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A NEW q-ANALOGUE OF VAN HAMME'S (G.2) SUPERCONGRUENCE FOR PRIMES $p \equiv 3 \pmod 4$

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Abstract. Van Hamme's (G.2) supercongruence modulo p^4 for primes $p \equiv 3 \pmod 4$ and $p > 3$ was first established by Swisher. A q-analogue of this supercognruence was implicitly given by the first author and Schlosser. In this paper, we present a new q-analogue of Van Hamme's (G.2) supercongruence for $p \equiv 3 \pmod 4$.

1. Introduction

In his first letter to Hardy in 1913, Ramanujan asserted that (see [2, p. 25, eq. (2)]):

$$\sum_{k=0}^{\infty}(8k+1)\frac{(\frac14)_k^4}{k!^4}=\frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma(\frac34)^2} \tag{1}$$

without proof. Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol and $\Gamma(x)$ is the Gamma function. The formula (1) was later proved by Hardy [10]. In 1997, Van Hamme [9] listed thirteen p-adic analogues of Ramanujan-type series, such as: for $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/4}(8k+1)\frac{(\frac14)_k^4}{k!^4}\equiv p\frac{\Gamma_p(\frac12)\Gamma_p(\frac14)}{\Gamma_p(\frac34)} \pmod{p^3} \tag{2}$$

(tagged (G.2) in Van Hamme's list). Here and in what follows, p is an odd prime and $\Gamma_p(x)$ denotes the p-adic Gamma function [19]. Swisher [20] and He [11] proved that (2) is true modulo the higher power p^4. Swisher [20, (3)] also proved the following generalization of Van Hamme's (G.2) supercongruence: for $p \equiv 3 \pmod 4$ and $p > 3$,

$$\sum_{k=0}^{(3p-1)/4}(8k+1)\frac{(\frac14)_k^4}{k!^4}\equiv -\frac{3p^2\Gamma_p(\frac12)\Gamma_p(\frac14)}{2\Gamma_p(\frac34)} \pmod{p^4}. \tag{3}$$

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(The factor (-1) was neglected by Swisher in her original supercongruence.)

In the past few years, q-analogues of Van Hamme's supercongruences have been widely studied. For example, the first author and Schlosser [5, Corollary 1.2 with $d=4$] gave the following q-analogue of (3): for $n\equiv 3 \pmod 4$,

$$\begin{aligned}\sum_{k=0}^{M}&[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\\ &\equiv \frac{(q^2;q^4)_{(3n-1)/4}}{(q^4;q^4)_{(3n-1)/4}}[3n]q^{(1-3n)/4} \pmod{[n]\Phi_n(q)^3},\end{aligned}\tag{4}$$

where $M=(3n-1)/4$ or $n-1$. Here, the *q-shifted factorial* is defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n=1,2,\dots$. For convenience, we also adopt the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$. Moreover, the *$q$-integer* is defined as $[n]=[n]_q=(1-q^n)/(1-q)$, and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial*, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Liu and Wang [15] showed that Van Hamme's original (G.2) supercongruence can be deduced from the following q-supercongruence: for $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{M}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4} \pmod{[n]\Phi_n(q)^2},\tag{5}$$

where $M=(n-1)/4$ or $n-1$. Very recently, Liu and Wang [17] gave a generalization of (5) modulo $[n]\Phi_n(q)^3$. For another generalization of (5), see [5, Theorem 4.3]. Liu and Wang [15] also established the following q-supercongruence: for $n\equiv 1 \pmod 4$,

$$\begin{aligned}\sum_{k=0}^{M}&[8k+1]_{q^2}[8k+1]^2\frac{(q^2;q^8)_k^4}{(q^8;q^8)_k^4}q^{-4k}\\ &\equiv -\frac{2(q^4;q^8)_{(n-1)/4}}{(1+q^2)(q^8;q^8)_{(n-1)/4}}[n]_{q^2}q^{(3-n)/2} \pmod{[n]_{q^2}\Phi_n(q^2)^2},\end{aligned}\tag{6}$$

where $M=(n-1)/4$ or $n-1$.

It is easy to see that the $n=p$ and $q\to -1$ case of (6) reduces to (2). Moreover, letting $n=p$ and $q\to 1$ in (6), Liu and Wang obtained the following new supercongruence: for $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/4}(8k+1)^3\frac{(\frac14)_k^4}{k!^4}\equiv -p\frac{\Gamma_p(\frac12)\Gamma_p(\frac14)}{\Gamma_p(\frac34)} \pmod{p^3}.$$

In this paper, we shall establish the following new q-analogue of (3).

Theorem 1.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{M}[8k+1]_{q^2}[8k+1]^2\frac{(q^2;q^8)_k^4}{(q^8;q^8)_k^4}q^{-4k} \equiv -\frac{2(q^4;q^8)_{(3n-1)/4}}{(1+q^2)(q^8;q^8)_{(3n-1)/4}}[3n]_{q^2}q^{(3-3n)/2} \pmod{[n]_{q^2}\Phi_n(q^2)^3}, \tag{7}$$

where $M=(3n-1)/4$ or $n-1$.

For some other recent work on q-supercongruenes, see [1,6–8,12–14,16,21,22].

To see that the q-supercongruences (4) and (7) are indeed q-analogues of (3), we need to prove the following result.

Proposition 1.2. *Let $p \equiv 3 \pmod 4$ and $p>3$. Then*

$$\frac{(\frac{1}{2})_{(3p-1)/4}}{(1)_{(3p-1)/4}} \equiv -\frac{p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{2\Gamma_p(\frac{3}{4})} \pmod{p^3}. \tag{8}$$

It is easy to see that the $n=p$ and $q\to -1$ case of (7) reduces to (3). Meanwhile, taking $n=p$ and $q\to 1$ in (7), we get the following new result: for $p\equiv 3 \pmod 4$ and $p>3$,

$$\sum_{k=0}^{(3p-1)/4}(8k+1)^3\frac{(\frac{1}{4})_k^4}{k!^4} \equiv \frac{3p^2\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{2\Gamma_p(\frac{3}{4})} \pmod{p^4}.$$

We shall prove Theorem 1.1 in the next section employing the method of 'creative microscoping', introduced by the first author and Zudilin [7]. A simple proof of Proposition 1.2 using properties of the p-adic Gamma function will be given in Section 3.

2. Proof of Theorem 1.1

We will make use of Watson's ${}_8\phi_7$ transformation formula (see [3, Appendix (III.18)]):

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\ &= \frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right],\end{aligned} \tag{9}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_r\cdots,a_{r+1};q)_k}{(q,b_1,\dots,b_r;q)_k}z^k.$$

We shall also utilize the following easily proved q-congruence due to the first author and Schlosser [4, Lemma 3].

Lemma 2.1. *Let d, m and n be positive integers with $m \le n-1$ and $dm \equiv -1 \pmod{n}$. Then, for $0 \le k \le m$, we have*

$$\frac{(aq;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k} \frac{(aq;q^d)_k}{(q^d/a;q^d)_k} q^{m(dm-d+2)/2+(d-1)k} \pmod{\Phi_n(q)}.$$

We first present the following q-congruence with two parameters a and b.

Theorem 2.2. *Let $n \equiv 3 \pmod 4$ be a positive integer, and let a, b be indeterminates. Then, modulo $\Phi_n(q^2)(1-aq^{6n})(a-q^{6n})$,*

$$\begin{aligned}&\sum_{k=0}^{(3n-1)/4} [8k+1]_{q^2}[8k+1]^2 \frac{(aq^2,q^2/a,q^2/b,q^2;q^8)_k}{(aq^8,q^8/a,bq^8,q^8;q^8)_k}\left(\frac{b}{q^4}\right)^k \\ &\equiv b^{(3n-1)/4} q^{(1-3n)/2}[3n]_{q^2}\frac{(q^4/b;q^8)_{(3n-1)/4}}{(bq^8;q^8)_{(3n-1)/4}} \\ &\quad\times\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right).\end{aligned} \tag{10}$$

Proof. For $a=q^{-6n}$ or $a=q^{6n}$, the left-hand side of (10) is equal to

$$\begin{aligned}&\sum_{k=0}^{(3n-1)/4} [8k+1]_{q^2}[8k+1]^2 \frac{(q^{2-6n},q^{2+6n},q^2/b,q^2;q^8)_k}{(q^{8-6n},q^{8+6n},bq^8,q^8;q^8)_k} b^k q^{-4k} \\ &= {}_8\phi_7\left[\begin{matrix} q^2, & q^9, & -q^9, & q^9, & q^9, & q^2/b, & q^{2+6n}, & q^{2-6n} \\ & q, & -q, & q, & q, & bq^8, & q^{8-6n}, & q^{8+6n}\end{matrix}; q^8, bq^{-4}\right].\end{aligned} \tag{11}$$

By Watson's ${}_8\phi_7$ transformation formula (9), the right-hand side of (11) can be written as

$$\begin{aligned}&\frac{(q^{10},bq^{6-6n};q^8)_{(3n-1)/4}}{(bq^8,q^{8-6n};q^8)_{(3n-1)/4}}\,{}_4\phi_3\left[\begin{matrix} q^{-8}, & q^2/b, & q^{2+2n}, & q^{2-2n} \\ & q,\ q,\ q^4/b & & \end{matrix}; q^8, q^8\right] \\ &= b^{(3n-1)/4} q^{(1-3n)/2}[3n]_{q^2}\frac{(q^4/b;q^8)_{(3n-1)/4}}{(bq^8;q^8)_{(3n-1)/4}} \\ &\quad\times\left(1-\frac{(1-q^{2-2n})(1-q^{2+2n})(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right).\end{aligned} \tag{12}$$

This means that (10) holds modulo $1-aq^{6n}$ and $a-q^{6n}$.

Moreover, setting $q \mapsto q^2$, $d=4$, and $m=(3n-1)/4$ in Lemma 2.1, for $0 \le k \le m$, we have

$$\frac{(aq^2;q^8)_{m-k}}{(q^8/a;q^8)_{m-k}} \equiv (-a)^{m-2k} \frac{(aq^2;q^8)_k}{(q^8/a;q^8)_k} q^{2m(2m-1)+6k} \pmod{\Phi_n(q^2)}.$$

Using this q-congruence, we can easily verify that the k-th and $((3n-1)/4-k)$-th summands on the left-hand side of (10) modulo $\Phi_n(q^2)$ cancel each other for $0 \le k \le (3n-1)/4$. This proves that the left-hand side of (10) is congruent to 0 modulo $\Phi_n(q^2)$, and so (10) is true modulo $\Phi_n(q^2)$.

The proof then follows from the fact that $1-aq^{6n}$, $a-q^{6n}$, and $\Phi_n(q^2)$ are pairwise coprime polynomials in q. $\square$

We also need a simpler q-congruence as follows.

Theorem 2.3. *Let $n\equiv 3 \pmod 4$ be a positive integer, and let a, b be indeterminates. Then, modulo $b-q^{6n}$,*

$$\begin{aligned}\sum_{k=0}^{(3n-1)/4}&[8k+1]_{q^2}[8k+1]^2\frac{(aq^2,q^2/a,q^2/b,q^2;q^8)_k}{(aq^8,q^8/a,bq^8,q^8;q^8)_k}\left(\frac{b}{q^4}\right)^k\\ &\equiv\frac{[3n]_{q^2}(q^2,q^6;q^8)_{(3n-1)/4}}{(aq^8,q^8/a;q^8)_{(3n-1)/4}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right).\end{aligned}\tag{13}$$

Proof. For $b=q^{6n}$, the left-hand side of (13) is equal to

$$\begin{aligned}\sum_{k=0}^{(3n-1)/4}&[8k+1]_{q^2}[8k+1]^2\frac{(aq^2,q^2/a,q^{2-6n},q^2;q^8)_k}{(aq^8,q^8/a,q^{8+6n},q^8;q^8)_k}q^{(6n-4)k}\\ &={}_8\phi_7\left[\begin{matrix}q^2, & q^9, & -q^9, & q^9, & q^9, & aq^2, & q^2/a, & q^{2-6n}\\ & q, & -q, & q, & q, & q^8/a, & aq^8, & q^{8+6n}\end{matrix};q^8,\,q^{6n-4}\right].\end{aligned}\tag{14}$$

In view of Watson's transformation (9), we can write the right-hand side of (14) as

$$\begin{aligned}&\frac{(q^{10},q^6;q^8)_{(3n-1)/4}}{(aq^8,q^8/a;q^8)_{(3n-1)/4}}\,{}_4\phi_3\left[\begin{matrix}q^{-8}, & aq^2, & q^2/a, & q^{2-6n}\\ & q, & q, & q^{4-6n}\end{matrix};q^8,\,q^8\right]\\ &=\frac{[3n]_{q^2}(q^2,q^6;q^8)_{(3n-1)/4}}{(aq^8,q^8/a;q^8)_{(3n-1)/4}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^{2-6n})}{(1-q)^2(1-q^{4-6n})}\right).\end{aligned}\tag{15}$$

This proves that the congruence (13) is true modulo $b-q^{6n}$. $\square$

We are now able to establish the following parametric generalization of Theorem 1.1.

Theorem 2.4. *Let $n\equiv 3 \pmod 4$ be a positive integer, and let a be an indeterminate. Then, modulo $\Phi_n(q^2)^2(1-aq^{6n})(a-q^{6n})$,*

$$\begin{aligned}\sum_{k=0}^{(3n-1)/4}&[8k+1]_{q^2}[8k+1]^2\frac{(aq^2,q^2/a,q^2,q^2;q^8)_k}{(aq^8,q^8/a,q^8,q^8;q^8)_k}q^{-4k}\\ &\equiv q^{(1-3n)/2}[3n]_{q^2}\frac{(q^4;q^8)_{(3n-1)/4}}{(q^8;q^8)_{(3n-1)/4}}\left(1-\frac{(1-aq^2)(1-q^2/a)}{(1-q)^2(1+q^2)}\right).\end{aligned}\tag{16}$$

Proof. It is obvious that $\Phi_n(q^2)(1-aq^{6n})(a-q^{6n})$ and $b-q^{6n}$ are relatively prime polynomials. Employing the Chinese reminder theorem for coprime polynomials, we can determine the remainder of the left-hand side of (10) modulo

$\Phi_n(q^2)(1-aq^{6n})(a-q^{6n})(b-q^{6n})$ from (10) and (13):

$$\begin{aligned}
&\sum_{k=0}^{(3n-1)/4}[8k+1]_{q^2}[8k+1]^2\frac{(aq^2,q^2/a,q^2/b,q^2;q^8)_k}{(aq^8,q^8/a,bq^8,q^8;q^8)_k}\left(\frac{b}{q^4}\right)^k\\
&\equiv b^{(3n-1)/4}q^{(1-3n)/2}[3n]_{q^2}\frac{(q^4/b;q^8)_{(3n-1)/4}}{(bq^8;q^8)_{(3n-1)/4}}\\
&\quad\times\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right)\frac{(b-q^{6n})(ab-1-a^2+aq^{6n})}{(a-b)(1-ab)}\\
&\quad+\frac{[3n]_{q^2}(q^2,q^6;q^8)_{(3n-1)/4}}{(aq^8,q^8/a;q^8)_{(3n-1)/4}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right)\\
&\quad\times\frac{(1-aq^{6n})(a-q^{6n})}{(a-b)(1-ab)}\pmod{\Phi_n(q^2)(1-aq^{6n})(a-q^{6n})(b-q^{6n})}.
\end{aligned}\tag{17}$$

Here we have used the following q-congruences:

$$\frac{(b-q^{6n})(ab-1-a^2+aq^{6n})}{(a-b)(1-ab)}\equiv 1\pmod{(1-aq^{6n})(a-q^{6n})},$$

$$\frac{(1-aq^{6n})(a-q^{6n})}{(a-b)(1-ab)}\equiv 1\pmod{b-q^{6n}}.$$

Note that $1-q^{6n}$ contains the factor $\Phi_n(q^2)$ and so do $(q^4;q^8)_{(3n-1)/4}$ and $(q^6;q^8)_{(3n-1)/4}$ since they have the factors $1-q^{4n}$ and $1-q^{2n}$, respectively. Moreover, the factor $(bq^8;q^8)_{(3n-1)/4}$ in the denominators of both sides of (17) is relatively prime to $\Phi_n(q^2)$ when $b=1$. Thus, letting $b=1$ in (17) and observing that

$$(1-q^{6n})(1+a^2-a-aq^{6n})=(1-a)^2+(1-aq^{6n})(a-q^{6n}),$$

we see that the right-hand of (17) reduces to

$$q^{(1-3n)/2}[3n]_{q^2}\frac{(q^4;q^8)_{(3n-1)/4}}{(q^8;q^8)_{(3n-1)/4}}\left(1-\frac{(1-aq^2)(1-q^2/a)}{(1-q)^2(1+q^2)}\right)\pmod{\Phi_n(q^2)^2(1-aq^{6n})(a-q^{6n})},$$

as desired. □

Proof of Theorem 1.1. Taking $a=1$ in (16), we know that the q-congruence (7) holds modulo $\Phi_n(q^2)^4$ for $M=(3n-1)/4$. It is easy to see that $(q^2;q^8)_k^4/(q^8;q^8)_k^4$ is congruent to 0 modulo $\Phi_n(q^2)^4$ for any k in the range $(3n-1)/4<k\le n-1$. Therefore, the q-congruence (7) also holds modulo $\Phi_n(q^2)^4$ for $M=n-1$.

Moreover, similarly to the proof of [5, Lemma 2.2], we can prove that (7) holds modulo $[n]_{q^2}$. Since the least common multiple of $[n]_{q^2}$ and $\Phi_n(q^2)^4$ is $[n]_{q^2}\Phi_n(q^2)^3$, we complete the proof of the theorem. □

3. Proof of Proposition 1.2

We first list some basic properties of Morita's p-adic Gamma function. Let p be an odd prime. Set $\Gamma_p(0)=1$, and for all integers $n\geq 1$, the p-adic Gamma function is defined as

$$\Gamma_p(n)=(-1)^n\prod_{\substack{0<k<n\\ p\nmid k}}k.$$

Let $\mathbb{Z}_p$ denote the ring of all p-adic integers. Extend Γ_p to all $x\in\mathbb{Z}_p$ by defining

$$\Gamma_p(x)=\lim_{x_n\to x}\Gamma_p(x_n),$$

where x_n is any sequence of positive integers p-adically approaching x. The following facts can be found in [18]: for any $x\in\mathbb{Z}_p$,

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases}-x, & p\nmid x,\\ -1, & p\mid x.\end{cases}\tag{18}$$

$$\Gamma_p(x)\Gamma_p(1-x)=(-1)^{a_0(x)},\tag{19}$$

where $a_0(x)\in\{1,2,\ldots,p\}$ satisfies $a_0(x)\equiv x \pmod p$.

In order to prove Proposition 1.2, we also need the following result (see [18, Theorem 14]).

Lemma 3.1. *For any odd prime p and $a,m\in\mathbb{Z}_p$, we have*

$$\Gamma_p(a+mp)\equiv\Gamma_p(a)+\Gamma_p'(a)mp \pmod{p^2}.\tag{20}$$

Proof of Proposition 1.2. By the properties (18)–(20), for $p\equiv 3 \pmod 4$ and $p>3$,

$$\begin{aligned}\frac{(\frac12)_{(3p-1)/4}}{(1)_{(3p-1)/4}}=\frac{p}{2}\frac{\Gamma_p(1)\Gamma_p(\frac{3p+1}{4})}{\Gamma_p(\frac12)\Gamma_p(\frac{3p+3}{4})}&=(-1)^{(3p+3)/4}\frac{p\Gamma_p(1)\Gamma_p(\frac{3p+1}{4})\Gamma_p(\frac{1-3p}{4})}{2\Gamma_p(\frac12)}\\ &\equiv(-1)^{(3p+3)/4}\frac{p\Gamma_p(1)\Gamma_p(\frac14)^2}{2\Gamma_p(\frac12)}\\ &\equiv\frac{p\Gamma_p(1)\Gamma_p(\frac14)}{2\Gamma_p(\frac12)\Gamma_p(\frac34)} \pmod{p^3}.\end{aligned}$$

Noticing that $\Gamma_p(1)=-1$ and $\Gamma_p(\frac12)^2=(-1)^{\frac{p+1}{2}}=1$, we complete the proof. □

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