha > 0$. Let $\phi \in C_0^\infty(M)$. Multiplying both sides of (4.5) by $\phi^2 h^{2q\alpha}$ and integrating over M, we have

$$
\begin{aligned}
&\Big(1 - \frac{1-A}{\alpha}\Big)\int_M \phi^2 h^{2q\alpha}|\nabla h^\alpha|^2 e^{-f} \\
\leq& \int_M \phi^2 h^{(2q+1)\alpha}\Delta_f h^\alpha e^{-f} + \alpha B\int_M \phi^2 h^{2(q+1)\alpha}e^{-f} \\
=& -(2q+1)\int_M \phi^2 h^{2q\alpha}|\nabla h^\alpha|^2 e^{-f} - 2\int_M \phi h^{(2q+1)\alpha}\langle\nabla\phi, \nabla h^\alpha\rangle e^{-f} \\
&+ \alpha B\int_M \phi^2 h^{2(q+1)\alpha}e^{-f}. \qquad (4.6)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, for any $\varepsilon > 0$, we have

$$
-2\langle h^\alpha\nabla\phi, \phi\nabla h^\alpha\rangle \leq \varepsilon\phi^2|\nabla h^\alpha|^2 + \frac{1}{\varepsilon}h^{2\alpha}|\nabla\phi|^2.
$$

Hence, from (4.6), we obtain

$$
\begin{aligned}
&\Big[2(q+1) - \frac{1-A}{\alpha} - \varepsilon\Big]\int_M \phi^2 h^{2q\alpha}|\nabla h^\alpha|^2 e^{-f} \\
\leq& \frac{1}{\varepsilon}\int_M h^{2(q+1)\alpha}|\nabla\phi|^2 e^{-f} + \alpha B\int_M \phi^2 h^{2(q+1)\alpha}e^{-f}. \qquad (4.7)
\end{aligned}
$$

On the other hand, inequality

$$
\int_M \phi^2 h^{2(q+1)\alpha}e^{-f} \leq \frac{1}{\lambda_{1,f}(M)}\int_M \big|\nabla(\phi h^{(q+1)\alpha})\big|^2 e^{-f}
$$

and the Hölder inequality $(a+b)^2 \leq (1+\beta)a^2 + (1+\frac{1}{\beta})b^2$ assert that

$$
\begin{aligned}
\int_M \phi^2 h^{2(q+1)\alpha}e^{-f} &\leq \frac{1}{\lambda_{1,f}(M)}\int_M \big|h^{(q+1)\alpha}\nabla\phi + (q+1)\phi h^{q\alpha}\nabla h^\alpha\big|^2 e^{-f} \\
&\leq \frac{(1+\beta)(q+1)^2}{\lambda_{1,f}(M)}\int_M \phi^2 h^{2q\alpha}|\nabla h^\alpha|^2 e^{-f} \\
&\quad + \frac{1+\frac{1}{\beta}}{\lambda_{1,f}(M)}\int_M h^{2(q+1)\alpha}|\nabla\phi|^2 e^{-f} \qquad (4.8)
\end{aligned}
$$

for any $\beta > 0$. Substituting (4.8) into (4.7), we find

$$
\Big[2(q+1) - \frac{1-A}{\alpha} - \varepsilon - \frac{\alpha B(1+\beta)(q+1)^2}{\lambda_{1,f}(M)}\Big]\int_M \phi^2 h^{2q\alpha}|\nabla h^\alpha|^2 e^{-f}
$$

$$\leq \Big[\frac{1}{\varepsilon}+\frac{\alpha B(1+\frac{1}{\beta})}{\lambda_{1,f}(M)}\Big]\int_M h^{2(q+1)\alpha}|\nabla\phi|^2e^{-f}. \tag{4.9}$$

Set $\sigma=(q+1)\alpha$. Then the assumption (4.4) ensures that $2(q+1)-\frac{1-A}{\alpha}-\varepsilon-\frac{\alpha B(1+\beta)(q+1)^2}{\lambda_{1,f}(M)}>0$ by choosing ε and β small enough. Let us choose a cut-off function $\phi_r(x)\in C_0^\infty(M)$ satisfying

$$\phi_r(x)=\begin{cases}1 & \text{on } B_{x_0}(r);\\ 0 & \text{on } M\backslash B_{x_0}(2r)\end{cases} \tag{4.10}$$

and

$$|\nabla\phi_r|(x)\leq\frac{2}{r}\quad \text{on } B_{x_0}(2r)\backslash B_{x_0}(r).$$

Substituting $\phi=\phi_r$ into (4.9) yields

$$\int_{B_{x_0}(r)}h^{2\sigma-2}|\nabla h|^2e^{-f}dv\leq\frac{4C_1}{r^2}\int_{B_{x_0}(2r)}h^{2\sigma}e^{-f}dv$$

from some positive constant C_1. Letting $r\to\infty$, the assumption (4.3) implies that h is a constant. If h is not identically zero, we deduce that

$$\mathrm{Vol}_f(B_{x_0}(r))=o(r^2)\quad \text{as } r\to\infty.$$

By (4.4), we have $\lambda_{1,f}(M)>\frac{B\sigma^2}{2\sigma-1+A}$. Hence,

$$\begin{aligned}\mathrm{Vol}_f(M)=\lim_{r\to\infty}\int_{B_{x_0}(r)}e^{-f}dv&=\lim_{r\to\infty}\int_{B_{x_0}(r)}\phi_r^2e^{-f}dv\\ &\leq\lim_{r\to\infty}\int_{B_{x_0}(2r)}\frac{1}{\lambda_{1,f}(M)}|\nabla\phi_r|^2e^{-f}dv\\ &\leq\lim_{r\to\infty}\frac{4C_2\mathrm{Vol}_f(B_{x_0}(2r))}{r^2}=0\end{aligned}$$

for some constant $C_2>0$. Therefore, this contradiction inserts that h is identically zero. The proof is complete. □

Theorem 4.2. *Let $(M^n,g,e^{-f}\mathrm{d}v)$ be a complete smooth metric measure space. Suppose*

$$Ric_f^m\geq-\delta$$

and $\lambda_{1,f}(M)>\frac{(m-1)\delta\sigma^2}{2(m-1)\sigma-(m-2)}$ with two constants $\delta\geq0$ and $\sigma>\frac{m-2}{2(m-1)}$. Then every f-harmonic 1-form ω on M with $\lim_{r\to\infty}\frac{1}{r^2}\int_{B_{x_0}(r)}|\omega|^{2\sigma}e^{-f}dv=0$ vanishes identically. In particular, $\mathcal{H}_f^1(L^{2\sigma}(M))=0$.

Proof. By (4.1) and the curvature condition, $|\omega|$ satisfies

$$|\omega|\Delta_f|\omega|\geq\frac{1}{m-1}|\nabla|\omega||^2+Ric_f^m(\omega^\#,\omega^\#)\geq\frac{1}{m-1}|\nabla|\omega||^2-\delta|\omega|^2.$$

According to Lemma 4.1, $h=|\omega|$ and $|\omega|$ is identically zero. Therefore, $\omega=0$. □

Theorem 4.3. *Let $(M^n, g, e^{-f}\mathrm{d}v)$, $n \geq 3$ be a complete smooth metric measure space. Assume that*

$$Ric_f^m \geq -\delta,$$

where $0 \leq \delta < \frac{(m-1)\lambda_{1,f}(M)}{m-2}$. If the weighted volume of M satisfies

$$\mathrm{Vol}_f(B(r)) \leq Cr^{\frac{2q(m-1)}{m(q-2)-q+4}} \tag{4.11}$$

for some $C > 0$ and $q > \frac{2(m-2)}{(m-1)}$, then $\mathcal{H}_f^1(L^q(M)) = 0$.

Proof. Given $\omega \in \mathcal{H}_f^1(L^q(M))$, suppose that ω is not identically zero. Denote by $h = |\omega|^{\frac{m-2}{m-1}}$, then, by (4.1) and the curvature condition, we have

$$\begin{aligned}
\Delta_f h &= \frac{m-2}{m-1}|\omega|^{-\frac{1}{m-1}}\Delta_f|\omega| - \frac{m-2}{(m-1)^2}|\omega|^{-\frac{m}{m-1}}|\nabla|\omega||^2 \\
&\geq \frac{m-2}{m-1}|\omega|^{-\frac{m}{m-1}}\Big(\frac{1}{m-1}|\nabla|\omega||^2 - \delta|\omega|^2\Big) - \frac{m-2}{(m-1)^2}|\omega|^{-\frac{m}{m-1}}|\nabla|\omega||^2 \\
&= -\frac{(m-2)\delta}{m-1}h. \qquad (4.12)
\end{aligned}$$

Let $0 \leq \phi \in C_0^\infty(M)$. Using (4.12), we compute

$$\begin{aligned}
&\int_M |\nabla(\phi h)|^2 e^{-f}\mathrm{d}v \\
= &\int_M |\nabla\phi|^2 h^2 e^{-f}\mathrm{d}v + \int_M |\nabla h|^2\phi^2 e^{-f}\mathrm{d}v + \frac{1}{2}\int_M \langle\nabla\phi^2, \nabla h^2\rangle e^{-f}\mathrm{d}v \\
= &\int_M |\nabla\phi|^2 h^2 e^{-f}\mathrm{d}v - \int_M \phi^2 h\Delta_f h e^{-f}\mathrm{d}v \\
\leq &\int_M |\nabla\phi|^2 h^2 e^{-f}\mathrm{d}v + \frac{(m-2)\delta}{m-1}\int_M \phi^2 h^2 e^{-f}\mathrm{d}v.
\end{aligned}$$

From the inequality $\int_M \phi^2 h^2 e^{-f}\mathrm{d}v \leq \frac{1}{\lambda_{1,f}(M)}\int_M |\nabla(\phi h)|^2 e^{-f}\mathrm{d}v$, we have

$$\Big[1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\Big]\int_M |\nabla(\phi h)|^2 e^{-f}\mathrm{d}v \leq \int_M |\nabla\phi|^2 h^2 e^{-f}\mathrm{d}v$$

which can be rewritten as

$$\begin{aligned}
&\Big[1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\Big]\int_M |\nabla h|^2\phi^2 e^{-f}\mathrm{d}v \\
\leq &\frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\int_M |\nabla\phi|^2 h^2 e^{-f}\mathrm{d}v \\
&- 2\Big[1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\Big]\int_M \phi h\langle\nabla\phi, \nabla h\rangle e^{-f}\mathrm{d}v.
\end{aligned}$$

Since $1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)} > 0$, using the Cauchy-Schwarz inequality gives

$$\Big[1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\Big](1-\varepsilon)\int_M |\nabla h|^2\phi^2 e^{-f}\mathrm{d}v$$

$$\leq \Big[\frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)} + \frac{1}{\varepsilon}\Big(1 - \frac{(m-2)\delta}{(m-1)\lambda_{1,f}(M)}\Big)\Big] \int_M |\nabla\phi|^2 h^2 e^{-f} \mathrm{d}v \tag{4.13}$$

for any $\varepsilon > 0$. On the other hand, applying the Cauchy-Schwarz inequality and using (4.11), we get

$$\begin{aligned}\int_{B(2r)} h^2 e^{-f} \mathrm{d}v &= \int_{B(2r)} |\omega|^{\frac{2(m-2)}{m-1}} e^{-f} \mathrm{d}v \\ &\leq \left(\int_{B(2r)} |\omega|^q e^{-f} \mathrm{d}v\right)^{\frac{2(m-2)}{q(m-1)}} \Big(\mathrm{Vol}_f(B(2r))\Big)^{\frac{m(q-2)-q+4}{q(m-1)}} \\ &\leq Cr^2 \left(\int_{B(2r)} |\omega|^q e^{-f} \mathrm{d}v\right)^{\frac{2(m-2)}{q(m-1)}}\end{aligned}$$

which means $\int_{B(2r)} h^2 e^{-f} \mathrm{d}v = o(r^2)$, since $\int_{B(2r)} |\omega|^q e^{-f} \mathrm{d}v < \infty$. Choose ϕ to be the cut-off function defined in (4.10), and substitute it into (4.13), then

$$\int_{B(r)} |\nabla h|^2 e^{-f} \mathrm{d}v \leq \frac{C'}{r^2} \int_{B(2r)} h^2 e^{-f} \mathrm{d}v$$

for some $C' > 0$. Letting $r \to \infty$, we have h is a constant on M. The remaining argument is the same as that of Theorem 4.2. $\square$

References

[1] K. Brighton, *A Liouville-type theorem for smooth metric measure spaces*, J. Geom. Anal. **23** (2013), no. 2, 562–570. https://doi.org/10.1007/s12220-011-9253-5

[2] E. L. Bueler, *The heat kernel weighted Hodge Laplacian on noncompact manifolds*, Trans. Amer. Math. Soc. **351** (1999), no. 2, 683–713. https://doi.org/10.1090/S0002-9947-99-02021-8

[3] D. M. J. Calderbank, P. Gauduchon, and M. Herzlich, *Refined Kato inequalities and conformal weights in Riemannian geometry*, J. Funct. Anal. **173** (2000), no. 1, 214–255. https://doi.org/10.1006/jfan.2000.3563

[4] M. P. Cavalcante, H. Mirandola, and F. A. Vitório, *L^2-harmonic 1-forms on submanifolds with finite total curvature*, J. Geom. Anal. **24** (2014), no. 1, 205–222. https://doi.org/10.1007/s12220-012-9334-0

[5] X. Chao, A. Hui, and M. Bai, *Vanishing theorems for p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality*, Differential Geom. Appl. **76** (2021), Paper No. 101741, 13 pp. https://doi.org/10.1016/j.difgeo.2021.101741

[6] X. Chao and Y. Lv, *L^2 harmonic 1-forms on submanifolds with weighted Poincaré inequality*, J. Korean Math. Soc. **53** (2016), no. 3, 583–595. https://doi.org/10.4134/JKMS.j150190

[7] X. Chao and Y. Lv, *L^p harmonic 1-form on submanifold with weighted Poincaré inequality*, Czechoslovak Math. J. **68(143)** (2018), no. 1, 195–217. https://doi.org/10.21136/CMJ.2018.0415-16

[8] D. F. Cibotaru and P. Zhu, *Refined Kato inequalities for harmonic fields on Kähler manifolds*, Pacific J. Math. **256** (2012), no. 1, 51–66. https://doi.org/10.2140/pjm.2012.256.51

[9] N. T. Dung, *p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality*, Nonlinear Anal. **150** (2017), 138–150. https://doi.org/10.1016/j.na.2016.11.008

[10] N. T. Dung, N. V. Duc, and J. Pyo, *Harmonic 1-forms on immersed hypersurfaces in a Riemannian manifold with weighted Bi-Ricci curvature bounded from below*, J. Math. Anal. Appl. **484** (2020), no. 1, 123693, 29 pp. https://doi.org/10.1016/j.jmaa.2019.123693

[11] N. T. Dung, N. T. Le Hai, and Nguyen Thi Thanh, *Eigenfunctions of the weighted Laplacian and a vanishing theorem on gradient steady Ricci soliton*, J. Math. Anal. Appl. **416** (2014), no. 2, 553–562. https://doi.org/10.1016/j.jmaa.2014.02.054

[12] N. T. Dung and K. Seo, *Vanishing theorems for L^2 harmonic 1-forms on complete submanifolds in a Riemannian manifold*, J. Math. Anal. Appl. **423** (2015), no. 2, 1594–1609. https://doi.org/10.1016/j.jmaa.2014.10.076

[13] N. T. Dung and K. Seo, *p-harmonic functions and connectedness at infinity of complete submanifolds in a Riemannian manifold*, Ann. Mat. Pura Appl. (4) **196** (2017), no. 4, 1489–1511. https://doi.org/10.1007/s10231-016-0625-0

[14] N. T. Dung and C.-J. A. Sung, *Analysis of weighted p-harmonic forms and applications*, Internat. J. Math. **30** (2019), no. 11, 1950058, 35 pp. https://doi.org/10.1142/s0129167x19500587

[15] H.-P. Fu, *Rigidity theorems on smooth metric measure spaces with weighted Poincaré inequality*, Nonlinear Anal. **98** (2014), 1–12. https://doi.org/10.1016/j.na.2013.12.002

[16] Y. Han and H. Lin, *Vanishing theorems for f-harmonic forms on smooth metric measure spaces*, Nonlinear Anal. **162** (2017), 113–127. https://doi.org/10.1016/j.na.2017.06.012

[17] K.-H. Lam, *Results on a weighted Poincaré inequality of complete manifolds*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5043–5062. https://doi.org/10.1090/S0002-9947-10-04894-4

[18] P. Li and J. P. Wang, *Complete manifolds with positive spectrum*, J. Differential Geom. **58** (2001), no. 3, 501–534. http://projecteuclid.org/euclid.jdg/1090348357

[19] J. Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. **78** (2003), no. 4, 865–883. https://doi.org/10.1007/s00014-003-0775-8

[20] O. Munteanu and J. P. Wang, *Smooth metric measure spaces with non-negative curvature*, Comm. Anal. Geom. **19** (2011), no. 3, 451–486. https://doi.org/10.4310/CAG.2011.v19.n3.a1

[21] N. D. Sang and N. T. Thanh, *Stable minimal hypersurfaces with weighted Poincaré inequality in a Riemannian manifold*, Commun. Korean Math. Soc. **29** (2014), no. 1, 123–130. https://doi.org/10.4134/ckms.2014.29.1.123

[22] K. Seo and G. Yun, *Liouville-type theorems for weighted p-harmonic 1-forms and weighted p-harmonic maps*, Pacific J. Math. **305** (2020), no. 1, 291–310. https://doi.org/10.2140/pjm.2020.305.291

[23] Y. Shen and R. Ye, *On stable minimal surfaces in manifolds of positive bi-Ricci curvatures*, Duke Math. J. **85** (1996), no. 1, 109–116. https://doi.org/10.1215/S0012-7094-96-08505-1

[24] M. Vieira, *Harmonic forms on manifolds with non-negative Bakry-Émery-Ricci curvature*, Arch. Math. (Basel) **101** (2013), no. 6, 581–590. https://doi.org/10.1007/s00013-013-0594-0

[25] M. Vieira, *Vanishing theorems for L^2 harmonic forms on complete Riemannian manifolds*, Geom. Dedicata **184** (2016), 175–191. https://doi.org/10.1007/s10711-016-0165-1

[26] P. Wang, X. Chao, Y. Wu, and Y. Lv, *Harmonic p-forms on Hadamard manifolds with finite total curvature*, Ann. Global Anal. Geom. **54** (2018), no. 4, 473–487. https://doi.org/10.1007/s10455-018-9609-1

[27] G. Wei and W. C. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405. https://doi.org/10.4310/jdg/1261495336

[28] J. Y. Wu, *L^p-Liouville theorems on complete smooth metric measure spaces*, Bull. Sci. Math. **138** (2014), no. 4, 510–539. https://doi.org/10.1016/j.bulsci.2013.07.002

[29] G. Yun and K. Seo, *Weighted volume growth and vanishing properties of f-minimal hypersurfaces in a weighted manifold*, Nonlinear Anal. **180** (2019), 264–283. https://doi.org/10.1016/j.na.2018.10.015

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