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THERE ARE NO NUMERICAL RADIUS PEAK n-LINEAR MAPPINGS ON c_0

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Abstract. For $n \geq 2$ and a real Banach space E, $\mathcal{L}(^nE : E)$ denotes the space of all continuous n-linear mappings from E to itself. Let

$$\Pi(E) = \{[x^*, (x_1, \ldots, x_n)] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n\ \}.$$

An element $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^nE : E)$ if $|x^*(T(x_1, \ldots, x_n))| = v(T)$, where the numerical radius $v(T) = \sup_{[y^*, y_1, \ldots, y_n] \in \Pi(E)} \Big| y^*\Big(T(y_1, \ldots, y_n)\Big)\Big|$. For $T \in \mathcal{L}(^nE : E)$, we define

$$\text{Nradius}(T) = \{[x^*, (x_1, \ldots, x_n)] \in \Pi(E) : [x^*, (x_1, \ldots, x_n)] \text{ is a numerical radius point of } T\}.$$

T is called a *numerical radius peak n-linear mapping* if there is a unique $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ such that $\text{Nradius}(T) = \{\pm[x^*, (x_1, \ldots, x_n)]\}$.

In this paper we present explicit formulae for the numerical radius of T for every $T \in \mathcal{L}(^nE : E)$ for $E = c_0$ or l_∞. Using these formulae we show that there are no numerical radius peak mappings of $\mathcal{L}(^nc_0 : c_0)$.

1. Introduction

Let us sketch a brief history of norm or numerical radius attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [4] showed that the Radon-Nikodym property is also sufficient for the denseness of norm or numerical radius attaining polynomials.

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Jiménez-Sevilla and Payá [13] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Choi, Domingo, Kim and Maestre [3] showed that for a scattered compact Hausdorff space K, every continuous n-homogeneous polynomial on $\mathcal{C}(K:\mathbb{C})$ can be approximated by norm attaining ones at extreme points and also that the set of all extreme points of the unit ball of $\mathcal{C}(K:\mathbb{C})$ is a norming set for every continuous complex polynomial. The authors obtained similar results if "norm" is replaced by "numerical radius."

Let $n\in\mathbb{N}$, $n\geq 2$. We write S_E for the unit sphere of a real Banach space E. We denote by $\mathcal{L}(^nE:E)$ the Banach space of all continuous n-linear mappings from E into itself endowed with the norm

$$\|T\|=\sup_{(x_1,\dots,x_n)\in S_E\times\cdots\times S_E}\|T(x_1,\dots,x_n)\|.$$

$\mathcal{L}_s(^nE:E)$ denotes the closed subspace of all continuous symmetric n-linear mappings on E. We let

$$\Pi(E)=\Big\{[x^*,x_1,\dots,x_n]:x^*(x_j)=\|x^*\|=\|x_j\|=1\text{ for }j=1,\dots,n\Big\}.$$

An element $[x^*,x_1,\dots,x_n]\in\Pi(E)$ is called a *numerical radius point* of $T\in\mathcal{L}(^nE:E)$ if $|x^*(T(x_1,\dots,x_n))|=v(T)$, where the numerical radius

$$v(T)=\sup_{[y^*,y_1,\dots,y_n]\in\Pi(E)}\Big|y^*\Big(T(y_1,\dots,y_n)\Big)\Big|.$$

We define

$$\begin{aligned}\text{Nradius}(T)=\{[x^*,(x_1,\dots,x_n)]\in\Pi(E):[x^*,(x_1,\dots,x_n)]\\ \text{is a numerical radius point of }T\}.\end{aligned}$$

Notice that $[x^*,x_1,\dots,x_n]\in\text{Nradius}(T)$ if and only if $[-x^*,-x_1,\dots,-x_n]\in\text{Nradius}(T)$.

Kim [12] classified Nradius(T) for every $T\in\mathcal{L}(^2l_1^2:l_1^2)$, where $l_1^2=\mathbb{R}^2$ with the l_1-norm.

An element $(x_1,\dots,x_n)\in E^n$ is called a *norming point* of $T\in\mathcal{L}(^nE)$ or $\mathcal{L}(^nE:E)$ if $\|x_1\|=\cdots=\|x_n\|=1$ and $\|T\|=\|T(x_1,\dots,x_n)\|$. We define

$$\text{Norm}(T)=\Big\{(x_1,\dots,x_n)\in E^n:(x_1,\dots,x_n)\text{ is a norming point of }T\Big\}.$$

Norm(T) is called the *norming set* of T.

Kim ([7–9]) classified Norm(T) for every $T\in\mathcal{L}_s(^2l_\infty^2)$, $\mathcal{L}(^2l_\infty^2)$ or $\mathcal{L}_s(^3l_1^2)$, respectively.

A mapping $P:E\to\mathbb{C}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E\times\cdots\times E$ such that $P(x)=L(x,\dots,x)$ for every $x\in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\|=\sup_{\|x\|=1}|P(x)|$.

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [5].

An element $[x^*, x] \in \Pi(E)$ is called a *numerical radius point* of $P \in \mathcal{P}(^nE : E)$ if $|x^*(P(x))| = v(P)$, where the numerical radius

$$v(P) = \sup_{[y^*,y]\in\Pi(E)} \Big|y^*(P(y))\Big|.$$

Kim [11] investigated the polynomial numerical index, the symmetric multilinear numerical index and the multilinear numerical index of l_p-spaces.

Similarly, we define

$$\text{Nradius}(P) = \Big\{[x^*, x] \in \Pi(E) : [x^*, x] \text{ is a numerical radius point of } P\Big\}.$$

Notice that $[x^*, x] \in \text{Nradius}(P)$ if and only if $[-x^*, -x] \in \text{Nradius}(P)$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ if $\|x\| = 1$ and $\|P\| = \|P(x)\|$. We define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P.

Kim [6] classified $\text{Norm}(P)$ for every $\mathcal{P}(^2l^2_\infty)$. If $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE : E)$ and $\text{Norm}(T) \neq \emptyset$, T is called a *norm attaining* and if $T \in \mathcal{L}(^nE : E)$ and $\text{Nradius}(T) \neq \emptyset$, T is called a *numerical radius attaining*. Similarly, if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and $\text{Norm}(P) \neq \emptyset$, P is called a *norm attaining* and if $P \in \mathcal{P}(^nE : E)$ and $\text{Nradius}(P) \neq \emptyset$, P is called a *numerical radius attaining* (See [4]).

Choi, Domingo, Kim and Maestre [3] showed that for a scattered compact Hausdorff space K and $n \in \mathbb{N}$, $P \in \mathcal{P}(^n\mathcal{C}(K : \mathbb{C}) : \mathcal{C}(K : \mathbb{C}))$ is norm attaining if and only if it is numerical radius attaining.

Let

$$\text{NA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is norm attaining}\}$$

and

$$\text{NRA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is numerical radius attaining}\}.$$

Kim [10] proved that $\text{NA}(\mathcal{L}(^nl_1 : l_1)) = \text{NRA}(\mathcal{L}(^nl_1 : l_1))$.

In this paper we present explicit formulae for the numerical radius of T for every $T \in \mathcal{L}(^nE : E)$ if E is the real space c_0 or the real space l_∞. Using these formulae we show that there are no numerical radius peak mappings of $\mathcal{L}(^nc_0 : c_0)$.

2. Properties of Nradius(T)

Throughout the paper, we let E be a real Banach space and $n \in \mathbb{N}$, $n \geq 2$. First, we present the following elementary remark:

Remark 2.1. Let a, b, v be real numbers such that $v \geq 0$. Suppose that $v \geq \max\{|a+b|, |a-b|, |a|\}$ and $|b| < |a|$. Then, $v = |a|$ if and only if $v = |a-b| = |a+b|$. Indeed, notice that

$$(*) \qquad v \geq \max\{|a+b|, |a-b|\} = |a| + |b| \geq |a|.$$

($\Rightarrow$) If $v = |a|$, then $b = 0$ by $(*)$ and so $v = |a-b| = |a+b|$.
($\Leftarrow$) If $v = |a-b| = |a+b|$ since $|b| < |a|$, then $b = 0$ and so $v = |a|$.

Proposition 2.2. *Let* $T \in \mathcal{L}(^n E : E)$. *Let* $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ *and* $y \in E$ *be such that* $[x^*, (x_1 \pm y, \ldots, x_n)] \in \Pi(E)$ *and* $|x^*(T(y, x_2, \ldots, x_n))| < |x^*(T(x_1, x_2, \ldots, x_n))|$. *Then* $[x^*, (x_1, \ldots, x_n)] \in \mathrm{Nradius}(T)$ *if and only if* $[x^*, (x_1 \pm y, x_2, \ldots, x_n)] \in \mathrm{Nradius}(T)$.

Proof. ($\Rightarrow$) Notices that $v(T) = |x^*(T(x_1, \ldots, x_n))|$. It follows that

$$\begin{aligned} v(T) \geq\ & \max\Big\{|x^*(T(x_1+y, x_2, \ldots, x_n))|, |x^*(T(x_1-y, x_2, \ldots, x_n))|\Big\} \\ =\ & \max\Big\{|x^*(T(x_1, x_2, \ldots, x_n)) + x^*(T(y, x_2, \ldots, x_n))|, \\ & \qquad |x^*(T(x_1, x_2, \ldots, x_n)) - x^*(T(y, x_2, \ldots, x_n))|\Big\} \\ =\ & |x^*(T(x_1, x_2, \ldots, x_n))| + |x^*(T(y, x_2, \ldots, x_n))| \\ =\ & v(T) + |x^*(T(y, x_2, \ldots, x_n))|, \end{aligned}$$

which implies that $x^*(T(y, x_2, \ldots, x_n)) = 0$ and

$$|x^*(T(x_1 \pm y, x_2, \ldots, x_n))| = |x^*(T(x_1, x_2, \ldots, x_n))| = v(T).$$

Hence, $[x^*, (x_1 \pm y, x_2, \ldots, x_n)] \in \mathrm{Nradius}(T)$.
($\Leftarrow$) Notice that

$$\begin{aligned} v(T) &= |x^*(T(x_1 \pm y, x_2, \ldots, x_n))| \\ &= |x^*(T(x_1, x_2, \ldots, x_n)) \pm x^*(T(y, x_2, \ldots, x_n))|. \end{aligned}$$

Since $|x^*(T(y, x_2, \ldots, x_n))| < |x^*(T(x_1, x_2, \ldots, x_n))|$, we have

$$v(T) = |x^*(T(x_1, \ldots, x_n))|.$$

Hence, $[x^*, (x_1, \ldots, x_n)] \in \mathrm{Nradius}(T)$. □

Proposition 2.3. *Let* $T \in \mathcal{L}(^n E : E)$. *Then* $\mathrm{Nradius}(T)$ *is closed in* $\Pi(E)$ *on the* $(w^* \times \|\cdot\|)$*-topology.*

Proof. Without loss generality we may assume that $\mathrm{Nradius}(T)$ is nonempty.

Let $\Big([x_j^*, (x_1^{(j)}, \ldots, x_n^{(j)})]\Big)_{j \in \mathbb{N}}$ be a sequence in $\mathrm{Nradius}(T)$ which converges to $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ in the $(\mathrm{w}^* \times \|\cdot\|)$-topology. We will show that $v(T) = |x^*(T(x_1, \ldots, x_n))|$. Since $(x_j^*)_{j \in \mathbb{N}}$ converges to x^* in the w*-topology,

$$\lim_{j \to \infty} (x_j^* - x^*)(T(x_1, x_2, \ldots, x_n)) = 0.$$

It follows that

$$\begin{aligned}
&\lim_{j\to\infty} \left|x_j^*(T(x_1^{(j)},\dots,x_n^{(j)})) - x^*(T(x_1,\dots,x_n))\right| \\
\le& \lim_{j\to\infty} \Big[\left|x_j^*(T(x_1^{(j)},\dots,x_n^{(j)})) - x_j^*(T(x_1,x_2^{(j)},\dots,x_n^{(j)}))\right| \\
&+ \left|x_j^*(T(x_1,x_2^{(j)},\dots,x_n^{(j)})) - x_j^*(T(x_1,x_2,x_3^{(j)},\dots,x_n^{(j)}))\right| \\
&+ \dots + \left|x_j^*(T(x_1,x_2,\dots,x_{n-1},x_n^{(j)})) - x_j^*(T(x_1,x_2,\dots,x_n))\right| \\
&+ \left|x_j^*(T(x_1,x_2,\dots,x_n)) - x^*(T(x_1,x_2,\dots,x_n))\right|\Big] \\
=& \lim_{j\to\infty} \Big[\left|x_j^*(T(x_1^{(j)}-x_1,x_2,\dots,x_n^{(j)}))\right| + \left|x_j^*(T(x_1,x_2^{(j)}-x_2,x_3^{(j)},\dots,x_n^{(j)}))\right| \\
&+ \dots + \left|x_j^*(T(x_1,x_2,\dots,x_{n-1},x_n^{(j)}-x_n))\right| \\
&+ \left|(x_j^*-x^*)(T(x_1,x_2,\dots,x_n))\right|\Big] \\
\le& \|T\| \lim_{j\to\infty} (\sum_{k=1}^n \left\|x_k^{(j)} - x_k\right\|) = 0.
\end{aligned}$$

Hence,

$$\lim_{j\to\infty} x_j^*(T(x_1^{(j)},\dots,x_n^{(j)})) = x^*(T(x_1,\dots,x_n)).$$

Therefore,

$$v(T) = \lim_{j\to\infty} \left|x_j^*(T(x_1^{(j)},\dots,x_n^{(j)}))\right| = |x^*(T(x_1,\dots,x_n))|.$$

We complete the proof. □

The following theorem shows some relation between Nradius(T) and Norm(T).

Proposition 2.4. *Let $T \in \mathcal{L}(^nE : E)$ be such that $v(T) = \|T\|$. If* Nradius(T) *is nonempty, then* Norm(T) *is nonempty.*

Proof. Let $[x^*, (x_1,\dots,x_n)] \in \text{Nradius}(T)$. Then

$$\begin{aligned}
\|T\| &= v(T) \\
&= |x^*(T(x_1,\dots,x_n))| \\
&\le \|x^*\|\ \|T(x_1,\dots,x_n)\| \\
&= \|T(x_1,\dots,x_n)\| \\
&\le \|T\|\ \|x_1\|\cdots\|x_n\| = \|T\|,
\end{aligned}$$

which shows that $\|T\| = |T(x_1,\dots,x_n)\|$. Hence $(x_1,\dots,x_n) \in \text{Norm}(T)$. □

Proposition 2.5. *Let $T \in \mathcal{L}(^nE : E)$ and $[x^*, (x_1,\dots,x_n)] \in$* Nradius($T$). *Let $f_j \in E^*$ be such that $f_j(x_j) = \|f_j\| = 1$ for $j = 1,\dots,n$. Let A be a* *nonempty subset of* $\{1, \ldots, n\}$ *with* $\{j_1, \ldots, j_m\}$ *for some* $1 \leq m \leq n$. *We define* $T_{A,f_{j_1},\ldots,f_{j_m}} \in \mathcal{L}(^{m+n}E : E)$ *by*

$$T_{A,f_{j_1},\ldots,f_{j_m}}(y_1, \ldots, y_{m+n}) := \prod_{1\leq k\leq m} f_{j_k}(y_k)\ T(y_{m+1}, \ldots, y_{m+n}).$$

Then, $v(T_{A,f_{j_1},\ldots,f_{j_m}}) = v(T)$ *and*

$$\Big[x^*, (x_{j_1}, \ldots, x_{j_m}, x_1, \ldots, x_n)\Big] \in \text{Nradius}(T_{A,f_{j_1},\ldots,f_{j_m}}).$$

Proof. Obviously, $T_{A,f_{j_1},\ldots,f_{j_m}} \in \mathcal{L}(^{m+n}E : E)$. Let $y^* \in E^*, y_1, \ldots, y_{m+n} \in E$ be such that

$$y^*(y_j) = \|y^*\| = \|y_j\| = 1 \text{ for } j = 1, \ldots, m+n.$$

It follows that

$$\begin{aligned}
&\Big|y^*(T_{A,f_{j_1},\ldots,f_{j_m}}(y_1, \ldots, y_{m+n}))\Big| \\
&= \prod_{1\leq k\leq m} |f_{j_k}(y_k)|\ |y^*(T(y_{m+1}, \ldots, y_{m+n}))| \\
&\leq |y^*(T(y_{m+1}, \ldots, y_{m+n}))| \\
&\leq v(T) \\
&= |x^*(T(x_1, \ldots, x_n))| \\
&= |x^*(T_{A,f_{j_1},\ldots,f_{j_m}}(x_{j_1}, \ldots, x_{j_m}, x_1, \ldots, x_n))| \\
&\leq v(T_{A,f_{j_1},\ldots,f_{j_m}}),
\end{aligned}$$

which shows that $v(T_{A,f_{j_1},\ldots,f_{j_m}}) = v(T)$ and

$$\Big[x^*, (x_{j_1}, \ldots, x_{j_m}, x_1, \ldots, x_n)\Big] \in \text{Nradius}(T_{A,f_{j_1},\ldots,f_{j_m}}). \qquad \square$$

3. There are no numerical radius peak mappings of $\mathcal{L}(^n c_0 : c_0)$

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of l_∞ and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$.

We present explicit formulae for the numerical radius $v(T)$ for every $T \in \mathcal{L}(^nE : E)$, where $E = c_0$ or l_∞.

Theorem 3.1. *Let* $E = c_0$ *or* l_∞ *and* $T \in \mathcal{L}(^nE : E)$ *with* $T = (T_k)_{k\in\mathbb{N}}$. *For each* $k \in \mathbb{N}$, *the following equalities hold*:

$$\begin{aligned}
v(T) &= \|T\| = \sup_{k\in\mathbb{N}} \|T_k\| \\
&= \sup_{k\in\mathbb{N}} \Big(\sup \Big\{ |T_k(X_1, \ldots, X_n)| : e_k^*(X_j) = 1 = \|X_j\|, j = 1, \ldots, n \Big\} \Big).
\end{aligned}$$

Proof. We prove the theorem only for $E = l_\infty$. The proof for the case $E = c_0$ is similar.

Claim. $\|T_k\| = \sup\Big\{|T_k(X_1,\dots,X_n)| : e_k^*(X_j) = 1 = \|X_j\|,\ j = 1,\dots,n\Big\}$ for every $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and $M_k := \sup\Big\{|T_k(X_1,\dots,X_n)| : e_k^*(X_j) = 1 = \|X_j\|,\ j = 1,\dots,n\Big\}$. Let $X_i = (x_j^{(i)})_{j\in\mathbb{N}} \in S_{l_\infty}$ for $i = 1,\dots,n$. Then for $i = 1,\dots,n$, there are $l^{(i)}, s^{(i)} \in [0,1]$ such that $l^{(i)} + s^{(i)} = 1$ and $x_k^{(i)} = l^{(i)}(1) + s^{(i)}(-1)$. It follows that

$$
\begin{aligned}
&|T_k(X_1,\dots,X_n)| \\
&= \Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)}, l^{(1)}(1)+s^{(1)}(-1), x_{k+1}^{(1)},\dots),\dots,\\
&\qquad (x_1^{(n)},\dots,x_{k-1}^{(n)}, l^{(n)}(1)+s^{(n)}(-1), x_{k+1}^{(n)},\dots)\Big)\Big| \\
&\le l^{(1)}\cdots l^{(n)}\Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)},1,x_{k+1}^{(1)},\dots),\dots,(x_1^{(n)},\dots,x_{k-1}^{(n)},1,x_{k+1}^{(n)},\dots)\Big)\Big| \\
&\quad + l^{(1)}\cdots l^{(n-1)}s^{(n)}\Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)},1,x_{k+1}^{(1)},\dots),\dots,\\
&\quad (x_1^{(n-1)},\dots,x_{k-1}^{(n-1)},1,x_{k+1}^{(n-1)},\dots),(x_1^{(n)},\dots,x_{k-1}^{(n)},-1,x_{k+1}^{(n)},\dots)\Big)\Big| + \cdots \\
&\quad + s^{(1)}\cdots s^{(n)}\Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)},-1,x_{k+1}^{(1)},\dots),\dots,\\
&\quad (x_1^{(n)},\dots,x_{k-1}^{(n)},-1,x_{k+1}^{(n)},\dots)\Big)\Big| \\
&= l^{(1)}\cdots l^{(n)}\Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)},1,x_{k+1}^{(1)},\dots),\dots,(x_1^{(n)},\dots,x_{k-1}^{(n)},1,x_{k+1}^{(n)},\dots)\Big)\Big| \\
&\quad + l^{(1)}\cdots l^{(n-1)}s^{(n)}\Big|T_k\Big((x_1^{(1)},\dots,x_{k-1}^{(1)},1,x_{k+1}^{(1)},\dots),\dots,\\
&\quad (x_1^{(n-1)},\dots,x_{k-1}^{(n-1)},1,x_{k+1}^{(n-1)},\dots),(-x_1^{(n)},\dots,-x_{k-1}^{(n)},1,-x_{k+1}^{(n)},\dots)\Big)\Big| + \cdots \\
&\quad + s^{(1)}\cdots s^{(n)}\Big|T_k\Big((-x_1^{(1)},\dots,-x_{k-1}^{(1)},1,-x_{k+1}^{(1)},\dots),\dots,\\
&\quad (-x_1^{(n)},\dots,-x_{k-1}^{(n)},1,-x_{k+1}^{(n)},\dots)\Big)\Big| \\
&\le \prod_{i=1}^{n}(l^{(i)}+s^{(i)})M_k = M_k \le \|T_k\|,
\end{aligned}
$$

which shows the claim.

By the claim, it follows that for $k \in \mathbb{N}$,

$$
\begin{aligned}
v(T) &\ge \sup\Big\{|e_k^*(T(X_1,\dots,X_n))| : e_k^*(X_j) = 1 = \|X_j\|,\ j = 1,\dots,n\Big\} \\
&= \sup\Big\{|T_k(X_1,\dots,X_n)| : e_k^*(X_j) = 1 = \|X_j\|,\ j = 1,\dots,n\Big\} = \|T_k\|,
\end{aligned}
$$

which implies that

$$\|T\| \ge v(T) \ge \sup_{k\in\mathbb{N}}\|T_k\| \ge \|T\|.$$

We completes the proof. $\square$

We are in a position to show the main result of this paper.

Theorem 3.2. *Let* $T \in \mathcal{L}(^n c_0 : c_0)$. *If* Nradius$(T)$ *is nonempty, then it is infinite. Consequently, there are no numerical radius peak n-linear mappings of* $\mathcal{L}(^n c_0 : c_0)$.

Proof. Let $[z^*, (Y_1, \ldots, Y_n)] \in \text{Nradius}(T)$. It follows that by Theorem 3.1,

$$\|T\| = v(T) = \Big|z^*\Big(T(Y_1, \ldots, Y_n)\Big)\Big| \leq \Big\|T(Y_1, \ldots, Y_n)\Big\| \leq \|T\|,$$

which shows that $(Y_1, \ldots, Y_n) \in \text{Norm}(T)$. Since $Y_1 \in S_{c_0}$, there are $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\|Y_1 \pm \lambda e_N\| = 1 \text{ for all } 0 < \lambda \leq \delta.$$

Claim 1. $[z^*, (Y_1 \pm \lambda e_N, Y_2, \ldots, Y_n)] \in \Pi(c_0)$ for all $0 < \lambda \leq \delta$.

Indeed,

$$\begin{aligned} 1 &\geq \max\Big\{|z^*(Y_1 + \lambda e_N)|,\ |z^*(Y_1 - \lambda e_N)|\Big\} \\ &= \max\Big\{|z^*(Y_1) + \lambda z^*(e_N)|,\ |z^*(Y_1) - \lambda z^*(e_N)|\Big\} \\ &= |z^*(Y_1)| + \lambda|z^*(e_N)| = 1 + \lambda|z^*(e_N)|, \end{aligned}$$

so $z^*(e_N) = 0$. Hence, $[z^*, (Y_1 \pm \lambda e_N, Y_2, \ldots, Y_n)] \in \Pi(c_0)$ for all $0 < \lambda \leq \delta$.

Claim 2. $[z^*, (Y_1 \pm \lambda e_N, Y_2, \ldots, Y_n)] \in \text{Nradius}(T)$ for all $0 < \lambda \leq \delta$.

It follows that

$$\begin{aligned} v(T) \geq\ &\max\Big\{\Big|z^*\Big(T(Y_1 + \lambda e_N, \ldots, Y_n)\Big)\Big|,\ \Big|z^*\Big(T(Y_1 - \lambda e_N, \ldots, Y_n)\Big)\Big|\Big\} \\ =\ &\max\Big\{\Big|z^*\Big(T(Y_1, \ldots, Y_n)\Big) + \lambda\ z^*\Big(T(e_N, Y_2, \ldots, Y_n)\Big)\Big|, \\ &\qquad \Big|z^*\Big(T(Y_1, \ldots, Y_n)\Big) - \lambda\ z^*\Big(T(e_N, Y_2, \ldots, Y_n)\Big)\Big|\Big\} \\ =\ &\Big|z^*\Big(T(Y_1, \ldots, Y_n)\Big)\Big| + \lambda\ \Big|z^*\Big(T(e_N, Y_2, \ldots, Y_n)\Big)\Big| \\ =\ &v(T) + \lambda\ \Big|z^*\Big(T(e_N, Y_2, \ldots, Y_n)\Big)\Big|, \end{aligned}$$

which shows that $\Big|z^*\Big(T(e_N, Y_2, \ldots, Y_n)\Big)\Big| = 0$. Therefore, for all $0 < \lambda \leq \delta$,

$$v(T) = \Big|z^*\Big(T(Y_1 \pm \lambda e_N, \ldots, Y_n)\Big)\Big|,$$

which concludes the claim 2. Therefore, we complete the proof. □

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